# Dynamic - Engineering Mechanics 131

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Winter of 2012

# Contents

# Chapter 1

# General Principles

## 1.1 Mechanics

Mechanics is a branch of the physical sciences concerned with the state of rest or motion of bodies that are subjected to the action of forces.

The mathematical models used can basically be divided into two categories:

- **Rigid Body Mechanics** All deformation is ignored. The special case of this is particle mechanics, as in no shape, only mass.

- **Continuum — Deformable Body Mechanics** Concerned with elastic bodies, fluids, solids, etc.

Here, we will be primarily concerned with rigid body mechanics. Thus we can divide it further as follows.

- **Statics** Equilibrium of bodies at rest or moving with constant velocity.

- **Dynamics** Bodies having accelerated motion.

  - **Kinematics** Concerned with the geometric aspects of motion — The study of motion without reference to forces causing the motion. Involves the position, velocity and acceleration at an instant of time $t$ of a particle or object.
  - **Kinetics** Analysis of forces causing the motion of a particle or object — Relates the action of forces (and moments) to the resulting motion.

## 1.2 Fundamental Concepts

**Basic Quantities** Length, Time, Mass, Force.

**Idealizations** Particle, Rigid Body, Concentrated Force

### 1.2.1 Newton's Laws of Motion

- **First Law** (inertia)

  "A particle originally at rest, or moving in a straight line, with constant velocity, tends to remain in this state unless acted upon by an unbalanced force."

- **Second Law** ($F = m\vec{a}$)

  "A particle acted upon by an unbalanced force $\vec{F}$ experienced an acceleration $\vec{a}$ that has the same direction as the force and a magnitude directly proportional to the force"

**Vector Equation**

$$\boxed{\vec{F} = \frac{d}{dt}(m\vec{c}) = m \cdot \frac{d}{dt}\vec{v} = ma} \tag{1.1}$$

- **Third Law** (action = −reaction)

  The natural forces of action and reaction between two particles are equal, opposite and collinear.

Newton's laws require an *inertial frame.*
The first law is violated inside a non-inertial frame.
The second law makes no mention of speed or velocity, however it fails for speeds near the speed of light.

**Einstein's Theory of Spacial (inertial frame) Relativity**

$$\boxed{\vec{F} = m\frac{d}{dt}(\gamma\vec{v});\ \gamma \text{ is the Lorentz Factor}} \tag{1.2}$$

The third law is always true.

**Interesting Consequence**

Newton's third law predicts that the earth will "fall" towards an object falling toward the earth.

### Example (1.1)

The force on Earth from a ball is $F = mg = 5 (g \sim 10)$. How long does it take the ball to hit earth?

$$\tfrac{1}{2}gt^2 = 100 \therefore t = 4\tfrac{1}{2}\text{sec}$$

According to the third law, earth experiences the same force in an opposite direction.

$$F_E = M_E a_E \therefore a_E = \tfrac{5}{6\times10^24} = 8 \times 10^{-25}\text{m/s}^2$$

The earth will fall for $4\frac{1}{2}$s towards the ball. Hence the distance travelled be the earth is:

$$\tfrac{1}{2}a_E t^2 = 8 \times 10^{-24}$$

We cannot ever guarantee an inertial frame — so these laws cannot be proved.

### 1.2.2 Newton's Law of Gravitational Attraction

The gravitational attraction between any two particles is described by:

$$\boxed{F = F\frac{m_1 m_2}{r^2}} \tag{1.3}$$

$F$ = magnitude of force of gravitation between the two particles.

$G$ = universal constant of gravitation. $G = 66.73 \times 10^{-12}$ $\text{m}^3/(\text{kg}\cdot\text{s}^2)$

$m_1$, $m_2$ = mass of each of the two particles.

$r$ = distance between the two particles.

### 1.2.3 Weight

A particle located at or near the surface of the Earth is attracted to the Earth according to *Equation (1.3)*.

$$W = mg, \text{ where } g = \frac{GM_E}{r^2}$$

Comparing this with *Equation (1.1)*, we can deduce that $g$ is the acceleration due to the mutual attractive (gravitational) force acting between the particle and the earth.
Since $g$ depends on $r$m the 'weight' is not an absolute quantity. For most engineering calculations, $g$ is determined at sea level and latitude 45°.

### 1.2.4 Units of Measurement (SI System)

Table 1.1 — Systems of Units

| **Name** | Length | Time | Mass | Force |
|---|---|---|---|---|
| — | meter | second | kilogram | newton |
| — | m | s | kg | $N = \left(\frac{kg \cdot m}{s^2}\right)$ |

Table 1.2 — Prefixes

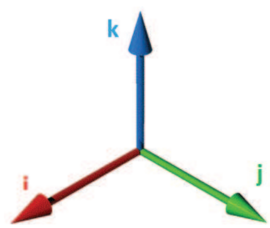
| — | Exponential Form | Prefix | SI Symbol |
|---|---|---|---|
| *Multiple* | | | |
| 1 000 000 000 | $10^9$ | giga | G |
| 1 000 000 | $10^6$ | mega | M |
| 1 000 | $10^3$ | kilo | k |
| *Submultiple* | | | |
| 0.001 | $10^{-3}$ | milli | m |
| 0.000 001 | $10^{-6}$ | micro | $\mu$ |
| 0.000 000 001 | $10^{-9}$ | nano | n |

# Chapter 2

# Kinematics of a Particle

## 2.1 Introduction

Suppose we want to analyse a rigid body moving through space. There are six different vector values that are necessary to accurately describe the movement of that object.




$$\vec{s}(t) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}} + z(t)\hat{\boldsymbol{k}} + \underbrace{\theta}_{\text{YAW}} + \underbrace{\psi}_{\text{PITCH}} + \underbrace{\gamma}_{\text{ROLL}}$$

The vectors $x$,$y$ and $z$ describe the location of the particle. By analysing the location of the particle with respect to time, we can determine the velocity of the object.

Average speed is a scalar value concerned with distance, where as average velocity is a vector concerned with direction and displacement.

$$\text{Average Speed} = \frac{\text{Total Distance}}{\text{Total Time}} \tag{2.1}$$

$$\text{Average Velocity} = \frac{\text{Total Displacement}}{\text{Total Time}} \tag{2.2}$$

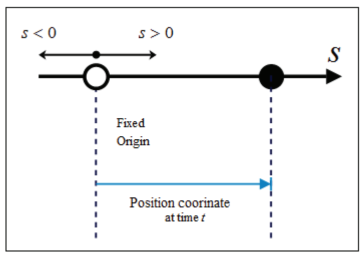
We begin with the simplest model of a moving body, a particle. That is, we focus on those objects whose dimensions, size and shape can be ignored for the purposes of the analysis. Consequently, the body's rotation is neglected and all the mass is assumed to be concentrated at the mass centre.

## 2.2 Rectilinear Kinematics — Continuous Motion

Consider the following assumptions of rectilinear kinematics.

- Particle moves along a straight line (rectilinear path)
- The kinematics of the particle are characterized by specifying, at any given instant, the particles
  - Position
  - Velocity
  - Acceleration

### 2.2.1 Position




Position is always a *vector* (magnitude and direction) but here, since the direction is always along the coordinate axis, we need only specify the coordinate $s$ as positive or negative.

### 2.2.2 Average Speed and Average Velocity

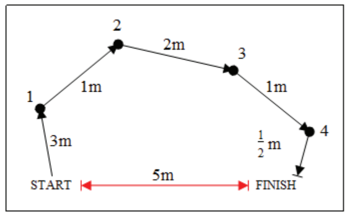
Average speed and average velocity are not the same thing — and they are calculated differently!

It's not just because velocity is a vector and speed is scalar (magnitude of velocity), but also because or the following key distinction.

- Average Velocity (Vector) deals with **displacement** (Vector).
- Average Speed (Scalar) deals with **distance** (Scalar)

The average speed **is not** the magnitude of the average velocity.

**Example (2.1)**




Suppose each part of the trip took 2 seconds. So the total time taken to travel 7.5m is 10sec.

The direction of travel, or the position of the final resting place is not important in the calculation of average speed.

Using *Equation (2.1)*

$$\text{Average Speed} = \frac{\text{Total Distance Travelled}}{\text{Total Time Taken}}$$

$$\text{The Average Speed is } \tfrac{7.5}{10} = 0.75\text{m/s}$$

Of the other hand, the average velocity is calculated using *Equation (2.2)* as follows.

Using *Equation (2.1)*

$$\text{Average Velocity} = \frac{\text{Total Displacement frim Origin}}{\text{Total Time Taken}}$$

$$\text{Thus the Average Velocity is } \tfrac{\text{5m to the } \textbf{right}}{\text{10sec}} = 0.5\text{m/s to the } \textbf{right}$$

Velocity does not depend on how you got to the finish — only **how far** you ended up from your original position and the total time taken to get there.

**Displacement**

Displacement of the particle ≡ change in the particles position

Again, displacement is vector. (different from 'total distance travelled' which is always a positive scalar)

**Velocity**

Average velocity during a time interval $\Delta t$ is the vector whose magnitude is:

$$v_{avg} = \frac{\Delta s}{\Delta t} = \frac{\text{Displacment of Particle in } \Delta t}{\Delta t}$$

The *instantaneous velocity* at time $t$ is a vector whose magnitude is defined by the following equation.

$$\boxed{v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}} \tag{2.3}$$

In each case, since $\Delta t$ and $dt$ are always positive, the sign of $\Delta s$ or $ds$ defines the sense or direction of the velocity, denoted as follows.

$\xrightarrow{+}$ Positive if the particle is moving to the right.

$\xleftarrow{-}$ Negative if the particle is moving to the left.

