Class February 7th

February 7, 2017 1:03 PM

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Audio recording started: 1:04 PM February 7, 2017

Where we are:

(1) Vector spaces
 ↳ 2 operations on a set
 & $\boxed{\text{10 axioms.}}$ Learn by heart
 "context"

(2) subspaces
 ↳ "the particular constraints in my problem"
 ↳ subspace test = 3 things to check

(3) span of some vectors
 If $v_1, v_2, \dots, v_m \in V$ a vector space

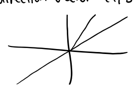
$$\operatorname{span}\underbrace{\{v_1, v_2, \dots v_m\}}_{\substack{\text{"spanning set"} \\ \text{finite set}}} = \underbrace{\{c_1v_1 + c_2v_2 + \dots + c_mv_m \mid c_i \in \mathbb{R}\}}_{\text{infinitely many vectors}}$$

__Theorem__ span$\{v_1, v_2, \dots v_m\}$ is a subspace of V

Applications of this theorem

(1) $W = \operatorname{span}\{(1,3)\} = \left\{c\begin{bmatrix}1\\3\end{bmatrix} \mid c \in \mathbb{R}\right\}$

line through (0,0) with direction vector (1,3)



Therefore, to show that W is a subspace, we don't have to do subspace test.

Since $W = \operatorname{span}\left\{\begin{bmatrix}1\\3\end{bmatrix}\right\}$

Theorem ⇒ W is a subspace

(2) $W_2 = \left\{\begin{bmatrix}a & b\\ a & b\end{bmatrix} \mid a, b \in \mathbb{R}\right\}$

Is this a subspace of $M_{2\times2}(\mathbb{R})$?

$$W_2 = \left\{\begin{bmatrix}a & 0\\ a & 0\end{bmatrix} + \begin{bmatrix}0 & b\\ 0 & b\end{bmatrix} \mid a, b \in \mathbb{R}\right\}$$

$$= \left\{a\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix} + b\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix} \mid a, b \in \mathbb{R}\right\}$$

$$= \operatorname{span}\left\{\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}\right\}$$

Yes, W_2 is a subspace of $M_{2\times2}(\mathbb{R})$.

want to be able to write finite entries.

(3) $W_3 = \{(x,y,z) \in \mathbb{R}^3 \mid x - y + 3z = 0\}$

Goal: rewrite this as the span of some vectors.

$x - y + 3z = 0 \Rightarrow x = y - 3z$

$$W_3 = \{(y-3z, y, z) \mid y, z \in \mathbb{R}\}$$
$$= \{(y, y, 0) + (-3z, 0, z) \mid -$$
$$\{y(1,1,0) + z(-3,0,1) \mid y, z \in \mathbb{R}\}$$
$$= \operatorname{span}\{(1,1,0), (-3,0,1)\}$$

W_3 is a subspace of $\mathbb{R}^3$

audio explanation

Theorem, span part 2

If U is a subspace of V and $v_1, v_2, \dots, v_m$ are in U then $\operatorname{span}\{v_1, v_2 \dots v_m\} \subseteq U$

This implies that $\operatorname{span}\{v_1, v_2 \dots v_m\}$ is the smallest subspace of U containing $v_1, v_2 \dots v_m$.

Proof

Since $v_1, \dots, v_m \in U$ and U is a subspace of V, we have $c_1v_1, \dots c_mv_m$ are in U (closure under scalar mult.) for any scalar $c_1 \dots c_m$

Therefore, since U is closed under addition $c_1v_1 + c_2v_2 + \dots c_mv_m \in U$

audio expl.

Applications

(1) Find all subspaces of $\mathbb{R}^2$

Let U be a subspace of $\mathbb{R}^2$

If $U = \{\vec{0}\}$, fine

If $U \neq \{\vec{0}\}$ then there is nonzero vector $u \in U$

(Ex) Claim: $\operatorname{span}\{u, v\} = \mathbb{R}^2$ ⇒ $U = \mathbb{R}^2$

To show this:

Solve $\begin{bmatrix}x\\y\end{bmatrix} = \underset{\uparrow}{c}\,u + \underset{\uparrow}{d}\,v$

show this system is consistent no matter how x, y are chosen (for a particular example)

If $U \neq \{\vec{0}\}$ then there is nonzero vector $u \in U$

thm $\Rightarrow$ span$\{\vec{u}\} \subseteq U$

If U = span$\{u\}$, then it is a line through $(0,0)$

If not then U contains a vector v that is not a multiple of u

show this system is consistent no matter how x, y are chosen (for a particular example)

2 Showing 2 spans are equal

Show that

span$\{(1,0,0), (0,1,0)\}$

= span$\{(1,1,0), (1,-1,0)\}$

If we show that

$(1,1,0) \in$ span$\{(1,0,0), (0,1,0)\}$

and

$(1,-1,0) \in$ span$\{(1,0,0), (0,1,0)\}$

then theorem part 2 gives

span$\{(1,1,0), (1,-1,0)\}$

** $\subseteq$ span$\{(1,0,0), (0,1,0)\}$

In fact this is true because

$(1,1,0) = 1\,(1,0,0) + 1\,(0,1,0)$

$(1,-1,0) = 1(1,0,0) - 1(0,1,0)$

Now notice;

