Class January 31st

January 31, 2017 1:00 PM

Class January 3...

Audio recording started: 1:01 PM January 31, 2017

Vector Spaces

- set V that has 2 operations
- addition & scalar multiplication such that 10 ax hold

Closure 1) if $\vec{u}, \vec{v} \in V$, then $\vec{u}+\vec{v} \in V$

big is V truth? 2) if $c \in \mathbb{R}$, and $\vec{u} \in V$, then $c\vec{u} \in V$

Existence 3) there is a $\vec{0}$ such that for all $\vec{v}$ in V, we have $\vec{0}+\vec{v}=\vec{v}$

4) for each $\vec{v}$ in V there is a vector $-\vec{v}$ in V such that $\vec{v}+(-\vec{v})=\vec{0}$

5) $u+v=v+u$

6) $(u+v)+w=u+(v+w)$

7) $(c+d)u=cu+du$

8) $c(d)u=c(du)$

9) $c(u+v)=cu+cv$

10) $1u=u$

Examples

1) $\mathbb{R}^n$ $n \geq 1$

2) $\mathcal{E}=\{$all

3) waveforms

4) $Mat_{n \times m}=\{$all matrices with n rows & m columns$\}$

Case $M=\{$all 2x2 matrices$\} = \{\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}^{\text{pattern}} \mid a,b,c,d \in \mathbb{R}\}$

$n=2$

$m=2$

addition rule:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix}$$

scalar multiplication

let $g \in \mathbb{R}$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M$ then $g\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ga & gb \\ gc & gd \end{bmatrix}$

Is this a vector space? 11:20

1) Since the sum of 2 matrices in M is a matrix in M, 1) holds

2) Since multiplying a matrix in M by a scalar gives a matrix in M, 2) holds

3) $\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in M$

Now check

let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M$ then

$$\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\vec{0} + \vec{v}} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\vec{v}}$$

3) holds

4) check that if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M$ then its opposite is $\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} \in M$

Axioms (5)-(10) ok

Conclusion: M is a vector space

5) $F(\mathbb{R})=\{$all functions from $\mathbb{R}$ to $\mathbb{R}\}$

examples of "vectors" in $F(\mathbb{R})$:

$\sin(x)$

e^x

$p(x)=x^2+2x+\cos(x)$ ← this is the "vector" p

Addition of functions

- if $f(x)$ & $g(x)$ are functions, so is $f(x)+g(x)$
- if $c \in \mathbb{R}$, then $cf(x)$ is also a function if $f(x)$ is.

Let's check that $z(x)$ is the $\vec{0}$, that is, satisfies 3).

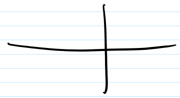
- $z(x) \in F(\mathbb{R})$ ✓ $z(x)$ is a function
- let $f(x) \in F(\mathbb{R})$, then $f(x)$ is another

• if $c \in \mathbb{R}$, then $cf(x)$ is also a function if $f(x)$ is.

Is $F(\mathbb{R})$ a vector space?

(1), (2), we just said this

3) let $z(x) = 0$ for all x



What is the zero function?

• $z(x) \in F(\mathbb{R})$ ✓ $z(x)$ is a function

• let $f(x) \in F(\mathbb{R})$, then $f(x)$ is another function

$z(x) + f(x) = 0 + f(x) = f(x)$

So 3) holds

4) let $f(x) \in F(\mathbb{R})$ then $-f(x) \in F(\mathbb{R})$ (is a function)

and $f(x) + (-f(x)) = f(x) - f(x) = 0 = z(x)$ ← the zero function

2 things checked
- is negative a function
- addition works?

5) 5)-10) hold because you checked this in high school

Therefore, $F(\mathbb{R})$ is a vector space.

Example of a non-standard vector space

$V = \{ (a) \boxed{b} \mid a, b \in \mathbb{R} \}$

with operation given by:

1) $(a)\boxed{b} + (c)\boxed{d} = (a+c)\boxed{b+d-1}$

2) if $c \in \mathbb{R}$: $c((a)\boxed{b}) = (ca)\boxed{cb-c+1}$

Is it a vector space?

Our formulas show that V is closed under addition and scalar multiplication

1 & 2 hold

③ What is $\vec{0}$? Use 2nd formula

Try $c = 0$: $0((a)\boxed{b}) = 0\boxed{0b - 0 + 1}$

$\vec{0}$: any number a & b

Our candidate for $\vec{0}$ is $(0)\boxed{1} \in V$ ✓

Check: let $\vec{v} \in V$

So $\vec{v} = (a)\boxed{b}$ for some $a, b \in \mathbb{R}$

Now check $\vec{0} + \vec{v} = (0)\boxed{1} + (a)\boxed{b}$

$= (a)\boxed{1+b-1} = (a)\boxed{b}$ ✓

don't know where that comes from

3) holds

4) let $\vec{v} \in V$

So $\vec{v} = (a)\boxed{b}$ for some $a, b \in \mathbb{R}$

Try $-\vec{v} = -1((a)\boxed{b}) = (-a)\boxed{(-1)b - (-1) + 1}$
($c = -1$)

$= (-a)\boxed{2-b}$

Check

$\vec{v} + (-\vec{v}) = (a)\boxed{b} + (-a)\boxed{2-b}$

$= (a-a)\boxed{b+2-b-1}$

$= (0)\boxed{1} = \vec{0}$ ✓

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5) Let $\vec{u}, \vec{v} \in V$