**Speed $\equiv$ Magnitude of Velocity**

**Recall** $\underbrace{\text{Average Speeed}}_{PositiveScalar} \equiv \frac{\text{Total Distance Travelled by Particle}}{\text{Elapsed Time}}$

**Note** Recall in contrast with definition of average velocity which takes into account direction of travel.

**So —**

- $(v_{sp})_{avg} = \frac{s_T}{\Delta t}$ — Always true since $s_T > 0$; $\Delta t > 0$
- $v_{avg} = -\frac{\Delta s}{\Delta t}$ — Since the particle moves to the left from position $p$ to position $p'$

**Acceleration**

Average acceleration has magnitude $a_{avg} = \frac{\Delta v}{\Delta t}$

Instantaneous acceleration has magnitude as follows.

$$\boxed{a = \lim_{\Delta t \to 0} a_{avg} = \frac{dv}{dt} = \frac{d^2 s}{dt^2}} \tag{2.4}$$

**Direction of Acceleration**

- If speed is **decreasing**, $a < 0$
- If speed is **increasing**, $a > 0$

From *Equation (2.3)* we find that $v \cdot dt = ds$, and from *Equation (2.4)* we find that $a \cdot dt = dv$. Thus we can deduce the following third equation.

$$a \cdot ds = v \cdot dv \tag{2.5}$$

So, *Equations (2.3) - (2.5)* are the three equations that can be used to decide on the kinematic properties of a particle in rectilinear motion.

### 2.2.3 Special Case — Constant Acceleration

Suppose $a = a_c =$ constant and $s = s_o$; $v = v_o$ when $t = 0$

**Case 1**

From *Equation (2.4)* we find $\int dv = \int a_c dt$. $\therefore v(t) = a_c t + c_1$ where $c_1$ is an arbitrary constant

But, $v = v_o$ when $t = 0$ thus $c_1 = v_o \therefore$ if acceleration is constant

$$v(t) = v_o + a_c t \overset{+}{\longrightarrow} \tag{2.6}$$

**Case 2**

From *Equation (2.3)* we find $\int v \cdot dt \overset{=}{(2.6)} \int (v_o + a_c t) dt = v_o t + a_c \frac{t^2}{2} + c_2$ where $c_2$ is an arbitrary constant

So we find $s(t) = a_c \frac{t^2}{2} + v_o t + v_2$

But when $t = 0$ we find $s = s_o$, $v = v_o$ thus $c_2 = s_o$. From this, we can deduce the following equation.

$$s(t) = s_o + v_o t + a_c \frac{t^2}{2} \overset{+}{\longrightarrow} \tag{2.7}$$

**Case 3**

From *Equation (2.5)* we find $\int a_c \cdot ds = \int v \cdot dv = \frac{v^2}{2} + c_3$ where $c_3$ is an arbitrary constant.

So that $a_c s = \frac{v^2}{2} + c_3$

But $v = v_o$ when $s = s_o$ thus $a_c s_o = \frac{v_o^2}{2} + c_3$ and so $c_3 = a_c s_o - \frac{v_o^2}{2}$

Therefore, $a_c s = \frac{v^2}{2} + a_c s_o - \frac{v_0^2}{2}$ so that $v^2 = f_o^2 + 2a_c s_o$. From this we can deduce the following equation.

$$v^2 = v_o^2 + 2a_c(s - s_o) \xrightarrow{+} \tag{2.8}$$

**Note** *Equation (2.6) - (2.8)* **only** apply if acceleration is constant. Otherwise, use *Equation (2.3) - (2.5)*

### 2.2.4 Summary

- Position is a vector.
- From *Equation (2.1)* Average speed $= \frac{\text{Total Distance Travelled}}{\text{Total Time Taken}}$ (Scalar)
- Displacement is the distance from the starting point (ie. change in position) (Vector)
- From *Equation (2.2)* Average velocity $= \frac{\text{Displacement}}{\text{Time Taken for Displacement}}$
- Instantaneous velocity at time $t = \frac{ds}{dt}$ with direction. (Vector)
- Instantaneous speed at time $t = \frac{ds}{dt}$. (Scalar)
- Average acceleration has a magnitude $\frac{\Delta v}{\Delta t}$ or $\frac{\text{Change in Speed}}{\text{Elapsed Time}}$
- Instantaneous acceleration $= \frac{dv}{dt} = \frac{d^2s}{dt^2} = \begin{cases} < 0 & \text{if speed is decreasing} \\ > 0 & \text{if speed is increasing} \end{cases}$
- Functional equations for rectilinear motion, *Equation (2.3) - (2.5)*

$$v = \frac{ds}{dt},\ a = \frac{dv}{dt} = \frac{d^2s}{dt^2},\ ads = vdv$$

- Special case of constant acceleration where $a = a_c$, $s = s_o$, $v = v_o$ when $t = 0$

$$v(t) = v_o + a_c t \xrightarrow{+}$$

$$s(t) = s_o + v_o t + \frac{a_c t^2}{2} \xrightarrow{+}$$

$$v^2(t) = v_o^2 + 2a_c(s - s_o) \xrightarrow{+}$$

### 2.2.5 Examples

#### Example (2.2)

A car starts from rest and with constant acceleration achieves a speed of 15m/s when it reaches a distance of 200m. Find the acceleration of the car and the time required to travel 200m

**Solution**

Given constant acceleration, use *Equation (2.6) - (2.8)*

$$v(t) = v_o + a_c t \xrightarrow{+}$$

$$s(t) = s_o + v_o t + \frac{a_c t^2}{2} \xrightarrow{+}$$

$$v^2(t) = v_o^2 + 2a_c(s - s_o) \xrightarrow{+}$$

We know: at $t = t_o = 0$, $s = s_o = 0$, $v = v_o = 0$;
We want: $t$ and $a_c$.

Using the equations above, we can deduce the following

$\xrightarrow{\text{at } t=t_1}$ $15 = 0 + a_c t_1$ therefore $a_c t_1 = 15$

$\xrightarrow{\text{at } t=t_1}$ $200 = \frac{a_c t_1^2}{2}$

$\xrightarrow{\text{at } t=t_1}$ $15^2 = 0^2 + 2a_c(200 - 0)$

It follows that $a_c = 0.5625\text{m/s}^2$ for all $t$.

**Note** Acceleration is $\vec{a}(t) = a_c \hat{\boldsymbol{\imath}} = 0.5625\hat{\boldsymbol{\imath}}$ for all $t$.

In order to find the time $t = t_1$ required to travel 200m, using the equations above, we find that:

From *Equation (2.6)* and the value for $a_c$ above, $t_1 = \frac{15}{a_c}\left(\frac{\text{m/s}}{\text{m/s}^2}\right) = 26.7\text{s}$

#### Example (2.3)

A particle is moving along a straight line with acceleration given by $\hat{\boldsymbol{a}}(t) = a(t)\hat{\boldsymbol{\imath}}$, where $a(t) = (12t - 3t^{1/2})\text{ft/s}$.

Find the velocity and position of the particle as functions of time. Note that when $t = 0$, $v = v_o = 0$ and $s = s_o = 15$

**Solution**

We will use *Equation (2.3) - (2.5)* as our math model because acceleration is not constant.

$$\underset{(2.3)}{v(t) = \frac{ds}{dt}} \text{ — } \underset{(2.4)}{a(t) = \frac{dv}{dt}} \text{ — } a \cdot ds \underset{(2.5)}{=} v \cdot dv$$

We want: $v(t)$, $s(t)$;
We know: $a(t)$.

Using *Equation (2.4)* we find that $\int dv = \int a(t) \cdot dt$

$$\iff v(t) = \int (12t - 3t^{1/2})dt = \tfrac{12t^2}{2} - \tfrac{3t^{3/2}}{3/2} + c \text{ where } c \text{ is an arbitrary constant.}$$

$$\text{Therefore } v(t) = 6t^2 - 2t^{3/2} + c$$

To evaluate the arbitrary constant $c$, recall that $v = 0$ when $t = 0$.

$$\text{So we know } 0 - 0 + c = 0 \iff c = 0$$

$$\text{Therefore } \hat{\boldsymbol{v}}(t) = (6t^2 - 2t^{3/2})\hat{\boldsymbol{\imath}} \text{ ft/s}$$

For position $s(t)$, based on *Equation (2.3)*

Because $v(t) = \frac{ds}{dt}$ we can deduce that $\int ds = \int v(t) \cdot dt$

$$\text{and so } s(t) = \int (6t^2 - 2t^{3/2})dt = 2t^3 = \tfrac{2t^{5/2}}{5/2} + c \text{ where } c \text{ is an arbitrary constant.}$$

$$\text{Therefore } \hat{\boldsymbol{s}}(t) = (2t^3 - \tfrac{4}{5}t^{5/2} + 15)\hat{\boldsymbol{\imath}} \text{ ft}$$

## Example (2.4)

A particle is moving along a straight line such that its acceleration is defined as $a = (-2v)\text{m/s}^2$ where $v$ is speed in m/s. If $v = 20$m/s when $s = 0$ and $t = 0$, find the particles position, velocity and acceleration as functions of time.