$(1,0,0) = \frac{1}{2}(1,1,0) + \frac{1}{3}(1,-1,0)$

$(0,1,0) = \frac{1}{2}(1,1,0) - \frac{1}{2}(1,-1,0)$

Therefore, $(1,0,0)$ and $(0,1,0)$

$\in$ span$\{(1,1,0), (1,-1,0)\}$

Therefore, theorem part 2 says

*** span$\{(1,0,0), (0,1,0)\} \subseteq$ span$\{(1,1,0), (1,-1,0)\}$

** & *** $\Rightarrow$ the spans are equal

Ex problem with span

(eg) Show that

span$\{(1,2)\}$ = span$\{(2,4), (0,0), (3,6)\}$

independant set — dependant set

$(2,4) = 2(1,2) \therefore (2,4) \in$ span$\{(1,2)\}$

$(0,0) = 0(1,2) \therefore (0,0)$ — 1, 2 –

$(3,6) = 3(1,2) \therefore (3,6)$ — 1, 2 –

Also

$(1,2) = \frac{1}{2}(2,4) + 0(0,0) + 0(3,6)$

$\Rightarrow (1,2) \in$ span$\{(2,4), (0,0), (3,6)\}$

$\Rightarrow \subseteq$

$\therefore$ the spans are equal

Next: linear dependance & linear independance

Case of 1 vector

$S = \{u\}$

If $\vec{u} = \vec{0}$, then span $S = \{c\}$

and we say S is linearly dependent

If $\vec{u} \neq \vec{0}$ then span S is a line

and we say $\{u\}$ is lin independent

Case 2 vectors

$S = \{u, v\}$

When are these colinear & linearly dependent

It could be

$u = cv$ for some $c \in \mathbb{R}$

$v = c'u$

In some

for some

(Ex) $(3,6)$ and $(2,4)$ are colinear because

$2(3,6) - 3(2,4) = (0,0)$

not both zero

$\{(3,6), (2,4)\}$ is linearly dependent

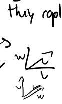
Case 3 vectors

$\{u, v, w\}$ ← linearly dependent

When are they coplanar?

Possibilities

$u = av + bv$



or

or $v = a'u + b'w$

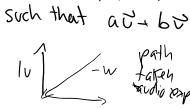
$w = a''u + b''v$

In sum:

they are coplanars if

scalars a, b, c NOT

such that $a\vec{u} + b\vec{v}$



(Ex) $\{(3,6), (0,0), ($

is linearly dependent

$2(3,6) + 5(0,0) - 3(2,$

$0(3,6) + 5(0,0)\ 0\ (2,$

All zeros can't be us

Any other way?

$av + bw = 0$

$a, b \in \mathbb{R}$, not both zero

therefore

ALL ZERO

$+ c\vec{w} = \vec{0}$

$(2,4)\}$

because

$4) = (0,0)$

$4) = (0,0)$ OR

eful.

Definition:

We call such an equation

a dependence relation on $\{v_1, \dots v_m\}$

linearly dependent

=

LD

Now the opposite notion

Definition: (unique solution, the only one)
If set $\{v_1, \dots v_n\}$ of vectors in vector space V, is linearly independent if the <u>only</u> solution to the dependence equation

$k_1 v_1 + k_2 v_2 \dots + k_m v_m = 0$

is $k_1 = 0$, $k_2 = 0$, $k_m = 0$

Ex Show that

$\{(1,1,0), (0,1,1), (0,0,1)\}$

are linearly independent.

Solution:

Solve:

$$a\begin{bmatrix}1\\1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\1\end{bmatrix} + c\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}a\\a+b\\b+c\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\therefore$ $a = 0$

$a + b = 0 \Rightarrow b = 0$

$b + c = 0 \Rightarrow c = 0$

$\therefore$ the <u>only</u> solution is the trivial solution $a = b = c = 0$

ex Show that

$$\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}\right\}$$

is LD.

<u>Solution:</u> Solve

$$a\begin{bmatrix}1\\0\\-1\end{bmatrix} + b\begin{bmatrix}0\\1\\-1\end{bmatrix} + c\begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\Leftrightarrow$
$a + c = 0$
$b - c = 0$
$-a - b = 0$

$$\left[\begin{array}{ccc|c}1&0&1&0\\0&1&-1&0\\-1&-1&0&0\end{array}\right] \underset{R_1+R_3}{\sim} \left[\begin{array}{ccc|c}1&0&1&0\\0&1&-1&0\\0&-1&1&0\end{array}\right] \underset{R_2+R_3}{\sim} \left[\begin{array}{ccc|c}1&0&1&0\\0&1&-1&0\\0&0&0&0\end{array}\right]$$

connecting variable non-pivot

$$\begin{bmatrix}a + 0b + c\\0a + b - c\\-a - b + 0c\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

the system ∞ many nontrivial solutions.

Set is <u>LD</u>