So $\vec{u} = (a)\boxed{b}$ and $\vec{v} = (c)\boxed{d}$

for some $a, b, c, d \in \mathbb{R}$

Then: $u + v = (a)\boxed{b} + (c)\boxed{d} = (a+c)\boxed{b+d-1}$

$v + u = (c)\boxed{d} + (a)\boxed{b} = (c+a)\boxed{d+b-1}$

they are equal, so **5) holds**

6) - 10): ex

This is a vector space

Subspaces

Next: **simpler test**

Suppose V is a vector space

Suppose W is a subset of V

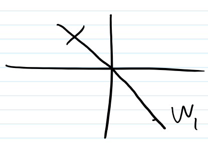
We write $W \subseteq V$

Q. Is W a vector space with same operations as on V?

Examples

$W_1 = \{ (3x, -x) \mid x \in \mathbb{R} \}$

$W_1 \subseteq \mathbb{R}^2$



$W_2 = \{ (x, y) \in \mathbb{R}^2 \mid xy = 0 \}$

$W_3 = \{ (x, y) \in \mathbb{R}^2 \mid x + y = 0 \}$

$W_2 = \{(x,y) \in \mathbb{R}^2 \mid xy = 0\}$
$W_3 = \{(x,y) \in \mathbb{R}^2 \mid x+y=0\}$
$W_4 = \{(x,1) \mid x \in \mathbb{R}\}$
$W_5 = \{\vec{v} \in \mathbb{R}^2 \mid \|\vec{v}\| = 1\}$

Def'n: A subset W of a vector space V is called a subspace (a subspace of V) if it is a vector space with the same addition & scalar mult as V

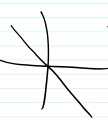
Ex) $\{\vec{0}\} \subseteq V$ is a subset and is a vector space, so it is a subspace of V

Ex) V is a subspace of V

Theorem: A subset W of a vector space V is a subspace iff & only if:
a) $\vec{0} \in W$
closure b) for every $u, v \in W$, we have $u+v \in W$. (closure under addition)
c) for every $c \in \mathbb{R}$, and $v \in W$ we have $cv \in W$

Examples Subspace test — understand your space

$W_1 = \{(3x, -x) \mid x \in \mathbb{R}\}$
Is this a subspace of $\mathbb{R}^2$?
$\iff$ is this a vector space with usual rules addition/scalar multiplication?



$W_1 = \{x(3,-1) \mid x \in \mathbb{R}\}$

a) Is $(0,0) \in W_1$?
Since $(0,0) = (3(0), -0)$
$(3x, -x)$ with $x = 0$
Yes, $\vec{0} \in W_1$

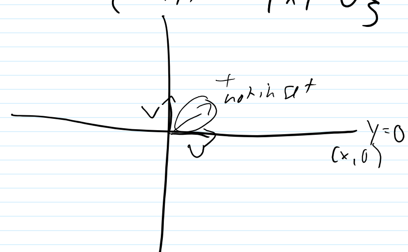
(b) Let $u, v \in W_1$
That means: $u = (3x, -x)$ and $v = (3y, -y)$ for some $x, y \in \mathbb{R}$

Then, $u + v = (3x, -x) + (3y, -y)$ — has to look like $(3x, -x)$
$= (3x + 3y, -x - y)$
$= (3\underline{(x+y)}, -\underline{(x+y)}) \in W_1$ — are equal

So W_1 is closed under addition

(c) Let $c \in \mathbb{R}$, $u \in W_1$
That means $u = (3x, -x)$ for some $x \in \mathbb{R}$
So $cu = c(3x, -x)$ — has to look like $(3x, -x)$
$= (c(3x), c(-x))$
$= (3(cx), -(cx))$ $c\vec{v} \in W_1$

Ex) $W_2 = \{(x,y) \in \mathbb{R}^2 \mid xy = 0\}$



No this is not a subspace of $\mathbb{R}^2$,

$= (3c(x), -c(x))$ $\quad c\vec{v} \in W_1$

So W_1 is closed under scalar mult.

W_1 is a subspace of $\mathbb{R}^2$ by the subspace test.

Notice: any line through the origin will be a subspace of $\mathbb{R}^n$

Subspace of $\mathbb{R}^3$?

No this is not a subspace of $\mathbb{R}^2$ because if $v = (1,0) \in W_2$, $v' = (0,1) \in W_2$ then $v + v' = (1,1) \notin W_2$

So closure (b) fails

Ex) $W_5 = \{\vec{v} \in \mathbb{R}^2 \mid \|\vec{v}\| = 1\}$ → circle

No, this is not a vector space. It is not closed under either addition or scalar mult.

Fails (c) because $c = 2 \in \mathbb{R}$, $v = (1,0) \in W_5$ but $cv = 2(1,0) = (2,0) \notin W_5$

1:09

Ex) Let P be the set of all polynomial functions on $\mathbb{R}$

Elements of P: examples

$1 + x^2 + x^5$

$\frac{3}{2}x - x^2$

Check that P is a subspace of $F(\mathbb{R})$. Do subspace test

a) $z(x) = 0 = 0 + 0x + 0x^2$ is a polynomial function, $\vec{0} \in P$

b) If we add 2 polynomials

$f(x) = a + bx + cx^2$

$g(x) = a' + b'x + c'x^2$

then $f(x) + g(x) = (a + a') + (b + b')x + (c + c')x^2$ is also a polynomial

c) multiplying poly by a scalar gives a poly

So P is closed under addition & scalar multiplication. P is a subspace

Ex) $P_n = \{\text{all poly of degree} \leq n\}$

$= \{a_0 + a_1x + \dots + a_nx^n\}$

1 more example

* The point is to check that addition & scalar multiplication can't give you a higher degree polynomial.

eg) $T = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \,\middle|\, a, b, c \in \mathbb{R} \right\}$