**Solution**

$$a = -2v \underset{(2.4)}{\Longrightarrow} \frac{dv}{dt} = -2v \implies \int \frac{dv}{v} = \int -2dt$$

$$\ln(v) = -2t + c, \text{ we assume } v > 0 \text{ for all } t > 0$$

$$\text{Apply given condition : } \ln|20| = c \iff c = \ln 20$$
$$\ln v = -2t + \ln 20 \iff \ln v - \ln 20 = -2t$$

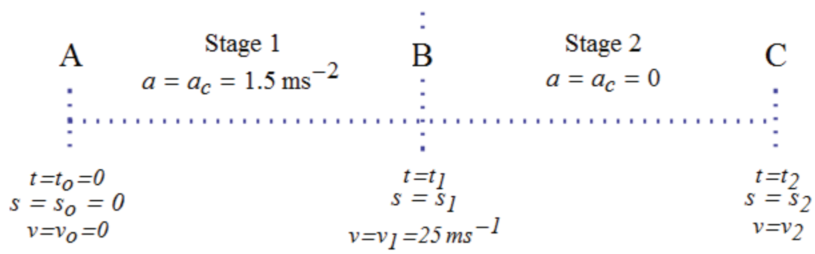
$$\implies \ln(\tfrac{v}{20}) = -2t, \text{ and so we deduce that } \tfrac{v}{20} = e^{-2t} \text{ therefore } \hat{\boldsymbol{v}}(t) = 20e^{-2t}\hat{\boldsymbol{\imath}} \text{ m/s}$$

### Example (2.5)

A car starts from rest and moves with a constant acceleration of $1.5\text{m/s}^2$ until it achieves a speed of 25m/s. It then travels with constant velocity for 60 seconds. Find the average speed of the car and the total distance it travels.

#### Solution

The motion of the car is in two stages : acceleration is constant in each stage. We will use *Equation (2.6)* and *Equation (2.8)*




$$\text{Average Speed} = \frac{\Delta s}{\Delta t} = \frac{\text{Total Distance}}{\text{Total Time}}$$

#### Stage 1 of Motion

Recall *Equation (2.6)*

$$\xrightarrow{+} v = v_o + a_c t$$
$$25 = 0 + 1.5t_1 \implies t_1 = 16.67\text{s}$$

To get $s_1$, we will use *Equation (2.8)*

$$\xrightarrow{+} v^2 = v_o^2 + 2a_c(s - s_o)$$
$$(25)^2 = 0 + 2(1.5)(s_1 - 0) \implies s_1 = 208.33 \text{ m}$$

#### Stage 2 of Motion

Take position $B$ and the initial conditions and position $C$ as the final conditions

$$s_o = 208.33 \text{ m};\ v_o = 25 \text{ ms}^{-1};\ t_1 = 16.67 \text{ s};\ a_c = 0$$

To get $s_1$, we will use *Equation (2.7)*

$$\xrightarrow{+} s = s_o + v_o t + \tfrac{1}{2}a_c t^2$$
$$s_1 = 208.33 + 25(60) + 0 \implies s_1 = 1708.33 \text{ m}$$

Not that the acceleration is clearly not constant from the initial state where $t_o = s_o = 0$ to the final state.

$$\text{Finally, average speed} = \frac{1708}{16.67+60} = 22.3 \text{ ms}^{-1}$$

## Example (2.6)

The acceleration of a particle travelling along a straight line is given by

$$a = (8.2s) \text{ ms}^{-2} \xrightarrow{+}$$

where $s$ is in meters. If $v = 0$ at $s = 0$, find the velocity of the particle at $s = 2$m and the position of the particle at it's maximum velocity.

### Solution

Clearly, the acceleration is not constant.
Thus we will use *Equation (2.5)*

$$\xrightarrow{+} a \cdot ds = v \cdot dv$$

$$\int (8 - 2s) \cdot ds = \frac{1}{2}v^2 + c$$

$$8s - s^2 = \frac{1}{2}v^2 + c$$

$$s(0) = s_o = 0 \text{ when } v(0) = v_o = 0 \text{ therefore } c = 0$$

$$\text{Hence, } v^2 = 16s - 2s^2 \implies v = \pm\sqrt{16s - 2s^2}$$

Then, at a distance of 2m, $s - 2$ and

$$v(2) = \pm\sqrt{24} = \pm 2\sqrt{6} \sim \pm 4.90 \text{ ms}^{-1}$$

$$\text{Finally, } \vec{v}\,|_{s=2} = \pm 4.90\hat{\boldsymbol{\imath}} \text{ ms}^{-1}$$

For the position where velocity is a maximum, not that

$$\vec{v} = |\vec{v}|(\pm\hat{\boldsymbol{\imath}})$$

$$\text{So we want to maximize } |\vec{v}| = \sqrt{16s - 2s^2}$$

$$\frac{d(\vec{v})}{ds} = \frac{1}{2}(16s - 2s^2)^{-1/2}(16 - 4s) = \frac{8-2s}{\sqrt{16s-2s^2}}$$

$$\text{Hence, } \frac{d(\vec{v})}{ds} = 0 \text{ when } s = 4\text{m}$$

To determine weather this is a maximum, we will use the second derivative test.

$$\frac{d^2|\vec{v}|}{ds^2} = \frac{-2\sqrt{16s-2s^2} - \frac{1}{2}(16s-2s^2)^{-1/2}(16-4s)(8-2s)}{\sqrt{16s-2s^2}}$$

Hence, at $s = 4$, we have $\frac{d^2|\vec{v}|}{ds^2} = -2 < 0 \implies s = 4$ is a maximum.

Finally , the maximum velocity of the particle occurs at $\vec{s} = \pm 4\hat{\boldsymbol{\imath}}$

## Example (2.7)

Ball $A$ is thrown vertically upwards with a speed of $v_o$. Ball $B$ is thrown vertically upwards from the same point with the same speed but $t$ seconds later. Find the elapsed time between the instant when ball $A$ is thrown to when the balls pass each other. Find the velocity of each ball at this instant.

### Solution

We are interested in time $t$ at when the balls have the common height $h$.

**Note** acceleration is constant.

We proceed on the idea that the sum of the time waited by ball $B$ and the time it takes for ball $B$ to reach height $h$ is equal to the time taken for ball $A$ to be at height $h$.

### Ball $A$

$$(v_A)_o = v_o;\ (s_A)_o = 0;\ s_A = h;\ t_A = t_1 \text{ (at } s = h);\ (a_c)_A = -g$$

Recall *Equation (2.7)*

$$+\uparrow s = s_o + v_o t + \frac{1}{2}a_c t^2$$

Hence $h = 0 + v_o t_1 + \frac{1}{2}(-g)(t_1)^2$

Then, using *Equation (2.6)*

$$+\uparrow v_= v_o + a_c t$$

So, $v_A = v_o - g t_1$

**Ball** $B$

$$(v_B)_o = v_o;\ (s_B)_o = 0;\ s_B = h;\ t_B = t_1 - t;\ (a_c)_B = -g \text{ Recall } \textit{Equation (2.7)}$$
$$+\uparrow s = s_o + v_o t + \frac{1}{2}a_c t^2$$

$$\text{Hence } h = 0 + v_o(t_1 - t) + \frac{1}{2}(-g)(t_1 - t)^2$$

$$\text{Then, using } \textit{Equation (2.6)}$$
$$+\uparrow v_= v_o + a_c t$$

$$\text{So, } v_B = v_o - g(t_1 - t)$$

$$v_o t_1 - \frac{1}{2}g t_1^2 = v_o(t_1 - t) - \frac{1}{2}g(t_1 - t)^2$$
$$\Longrightarrow\ t_1 = \frac{2v_o + gt}{2g}$$

Thus

- $v_A = \frac{-1}{2}gt$
- $v_B = \frac{1}{2}gt$

## 2.3 Rectilinear Kinematics — Erratic Motion

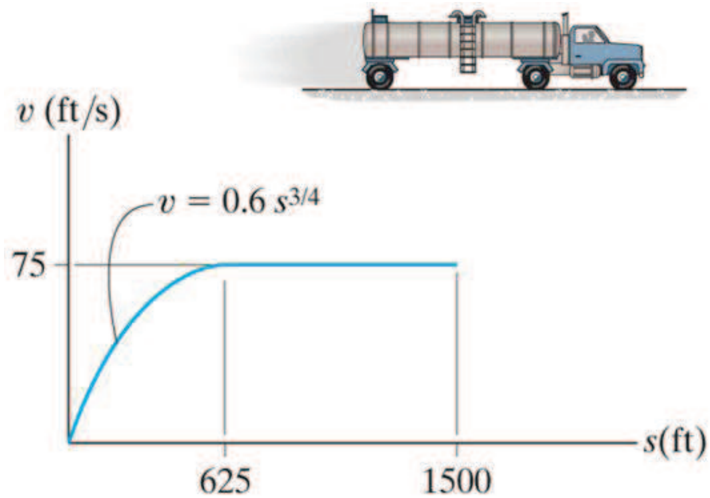
**Objectives** Determine the position, velocity and acceleration of a particle using graphs.

**Application** In many experiments, a speed versus position ($v$-$s$) profile is obtained.

If we have a $v$-$s$ graph for the tank truck, how can we determine its acceleration at position $s = 1500$feet? Graphing provides a good way to handle complex motions that would be difficult to describe with formulas.

Graphs also provide a visual description of motion and reinforce the calculus concepts of differentiation and integration as used in dynamics.

The approach builds on the facts that slope and differentiation are linked and that integration can be thought of as finding the area under a curve.




### 2.3.1 Position — Time Graphs

Plots of position versus time, also called motion curves, can be used to find speed versus time curves. Finding the slope of the line tangent to the motion curve at any point leads to the velocity at that point or $v = \frac{ds}{dt}$

Therefore, the $v$-$t$ graph can be constructed by finding the slope at various points along the $s$-$t$ graph.

### 2.3.2 Velocity — Time Graphs

Plots of speed versus time can be used to find the acceleration versus time curves. Finding the slope of the line tangent to the speed curve at any point leads to the acceleration at that point or $a = \frac{dv}{dt}$.

Therefore, the acceleration versus time or $a$-$t$ graph can be obtained by finding the slope at various points along the $v$-$t$ graph.

Also, the distance moved (displacement) of the particle is the area under the $v$-$t$ graph during a time interval $\Delta t$.

