Class Feb 14

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Linear Algebra

When you're in a vector space you have 2 fundamental questions:

① Is $\vec{v} \in \text{span}\{\vec{v}_1, \dots \vec{v}_n\}$?

Solution: $a_1\vec{v}_1 + \dots a_n\vec{v}_n = \vec{v}$

variables unknowns

inconsistent no solution NO

consistent at least 1 solution YES

② Is $\{v_1, v_2, \dots v_n\}$ LI?

Solution Solve

$a_1 v_1 \quad a_n v_n$

variables/unknowns

a unique solution YES

infinitely many solutions NO

A set $\{\vec{v}_1, \dots \vec{v}_n\}$ is LD if and only if at least one vector $\vec{v}_i$ is in the span of the rest

$\Rightarrow$ If your spanning set is LD, then you can remove redundant vectors (one at a time) <u>without changing the total span</u> until you have a LI spanning set.

Flip Side Theorem (Enlarging LI sets)

Suppose $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$ is a LI set in a vector space V. Then is ~~iff~~ only if

$v \notin \text{span}\{v_1, \dots \vec{v}_n\}$

eg $\left\{\begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}\right\} \subseteq M_{22}$

This is LI the matrix $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} \notin \text{span} \dots$

$\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\right\}$ is LI

eg $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\} \in \mathbb{R}^3$

Is this LI?

Can I find another vector to make a bigger LI set? (with a bigger span)

By theorem: need $\vec{v} \notin \text{span}\{\hat{i}, \hat{j}, \hat{k}\}$

Theorem:

$\text{span}\{\hat{i}, \hat{j}, \hat{k}\} = \left\{a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix} + c\begin{bmatrix}0\\0\\1\end{bmatrix} \middle| a, b \in \mathbb{R}\right\}$

$= \left\{\begin{bmatrix}a\\b\\c\end{bmatrix} \middle| a, b, c \in \mathbb{R}\right\}$

So no such $\vec{v}$ exists $\text{span}\{\hat{i}, \hat{j}, \hat{k}\} = \mathbb{R}^3 \to$ our spanning set is maximal

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$\{\hat{i}, \hat{j}, \hat{k}\}$ is a LI spanning set of $\mathbb{R}^3$

eg $W = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$

Find a vector $\vec{v}_1 \in W$

$\vec{v}_1 = \begin{bmatrix}1\\-1\\0\end{bmatrix} \in W$

$\{\vec{v}_1\}$ is LI

Find $\vec{v}_2 \notin \text{span}\{\vec{v}_1\}$ but $\vec{v}_2 \in W$

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Find $\vec{v}_2 \notin \text{span}\{\vec{v}_1\}$ but $\vec{v}_2 \in W$

Try $\vec{v}_2 = \begin{bmatrix}0\\1\\-1\end{bmatrix} \in W$

$\vec{v}_2 \notin \text{span}\{\vec{v}_2\}$

$\{\vec{v}_1, \vec{v}_2\}$ is LI

picture

and $\{\vec{v}_1, \vec{v}_2\}$ is a LI spanning set

(eg) We know that if $\{\vec{u}, \vec{v}\} \subseteq \mathbb{R}^2$ is LI then span $\{\vec{u}, \vec{v}\} = \mathbb{R}^2$

Consequence: If $\{\vec{u}, \vec{v}, \vec{w}\} \subseteq \mathbb{R}^2$ then this set is LD

$\mathbb{R}^2$ any spanning set ≥ 2 vectors
any LI set has ≤ 2 vectors

Recap from 2 theorems

- if $\{\vec{v}_1, \ldots, \vec{v}_m\}$ spans V then any BIGGER set is LD
- if $\{\vec{v}_1, \ldots, \vec{v}_m\}$ is LI in V then no SMALLER set could span V.

Thm: LI sets are never bigger than spanning sets.

If $V = \text{span}\{\vec{v}_1, \ldots, \vec{v}_n\}$ then any set $\{\vec{v}_1, \ldots, \vec{v}_k\}$ which is LI has $k \leq n$

Equivalently: If $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is LI in V then any spanning set for V has at least k vectors

(eg) $P_2 = \text{span}\{1, x, x^2\}$

So $\{1+x+x^2, 2-x+x^2, x-x^2, 4+4x\}$ is LD because this has 4 vectors and a spanning set has 3.

(eg) $W = \{(x,y,z) \mid x+y+z=0\}$
$= \text{span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}\right\}$

$\Rightarrow$ any set of 3 vectors in W is LD

But of course $\vec{n} = (1,1,1) \in \mathbb{R}^3$ and $\vec{n} \notin W$ and $\vec{n} \notin \text{span}\{$

So is LI in $\mathbb{R}^3$
pic

(a) $\begin{bmatrix}1\\0\\-1\end{bmatrix} \in W$ $\begin{bmatrix}1\\0\\-1\end{bmatrix} = \begin{bmatrix}1\\-1\\0\end{bmatrix} + \begin{bmatrix}0\\1\\-1\end{bmatrix}$ ✓

Definition: A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$ in a vector space V is called a basis of V if

1. $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is LI AND
2. $\text{span}\{\vec{v}_1, \ldots, \vec{v}_n\} = V$

So a basis is:

- a LI spanning set
- a maximal LI set
- a minimal spanning set

* there are ∞ many choices of bases

picture

(These are examples of standard bases)

(eg) 5th example picture — Copy pictures here, complete the notes later

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Def: If V has a finite basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ then the dimension of V is n and we write then $V = n$

In this case V is finite dimensional (otherwise V is infinite dimensional)

(eg) $\dim \mathbb{R}^2 = 2$
$\dim \mathbb{R}^3 = 3$
$\dim P_2 = 3$ $(1, x, x^2 = 3 \text{ people})$
$\dim M_{22} = 4$

(eg) $P = \{\text{all polynomials}\}$
$\supseteq \{1, x, x^2, x^3, x^4, \ldots\}$

digression $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^2 + \cdots \infty$
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(eg) Find a basis for

$W = \text{span}\left\{\begin{bmatrix}1\\1\\3\\4\end{bmatrix}, \begin{bmatrix}3\\1\\-1\\0\end{bmatrix}, \begin{bmatrix}2\\4\\6\\8\end{bmatrix}, \begin{bmatrix}-2\\1\\-1\\4\end{bmatrix}\right\}$

So reduce spanning set if it is LD
picture

$\therefore W =$

Is this set LD? ...

Is this set LD? pic

$$a\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + b\begin{bmatrix} \\ \\ \\ \end{bmatrix} + c\begin{bmatrix} \\ \\ \\ \end{bmatrix} =$$

Solve

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & -1 & 4 & 0 \\ 4 & 0 & 4 & 0 \end{array}\right]$$ pic

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more pics

$$\begin{matrix} 2 & 3 \\ 1 & 1 \\ 4 & -1 \\ 4 & 0 \end{matrix}$$

$\therefore$ W = span

Question

why dim W = 2

Can I have a 5 dimensional subspace in $\mathbb{R}$