### 2.3.3 Acceleration — Time Graphs

Given the acceleration versus time or *a*-*t* curve, the change in velocity $\Delta v$ during a time period is the area under the *a*-*t* curve.

Thus we can construct a velocity versus time graph from the acceleration versus time graph if we know the initial velocity of the particle.

### 2.3.4 Acceleration — Displacement Graphs

A more complex case is presented by the acceleration versus position or *a*-*s* graph. The area under the *a*-*s* curve characterizes the change in velocity (recall that $\int a \cdot ds = \int v \cdot dv$)

$$\frac{1}{2}(v_1^2 - v_o^2) = \int_{s_1}^{s_2} a \cdot ds \equiv \text{area under the } a\text{-}s \text{ graph}$$

This equation can be solved for $v_1$, allowing you to solve the velocity at a point. By doing this repeatedly, you can create a plot of velocity versus distance.

### 2.3.5 Velocity — Displacement Graphs

Another complex case is presented by the velocity versus distance or *v*-*s* graph. By reading the velocity $v$ at a point on the curve and multiplying it by the slope of the curve $\frac{dv}{ds}$ at this same point, we can obtain the acceleration at that point. Recall the formula

$$a = v \cdot \frac{dv}{ds}$$

Thus, we can obtain an *a*-*s* plot from the *v*-*s* curve.

### 2.3.6 Examples

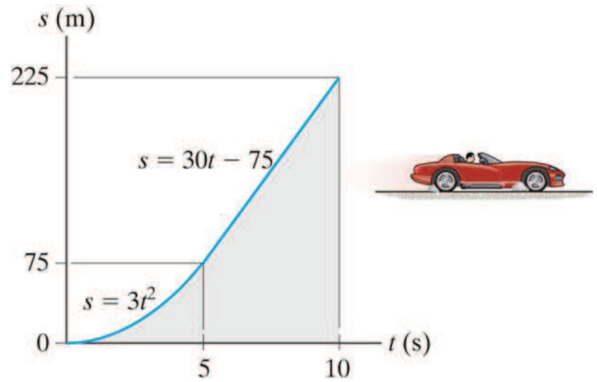
#### Example (2.8)

Given the *s*-*t* graph for a sports care moving along a straight line shown below, find the *v*-*t* graph and *a*-*t* graph over the time interval shown

**Solution**

The *v*-*t* graph can be constructed by finding the slope of the *s*-*t* graph at key points. What are those?

- when $0 < t < 5$s; $v_{0-5} = \frac{ds}{dt} = \frac{d}{dt}(3t^2) = 6t \text{ ms}^{-1}$
- when $f < t < 10$s; $v_{5-10} = \frac{ds}{dt} = \frac{d}{dt}(30t - 75) = 30 \text{ ms}^{-1}$

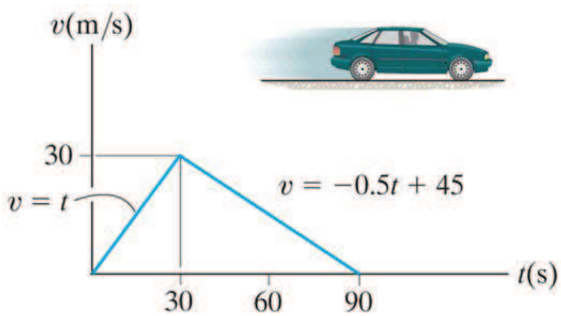



Similarly, the $a$-$t$ graph can be constructed by finding the slope at various points along the $v$-$t$ graph.

- when $0 < t < 5$s; $a_{0-5} = \frac{dv}{dt} = \frac{d}{dt}(6t) = 6 \text{ ms}^{-2}$
- when $5 < t < 10$s; $a_{5-10} = \frac{dv}{dt} = \frac{d}{dt}(30) = 0 \text{ ms}^{-2}$

## Example (2.9)

Given the $v$-$t$ graph shown below, find the $a$-$t$ graph, average speed and distance travelled for the 90 second interval.




### Solution

We need to find the slopes of the $v$-$t$ curve and draw the $a$-$t$ graph. Finding the are under the curve will tell us the distance travelled. Finally, we will calculate the average speed using basic definitions.

Find the $a$-$t$ graph:

- For $0 \leq t \leq 30$s; $a = \frac{dv}{dt} = 1.0 \text{ ms}^{-2}$
- For $30 \leq t \leq 90$s; $a = \frac{dv}{dt} = -0.5 \text{ ms}^{-2}$

Now find the distance travelled:

$$\Delta s_{0-30} = \int v \cdot dt = \frac{1}{2}(30)^2 = 450 \text{ m}$$

$$\Delta s_{30-90} = \int v \cdot dt = \frac{1}{2}(-0.5)(90)^2 + (45)(90) - \frac{1}{2}(-0.5)(30)^2 - (45)(30) - 900 \text{ m}$$

$$s_{0-90} = 450 + 900 = 1350 \text{ m}$$

$$v_{\text{avg}(0-90)} \equiv \frac{\text{total distance}}{\text{time}} = \frac{1350}{90} = 15 \text{ ms}^{-1}$$

## 2.4 General Curvilinear Motion

Curvilinear Motion ≡ particle moves along a cruved path

### 2.4.1 Kinematics

Consider a particle located at a point on a space curve defined by the path function $s(t)$.

### 2.4.2 Position

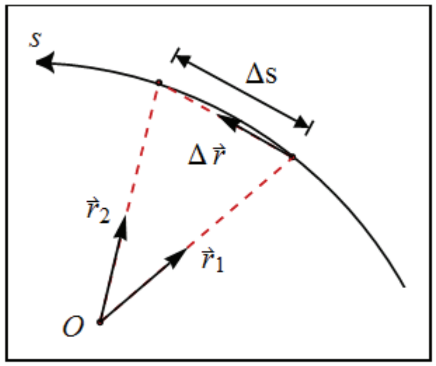
The position of the particle measured prom a fixed point $O$ is designated by the *position vector* $\vec{r}(t)$ which changes magnitude and direction with time $t$ as the particle moves along the curve.

### 2.4.3 Displacement

In a small time interval $\Delta t$, the particle moves a distance $\Delta s$ along the path to a new position $\vec{r'} = \vec{r} + \Delta \vec{r}$

Recall that the displacement of a particle is given by the change in the particles position, as in $\Delta \vec{r} = \vec{r'} - \vec{r}$

### 2.4.4 Velocity




In a time interval $\Delta t = t_2 - t_1$, the particle moves from the position $s_1$ to the position $s_2$.

#### Average Velocity

The average velocity of a particle during a time interval $\Delta t$ is:

$$\vec{v}_{\text{avg}} = \frac{\Delta \vec{r}}{\Delta t}$$

#### Instantaneous Velocity

The particles instantaneous velocity at a time $t = t_2$ at $s_2$ or $\vec{r}_2$ is given by

$$\lim_{\Delta t \to 0} \vec{v}_{avg} = \frac{d}{dt}(\vec{r}) \tag{2.9}$$

Note that as the time interval $\Delta t$ approaches zero,

- The magnitude $|\Delta \vec{r}|$ approaches the arc length $\Delta s = s_2 - s_1$
- The direction of $\Delta \vec{r}$ approaches the tangential direction $\vec{u}_t$ at the position $s_2$ where $\vec{u}$ is a unit vector.

Thus, we could express *Equation (2.9)* equivalently as

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} \vec{u}_t$$

where $\vec{v}$ has magnitude $|\vec{v}| = \frac{d}{dt}(s)$ (where speed $\equiv$ the derivative of the path at $r_n$ with respect to time $t$) and direction $\vec{u}_t \equiv$ the tangential unit vector.

### 2.4.5 Acceleration

$$\text{Average acceleration on a time interval } \Delta t \equiv \frac{\Delta\vec{v}}{\Delta t}$$

$$\text{Instaneous acceleration} \equiv \vec{a} = \frac{d}{dt}(\vec{v}) \overset{(12.9)}{=} \frac{d^2}{dt^2}(\vec{r}) \tag{2.10}$$

Acceleration is a vector with magnitude $|\vec{a}| = \left|\frac{d}{dt}\vec{v}\right|$ and a direction not necessarily tangent to the path of motion.

### 2.4.6 Summary

- Curvilinear motion can cause changes in both the magnitude and direction of the position, velocity and acceleration vectors.
- Velocity vector is always tangent to path.
- The acceleration vector is, in general, not tangent to the path.

$$\vec{v}(t) = v(t)\vec{u}_t$$

$$\vec{a}(t) = \frac{d}{dt}\vec{v}(t) = \frac{d}{dt}(\vec{u}_t) + v\frac{d}{dt}(\vec{u}_t)$$

## 2.5 Curvilinear Motion — Rectangular Components

The quantities $\vec{r}$, $\vec{v}$, $\vec{a}$ mentioned in the previous section are vectors and, as such,have different representations according to which basis (coordinate system) to which they are referred. We choose the easiest basis or coordinate system for calculation purpose. This choice depends on physics, path, etc.

The 'rectangular basis' is the easiest to implement and most useful when motion is rectilinear, although it can also be used on a curved path.

### 2.5.1 Position

$$\vec{r}(t)' = |\vec{r}(t)|\vec{u}_r = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k} \tag{2.11}$$

Where $|\vec{r}(t)| = \sqrt{x(t)^2 + y(t)^2 + z(t)^2}$
$\vec{u}_r$ is the unit vector $\frac{\vec{r}}{r}$

### 2.5.2 Velocity

$$\vec{v} = \frac{d}{dt}(\vec{r}) \overset{(2.11)}{=} \frac{d}{dt}\left[x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}\right]$$

Since $\vec{i}$, $\vec{j}$ and $\vec{k}$ are constant for all time $t$ in magnitude and direction.

$$\vec{v} = \underset{\dot{x}}{\underset{\uparrow}{v_x}}\vec{i} + \underset{\dot{y}}{\underset{\uparrow}{v_y}}\vec{j} + \underset{\dot{z}}{\underset{\uparrow}{v_z}}\vec{k} \tag{2.12}$$

$$\text{So } \vec{v} = |\vec{v}|\vec{u}_v \text{ where } |\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

Thus the tangential direction of the speed of a particle at time $t \equiv \vec{u}_v = \dfrac{\vec{v}}{v} = \vec{u}_t$

### 2.5.3 Acceleration

$$\vec{a} = \frac{d}{dt}(\vec{v}) = |\vec{a}|\vec{u}_a = a_x\vec{i} + a_y\vec{j} + a_z\vec{k} \tag{2.13}$$

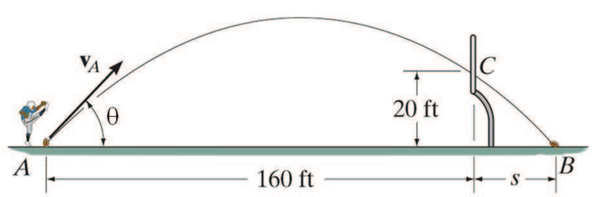
$$\text{Where } \begin{aligned} a_x &= \dot{v}_x = \ddot{x} \\ a_y &= \dot{v}_y = \ddot{y} \\ a_z &= \dot{a}_z = \ddot{z} \end{aligned} \tag{2.14}$$

Note that the direction of acceleration $\vec{u}_a = \frac{\vec{a}}{a}$ is not, in general, tangent to the path of the particle. Recall that $\vec{a}$ represents time rate of change in both magnitude and direction of velocity $\vec{v}$.

## 2.6 Curvilinear Motion — Motion of a Projectile

Our objective is to analyse the free-flight motion of a projectile. Basically a special case of the equations of constant acceleration.

Consider the following application: A good kicker instinctively knows at what angle,




$\theta$, and initial velocity, $\vec{v}_A$, he must kick the ball to make a field goal. For a given kick "strength", at what angle should the ball be kicked to get the maximum distance?

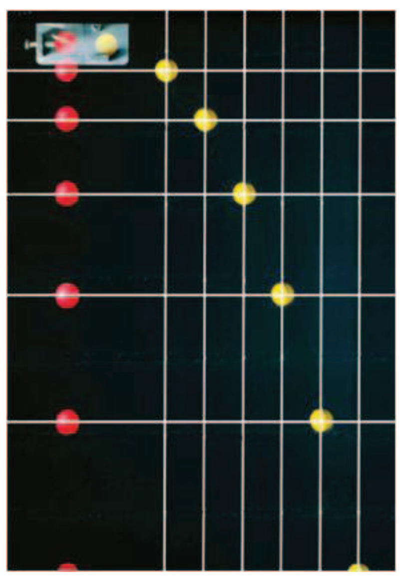
Consider another application: A basketball is shot at a certain angle. What parameters should the shooter consider in order for the basketball to pass through the basket?

A fire-fighter needs to know the maximum height on the wall she can project water from the hose. What parameters would you program into a wrist computer to find the angle at which she should aim the hose?

### 2.6.1 Theory

Projectile motion can be treated as two rectilinear motions, one in the horizontal direction experiencing zero acceleration and the other in the vertical direction experiencing constant acceleration from gravity.

Consider the following illustration. Consider the two balls on the left. The red ball falls from rest, whereas the yellow ball is given a horizontal velocity. Each picture in this sequence is taken after the same time interval. Notice both balls are subjected to the same downward acceleration since they remain at the same elevation at any instant. Also, notice that the horizontal distance between successive photos of the yellow ball is constant since the velocity in the horizontal direction is constant.



### 2.6.2 Kinematic Equations — Horizontal Motion

Recall the kinematic equations of rectilinear motion with constant acceleration. Applying these equations to the horizontal motion of a projectile.

$$\xrightarrow{+} v = v_o + a_c t \quad \text{becomes} \quad v_x = (v_o)_x$$
$$\xrightarrow{+} x = x_o + v_o t + \tfrac{1}{2} a_c t^2 \quad \text{becomes} \quad x = x_o + (v_o)_x t$$
$$\xrightarrow{+} v^2 = v_o^2 + 2a_c(x - x_o) \quad \text{becomes} \quad v_x = (v_o)_x$$

The first and second equation indicate that the horizontal component of velocity always remains constant during motion.

### 2.6.3 Kinematic Equations — Vertical Motion

Since the positive y-axis is directed upwards, $a_y = -g$. Application of the constant acceleration equations yields:

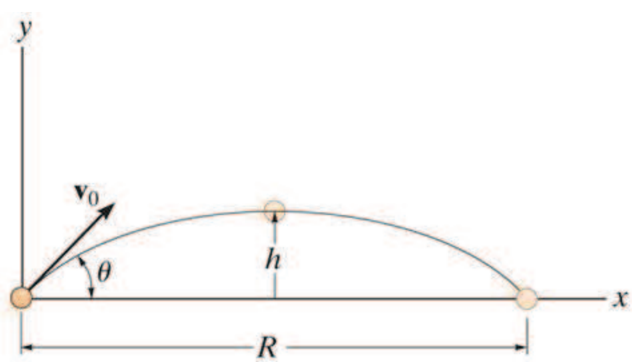
$$\uparrow + v_y = (v_o)_y - gt$$
$$\uparrow + y = y_o + (v_o)_y t - \tfrac{1}{2} g t^2$$
$$\uparrow + v_y^2 = (v_o)_y^2 - 2g(y - y_o)$$

For any given problem, only two of these three equations can be used because only two of the three are independent.

### 2.6.4 Examples

#### Example (2.10)

Given $v_o$ and $\theta$, find the equation that defines the $y$-coordinate of the particle as a function of $x$.



**Solution**

We want to eliminate time from the kinematic equations.

Using $v_x = v_o \cos(\theta)$ and $v_y = v_o \sin(\theta)$ we can write:

$$x = (v_o \cos(\theta))t \qquad \text{or} \qquad t = \frac{x}{v_o \cos(\theta)}$$

$$y = (v_o \sin(\theta)) \left( \frac{x}{v_o \cos(\theta)} \right) - \frac{g}{2} \left( \frac{x}{v_o \cos(\theta)} \right)^2$$

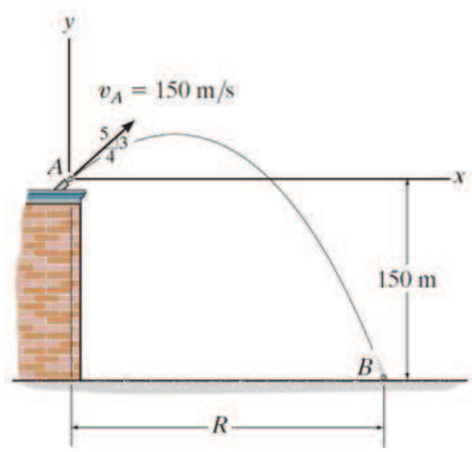
Simplifying this equation, we get:

$$y = x \tan(\theta) - \frac{g x^2}{2 v_o^2} (1 + \tan^2(\theta))$$

The above equation is called the 'path equation' which describes the path of a particle in projectile motion. The equation shows that the path is parabolic for each $\theta$.

## Example (2.11)

Suppose a projectile is fired with $v_A = 150$ ms$^-1$ at point $A$. Find the horizontal distance $R$ it travels and the time for which it is in the air.



**Solution**

We will establish a fixed $x$, $y$ coordinate system (in this solution, the origin of the coordinate system is places at $A$). Apply the kinematic equation in the $x$ and $y$ direction.

Place the coordinate system at point $A$ then write the equation for horizontal motion.

$$\xrightarrow{+} x_B = x_A + v_{Ax} t_{AB}$$

$$\text{Where} \quad \begin{aligned} x_B &= R \\ x_A &= 0 \\ v_{Ax} &= 150\left(\tfrac{4}{5}\right)\ \text{ms}^{-1} \end{aligned}$$

Thus the distance $R$ will be $R = 120t_{AB}$

Now, we will use the vertical motion equations.

$$\uparrow + y_B = y_A + v_{Ay} t_{AB} - 0.5gt_{AB}^2$$

$$\text{Where} \quad \begin{aligned} y_B &= -150 \\ y_A &= 0 \\ v_{Ay} &= 150\left(\tfrac{3}{5}\right)\ \text{ms}^{-1} \end{aligned}$$

Thus we get the following equation:

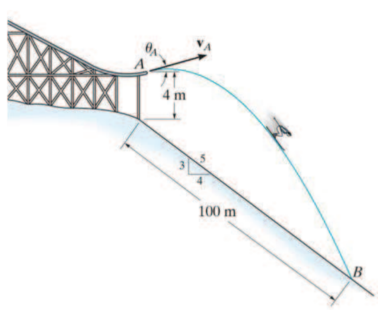
$$-150 = 90t_{AB} + 0.5(-9.81)t_{AB}^2$$

Thus we can find that $t_{AB} = 19.89$ s

$$\text{Then, } R = 120t_{AB} = 120(19.89) = 2387 \text{ m} \simeq 2.39 \text{ km}$$

## Example (2.12)

A skier leaves the ski jump ramp at $\theta_A = 25^o$ and hits the slope at $B$. Find the skier's initial speed $v_A$.



**Solution**

We will establish a fixed $x$,$y$ coordinate system which in this solution has its origin placed at $A$. Apply the kinematic relations in the $x$ and $y$-directions.

**Horizontal Motion**

$$x = x_o + (v_o)_x t \implies (\frac{4}{5})(100) = 0 + v_o \cos(25^o)t$$

$$t = \frac{80}{v_o \cos(25^o)} = \frac{88.27}{v_o}$$

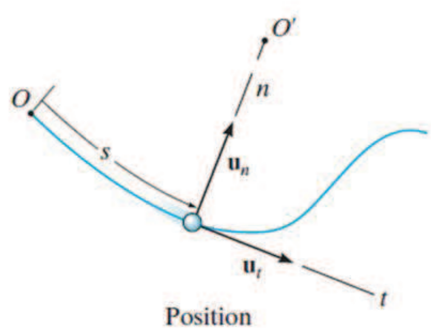
**Vertical Motion**

$$y = y_o + (v_o)_y t - \frac{1}{2} g t^2$$

$$-64 = 0 + v_o \sin(25^o)\left(\frac{88.27}{v_o}\right) - \frac{1}{2}(9.81)\left(\frac{88.27}{v_o}\right)^2$$

$$\text{Hence } v_o = 19.42 \text{ ms}^{-1}$$

## 2.7 Normal and Tangential Coordinate Systems




When a particle moves along a curved path, it is sometimes convenient to describe its motion using coordinates other than Cartesian. When the path of motion is known, normal $n$ and tangential $t$ coordinates are often used.

In the $n$-$t$ coordinate system, the origin is located on the particle such that the origin moves with the particle.

The $t$-axis is **tangent** to the path or curve at the instant considered and is positive in the direction of the particle's motion.
The $n$-axis is **perpendicular** to the $t$-axis with the positive direction towards the centre of curvature of the curve.

The positive $n$ and $t$ directions are defined by the unit vectors $\vec{u}_n$ and $\vec{u}_t$ respectively. The centre of curvature, $O'$, always lies of the concave side of the curve. The radius of curvature, $\rho$, is defined as the perpendicular distance from the curve to the centre of curvature at that point.

The position of the particle at any given instant is defined by the distance $s$ along the curve from a fixed reference point.

### 2.7.1 Velocity

The velocity vector is **always** tangent to the path of motion, that is the $t$-direction.

The magnitude of that velocity is determined by taking the time derivative of the path function $s(t)$ such that,

$$\vec{v} = |\vec{v}|\vec{u}_t \tag{2.15}$$

$$|\vec{v}| = \dot{s} = \frac{d}{dt}s \tag{2.16}$$

Where $|\vec{v}|$ defined the magnitude of the velocity, as in speed, and $\vec{u}_t$ defines the direction of the velocity vector.

### 2.7.2 Acceleration

Recall that acceleration is the time rate of change of velocity. That is,

$$\vec{a} = \frac{d}{dt}(|\vec{v}|\vec{u}_t) = \dot{v}\vec{u}_t + |\vec{v}|\dot{u}_t \tag{2.17}$$

Where $\dot{v}$ represents the change in the magnitude of velocity
$\dot{u}_t$ represents the change in the direction of $\vec{u}_t$

The above equation can be restated in turns of the radius of curvature $\rho$ of the particles path such that,

$$\vec{a} = \dot{v}\vec{u}_t + \left(\frac{v^2}{\rho}\right)\vec{u}_n = |\vec{a}|\vec{u}_t + |\vec{a}_n|\vec{u}_n \tag{2.18}$$

where there are two components to the acceleration vector.

- The tangential component is tangent to the curve and in the direction of increasing or decreasing velocity.

$$a_t = \dot{v} \quad \text{or} \quad a_t \cdot ds = v \cdot dv \tag{2.19}$$

- The normal or centripetal component is always directed toward the centre of curvature of the curve such that,

$$a_n = \frac{v^2}{\rho} \tag{2.20}$$

- The magnitude of the acceleration vector is,

$$|\vec{a}| = \sqrt{(a_t)^2 + (a_n)^2} \tag{2.21}$$

### 2.7.3 Special Cases of Motion

Consider the following special cases of motion.

- The particle moves along a straight line such that,

$$\rho \to \infty \implies a_n = \frac{v^2}{\rho} = 0 \implies a = a_t = \dot{v}$$

  The tangential component represents the time rate of change in the magnitude of the velocity.

- The particle moves along a curve with a constant speed.

$$a_t = \dot{v} = 0 \implies a = a_n = \frac{v^2}{\rho}$$

  The normal component represents the time rate of change in the direction of the velocity.

- The tangential component of acceleration is constant such that $a_t = (a_t)_c$. In this case,

$$s = s_o + v_o t + \frac{1}{2}(a_t)_c t^2$$
$$v = v_o + (a_t)_c t$$
$$v^2 = (v_o)^2 + 2(a_t)_c(s - s_o)$$

  As before, $s_o$ and $v_o$ are the initial position and velocity of the particle at $t = 0$ respectively.

- The particle moves along a path expressed as $y = f(x)$. Then the radius of curvature $\rho$ at any given point along that path can be calculated using the following equation.

$$\rho = \frac{\left(1 + \left(\frac{d}{dx}(y)\right)^2\right)^{\frac{3}{2}}}{\left|\frac{d^2}{dx^2}(y)\right|} \tag{2.22}$$

### 2.7.4 Three-Dimensional Motion

If a particle moves along a curve in a three dimensional space, or a *space curve*, the normal $n$ and tangential $t$ axes are defined as before. At any point, the $t$-axis is tangent to the path of motion and the $n$-axis points towards the centre of curvature. The plane, containing the normal and tangential axes is called the osculating plane.

A third axis can be defined, called the *binomial axis* $b$ such that the binomial unit vector $\vec{u}_b$ is directed perpendicular to the osculating plane, and its sense is defined by the following cross product.

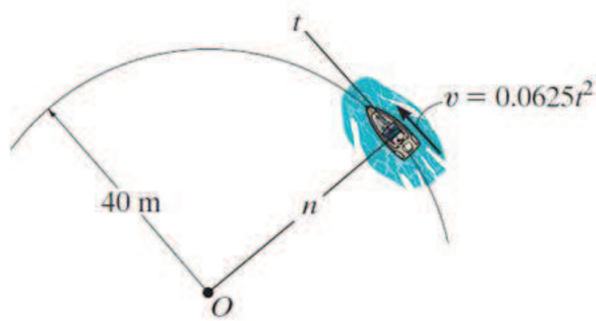
$$\vec{u}_b = \vec{u}_t \times \vec{u}_n$$

There is no motion, thus no velocity or acceleration in the binomial direction.

### 2.7.5 Examples

#### Example (2.13)

Consider a boat travelling around in a circular path such that the radius of curvature $\rho = 40$ m with a speed that increases with time defined by $v = (0.0625t^2)$ ms$^{-1}$.
Find the magnitudes of the boat's velocity and acceleration at the instant $t = 10$s.




#### Solution

The boat starts from rest where $v_o = 0$ and $t_o = 0$

- Calculate the velocity at $t = 10$s using $v(t)$
- Calculate the tangential and normal components of acceleration and then the magnitude of the acceleration vector.

The velocity vector is $\vec{v} = |\vec{v}|\vec{u}_t$ where the magnitude is given by $v = (0.0625t^2)$ms$^{-1}$. Then, at time $t = 10$s

$$v = 0.0625t^2 = 0.0625(10)^2 = 6.25 \text{ ms}^{-1}$$

The acceleration vector is $\vec{a} = |\vec{a}_t|\vec{u}_t + |\vec{a}_n|\vec{u}_n = \dot{v}\vec{u}_n + \left(\frac{v^2}{\rho}\right)\vec{u}_n$.

**Tangential Component**

$$a_t = \dot{v} = \frac{d}{dt}(0.0625t^2) = 0.125t \text{ ms}^-2$$

Where $t = 10$s such that $a_t = 0.125t = 0.125(10) = 1.25 \text{ ms}^{-2}$

**Normal Component**

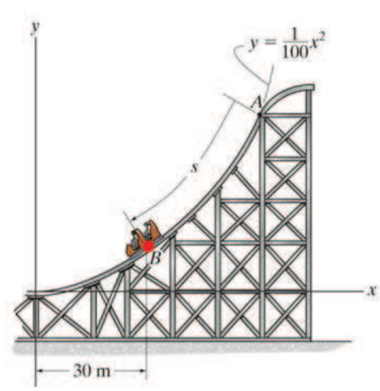
$$a_n = \frac{v^2}{\rho} \text{ ms}^{-2}$$

Where $t = 10$s such that $a_n = \frac{(6.25)^2}{40} = 0.9766 \text{ ms}^{-2}$

Thus, the magnitude of the acceleration is given by

$$|\vec{a}| = \sqrt{(a_t)^2 + (a_n)^2} = \sqrt{(1.25)^2 + (0.9766)^2} = 1.59 \text{ ms}^{-2}$$

## Example (2.14)




Consider a roller coaster travelling along a vertical parabolic path defined by the equation $y = 0.01x^2$. At point $b$, it has a speed of $25\text{ms}^{-1}$, which is increasing at the rate of $3\text{ms}^{-2}$. Find the magnitude of the roller coaster's acceleration when it is at point $B$.

### Solution

- The change in the speed of the car ($3\text{ms}^{-1}$) is the tangential component of the total acceleration.
- Calculate the radius of curvature of the path at $B$.

- Calculate the normal component of acceleration.

- Determine the magnitude of the acceleration vector.

The tangential component of acceleration is the rate of increase of the roller coaster's speed, so

$$\vec{a}t = \dot{v} = 3 \text{ ms}^{-2}$$

Determine the radius of curvature at point $B$ where $x = 30$m.

$$\frac{d}{dx}(y) = \frac{d}{dx}(0.01x^2) = 0.02x$$

$$\frac{d^2}{dx^2}(y) = \frac{d}{dx}(0.02x) = 0.02$$

Thus, where $x = 30$m, we find that $\frac{d}{dx}(y) = 0.02(30) = 0.6$ and $\frac{d^2}{dx^2}(y) = 0.02$.

Recall *Equation (2.22)* for radius of curvature.

$$\rho = \frac{\left(1 + \left(\frac{d}{dx}(y)\right)^2\right)^{\frac{3}{2}}}{\left|\frac{d^2}{dx^2}(y)\right|} = \frac{\left(1 + (0.6)^2\right)^{\frac{3}{2}}}{0.02} = 79.3 \text{ m}$$

The normal component of acceleration can be found using *Equation (2.20)* such that,

$$\vec{a}_n = \frac{v^2}{\rho} = \frac{(25)^2}{(79.3)} = 7.881 \text{ ms}^{-2}$$

The magnitude of the acceleration vector is found using *Equation (2.21)*.

$$|\vec{a}| = \sqrt{(a_t)^2 + (a_n)^2} = \sqrt{(3)^2 + (7.881)^2} = 8.43 \text{ ms}^{-2}$$

### Example (2.15)

Consider a van travelling over a hill described by the equation

$$s_y = \left[(-1.5 \cdot 10^{-3})(s_x)^2 + 15\right]\text{m} \xrightarrow{+}$$

If the van has a constant speed of $75\text{ms}^{-1}$, find the $x$ and $y$ components of the vans acceleration when $s_x = 50\text{m}$ and the van is travelling in the negative $x$ direction.

#### Solution

$$s_y = (-1.5 \cdot 10^{-3})(s_x)^2 + 15$$
$$\frac{d}{dt}(s_y) = 2(-1.5 \cdot 10^{-3})s_x \cdot \frac{d}{dt}(s_x)$$
$$\vec{v}_y = (-3 \cdot 10^{-3})s_x \cdot \vec{v}_x$$
$$\vec{v}_y\Big|_{s_x=50\text{m}} = (-3 \cdot 10^{-3})(50)\vec{v}_x = (-0.15)\vec{v}_x$$

The speed of the van has a constant magnitude of $75\text{ms}^{-1}$ such that,

$$75 = \sqrt{(\vec{v}_x)^2 + (\vec{v}_y)^2} = \sqrt{(\vec{v}_x)^2 + \big((-0.15)\vec{v}_x\big)^2} \implies \vec{v}_x\Big|_{s_x=50\text{m}} = \pm 74.2\text{ms}^{-1}$$

Recall that the van is travelling in the negative $x$ direction so we will choose $\vec{v}_x$ to be,

$$\vec{v}_x\Big|_{s_x=50\text{m}} = -74.2\text{ms}^{-1} \xrightarrow{+}$$

Thus, we can find the acceleration as follows,

$$\vec{a}_y = \frac{d}{dt}(\vec{v}_y) = \frac{d}{dt}\left((-3 \cdot 10^{-3})s_x \cdot \underbrace{\frac{d}{dt}(s_x)}_{\vec{v}_x}\right) = (-3 \cdot 10^{-3})\left(\left(s_x \cdot \underbrace{\frac{d}{dt}(\vec{v}_x)}_{\vec{a}_x}\right) + \left(\underbrace{\frac{d}{dt}(s_x) \cdot \frac{d}{dt}(s_x)}_{(\vec{v}_x)^2}\right)\right)$$

$$\therefore \vec{a}_y = (-3 \cdot 10^{-3})\Big((s_x \cdot \vec{a}_x) + (\vec{v}_x)^2\Big)$$

Note that the constant magnitude of velocity has the following implications on acceleration.

$$\vec{a} = \frac{d}{dt}(\vec{v}) = \frac{d}{dt}\Big(|\vec{v}| \cdot \vec{u}_t\Big) = \left(\left(|\vec{v}| \cdot \frac{d}{dt}(\vec{u}_t)\right) + \left(\vec{u}_t \cdot \frac{d}{dt}|\vec{v}|\right)\right)$$

Because the speed is constant, we can suppose that the tangential acceleration is zero.

$$\vec{a}_t = |\vec{a}_t| \cdot \vec{u}_t = \frac{d}{dt}|\vec{v}| \cdot \vec{u}_t = 0 \implies \vec{a} = |\vec{v}| \cdot \frac{d}{dt}(\vec{u}_t)$$

So, we need to find $\vec{u}_t$ when $s_x = 50\text{m}$

$$\tan\theta = \text{slope when } s_x \text{ is } 50\text{m} \equiv \frac{d}{ds_x}(s_y)\bigg|_{s_x=50\text{m}}$$

Recall that $s_y = (-1.5\cdot 10^{-3})(s_x)^2 + 15$ such that $\frac{d}{ds_x}(s_y) = (-3\cdot 10^{-3})s_x$

$$\text{Thus, } \theta = \tan^{-1}\big((-3\cdot 10^{-3})s_x\big) \implies \theta\big|_{s_x=50\text{m}} = -8.531^o$$

We know that $\vec{u}_t = \cos(\theta)\vec{i} + \sin(\theta)\vec{j}$ where $\theta$ is the angle relative to the $x$-axis such that,

$$\frac{d}{dt}(\vec{u}_t) = \left(-\sin(\theta)\cdot\frac{d}{dt}(\theta)\right)\vec{i} + \left(\cos(\theta)\cdot\frac{d}{dt}(\theta)\right)\vec{j}$$

$$\text{Where } \frac{d}{dt}(\theta) = \frac{d}{dt}\left(\tan^{-1}\big((-3\cdot 10^{-3})s_x\big)\right) = \frac{(-3\cdot 10^{-3})\cdot\frac{d}{dt}(s_x)}{1+\big(-3\cdot 10^{-3}(s_x)\big)^2} = \frac{(-3\cdot 10^{-3})\cdot\vec{v}_x}{1-\big(3\cdot 10^{-3}(s_x)\big)^2}$$

Recall that when $s_x = 50\text{m}$, we get $\vec{v}_x = -74.2 \xrightarrow{+}$ such that,

$$\frac{d}{dt}(\theta)\bigg|_{s_x=50\text{m}} = \frac{(-3\cdot 10^{-3})\cdot(-74.2)}{1-\big(3\cdot 10^{-3}(50)\big)^2} = 0.23$$

Thus we can find the following,

$$\frac{d}{dt}(\vec{u}_t)\bigg|_{s_x=50\text{m}} = \big(-\sin(-8.531)\cdot(0.23)\big)\vec{i} + \big(\cos(-8.531)\cdot(0.23)\big)\vec{j} = 0.034\vec{i} + 0.227\vec{j}$$

Recall that $\vec{a} = |\vec{v}|\cdot\frac{d}{dt}(\vec{u}_t)$ such that,

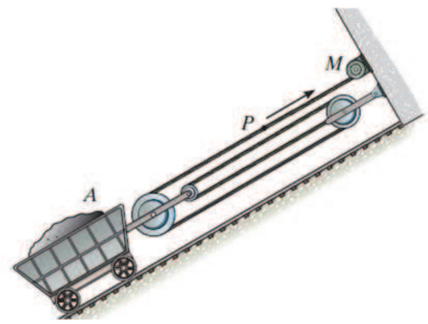
$$\vec{a} = (75)(0.034\vec{i} + 0.227\vec{j}) = [2.55\vec{i} + 17.03\vec{j}]\text{ms}^{-2}$$

## 2.8 Absolute Motion Analysis

**Objectives** We want to relate the positions, velocities and accelerations of two particles undergoing dependent motion.

**Applications** Consider a cable and puller system shown below which can be used to modify the speed of the mine car $A$ relative to the speed of the motor $M$.

We want to establish the relationships between the various motions in order to determine the power requirements for the motor and the tension in the cable.




For instance, if the speed of the cable $P$ is known because we know the motor characteristics, how can we determine the speed of the mine car? Will the slope of the track have any impact on the answer?

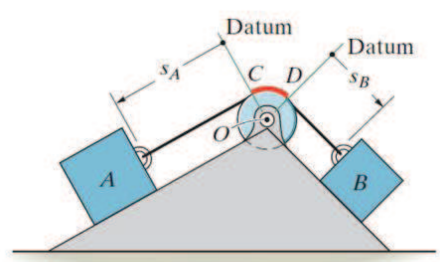
Rope and pulley arrangements are often used to assist in lifting heavy objects. The total lifting fore required from the truck depends on both the weight and the acceleration of the cabinet.

How can we determine the acceleration and velocity of the cabinet if the acceleration of the truck is known?

### 2.8.1 Dependent Motion

In many kinematics problems, the motion of one object will depend on the motion of another object.

The blocks in this figure are connected by an inextensible cord wrapped around a pulley. If block $A$ moves downward along the inclined plane, then block $B$ will move up the other inclined plane.




The motion of each block can be related mathematically by defining the position coordinates $s_A$ and $s_B$. Each coordinate axis is defined from a fixed point or datum line,measured positive along each plane in the direction of motion of each block.

In this example, the position coordinates $s_A$ and $s_B$ can be defined from the fixed datum lines extending from the centre of the pulley along each inclined plane to the blocks $A$ and $B$ respectively such that the datums extend in the direction of motion of each block.

If the cord has a fixed length, the position coordinates $s_A$ and $s_B$ are related mathematically by the following equation,

$$s_A + l_{CD} + s_B = l_T$$

Where $l_T$ is the total cord length
$l_{CD}$ is the length of cord passing over the arc $CD$ on the pulley.

Consider a more complicated example. The position coordinates $s_A$ and $s_B$ are defined from fixed datum lines measured along the direction of motion of each block.

The position coordinates are related by the following equation

$$2s_B + h + s_A = l_T$$

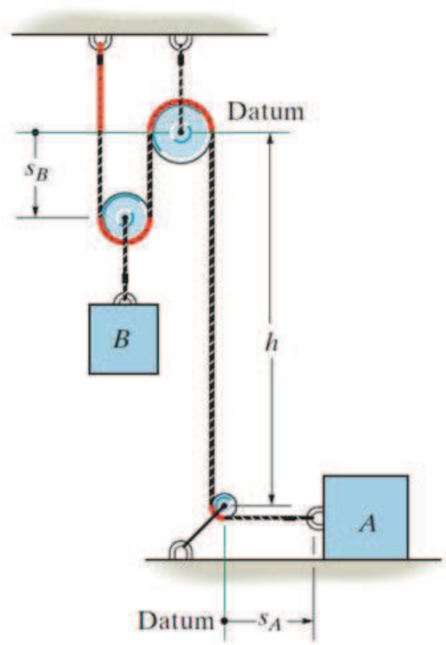
Where $l_T$ is the total cord length minus the lengths of the red segments

Since $l_T$ and $h$ remain constant during the motion, the velocities and accelerations can be related by two successive time derivatives such that,

$$2v_B = -v_A \qquad \text{and} \qquad 2a_B = -a_A$$

When block $B$ moves downwards such that $s_B$ increases, block $A$ moves to the left such that $s_A$ decreases.

This example can also be solved by defining the position coordinate for $B$ as $s_B$ from the bottom pulley instead of the top pulley.




The position, velocity and acceleration equations then become,

$$2(h - s_B) + h + s_A = l_T$$

$$2v_B = v_A$$

$$2a_B = a_A$$

Such that the results are the same, even if the sign conventions are different than the previous method.

### 2.8.2 Dependent Motion — Procedures

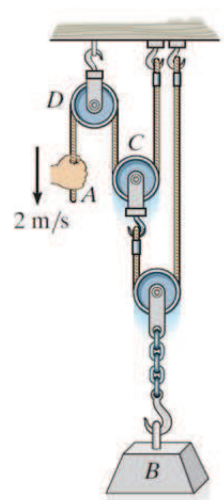
These procedures can be used to relate the dependent motion of particles moving along rectilinear paths. Not that only the magnitudes of velocity and acceleration change while their directions remain constant.

- Define the position coordinates from the fixed datum lines along the path of each particle. Note that different datum lines can be used for each particle.
- Relate the position coordinates to the cord length. Segments of the cord that do not change in length during the motion may be left out.
- If a system contains more than one cord, relate the position of a point on one cord to a point on another cord. Separate equations are written for each independent cord.
- Differentiate the position coordinate equations to relate velocities and accelerations. Note the sign conventions

### 2.8.3 Examples

#### Example (2.16)

In the figure below, the cord at $A$ is pulled downwards with a speed on $2\text{ms}^{-1}$. Find the speed of block $B$.



**Solution**

Note that there are two cords involved in the motion of this example. Thus there will be two position equations. Write these two equations, combine them, and then differentiate them.

Define the position coordinate from a fixed datum line. Three coordinates must be defined:

- $s_A$ for point $A$
- $s_B$ for point $B$
- $s_C$ for point $C$

Define the datum line through the top pullet which has a fixed position.

- Define $s_A$ from the datum line to the point $A$
- Define $s_B$ from the datum line to the centre of the pulley above $B$
- Define $s_C$ from the datum line to the centre of pulley $C$

Note that all coordinates are defined as positive downwards and along the direction of motion of each point or object.

Thus we will write the position length equations for each cord. Define $l_1$ as the length of the first cord minus any segments of constant length. Define $l_2$ in a similar manner for the second cord such that,

$$\text{For cord 1}: \; s_A + 2s_C = l_1$$

$$\text{For cord 2}: \; s_B + (s_B - s_C) = l_2$$

Eliminating $s_C$ between the two equations gives us the following.

$$s_A + 4s_B = l_1 + 2l_2$$

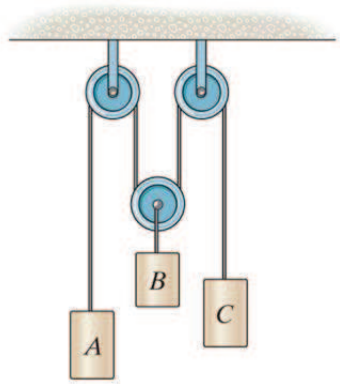
Relate velocities by differentiating the expression above. Note that $l_1$ and $l_2$ are constant.

$$v_A + 4_B = 0 \implies v_B = 1\frac{1}{4}v_A = -\frac{1}{4}(2) = -0.5 \text{ ms}^{-1}$$

The velocity of block $B$ is 0.5 $\text{ms}^{-1}$ up such that $v_B$ is in the negative $s_B$ direction.

### Example (2.17)

In the pulley system shown below, block $A$ is moving downwards with a speed of $4\frac{\text{ft}}{\text{s}}$ while the block $C$ is moving up at $2\frac{\text{ft}}{\text{s}}$. Find the speed of block $B$.




### Solution

All blocks are connected to a single cable such that only one position/length equation will be required. We will define position coordinates for each block and write out the position relation then differentiate it to relate the velocities.

The datum line can be drawn through the upper fixed pulleys and position coordinates defined from this line to each block or the pulley above the block.

- Define $s_A$ from the datum line to block $A$.
- Define $s_B$ from the datum line to centre of the pulley above block $B$.
- Define $s_C$ from the datum line to block $C$.

Thus we obtain the following expressions.

$$s_A + 2s_B + s_C = 1$$

By differentiating with respect to time, we obtain the following relations concerning velocities.

$$v_A + 2v_B + v_C = 0 \implies 4 + 2v_B + (-2) = 0 \implies v_B = -1\frac{\text{ft}}{\text{s}}$$

Thus the velocity of block $B$ is $1\frac{\text{ft}}{\text{s}}$ upwards in the negative $s_B$ direction.

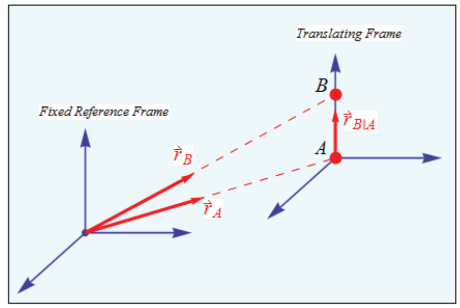
## 2.9 Relative Motion using Translating Axes

Consider two particles $A$ and $B$ in a space defined by a fixed reference frame or *observer*.

$\vec{r}_{B|A}$ Defines the position vector representing the position of $B$ relative to $A$.

$\vec{r}_A$ Defines the *absolute position* of $A$ as a position vector representing the position of $A$ with respect to the fixed reference frame.

$\vec{r}_B$ Defines the *absolute position* of $B$ as a position vector representing the position of $B$ with respect to the fixed reference frame.




Comparing these position vectors, we obtain the following equation.

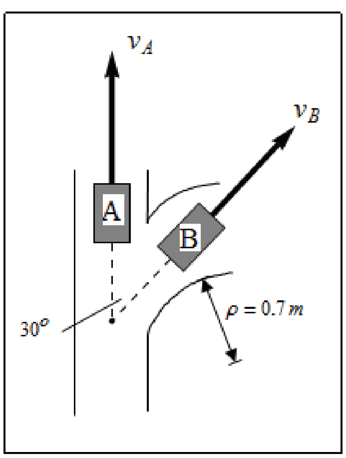
$$\vec{r}_{B|A} = \vec{r}_B - \vec{r}_A \tag{2.23}$$

By taking the derivative of this equation with respect to time, we obtain the velocity.

$$\vec{v}_{A|B} = \vec{v}_B - \vec{v}_A \tag{2.24}$$

Similarly, the next higher derivative gives us the following equation for acceleration,

$$\vec{a}_{A|B} = \vec{a}_B - \vec{a}_A \tag{2.25}$$




### Example (2.18)

At the instant shown, cars $A$ and $B$ travel at $70\text{ms}^{-1}$ and $50\text{ms}^{-1}$ respectively. If car $B$ is increasing its speed by $100\text{ms}^{-1}$ while car $A$ remains at a constant speed, find the velocity $\vec{v}_{B|A}$ and acceleration $\vec{a}_{B|A}$ of car $B$ with respect to car $A$.

**Solution**

Recall equation (2.24),

$$\vec{v}_{B|A} = \vec{v}_B - \vec{v}_A$$

Using the given velocity vectors we get,

$$\vec{v}_{B|A} = \big(\sin(30^o)\vec{i} + 50\cos(30^o)\vec{j}\big) - \big(70\vec{j}\big)$$

$$\vec{v}_{B|A} = [25.0\vec{i} - 26.7\vec{j}]\text{ms}^{-1}$$

For acceleration, we will use equation (2.25),

$$\vec{a}_{B|A} = \vec{a}_B - \vec{a}_A$$

Given that car $B$ has curvilinear motion, we can describe its absolute acceleration as,

$$\vec{a}_B \overset{(2.18)}{=} (\vec{a}_B)_t\vec{u}_t + (\vec{a}_B)_n\vec{u}_n$$

We know the following,

- $(\vec{a}_B)_t = 100\text{ms}^{-1}$ which is given.
- The tangent unit vector $\vec{u}_t = \sin(30^o)\vec{i} + \cos(30^o)\vec{j}$
- $(a_B)_n \overset{(2.20)}{=} \frac{(v_B)^2}{\rho} = \frac{(50)^2}{(0.7)} = 3571.43\text{ ms}^{-2}$

To find $\vec{u}_n$, we choose a vector such that $\vec{u}_n \cdot \vec{u}_t = 0$ (ie. $\vec{u}_n$ and $\vec{u}_t$ are orthogonal) and so $\vec{u}_n$ points towards the centre of curvature.

$$\vec{u}_t = \begin{bmatrix} \sin(30^o) \\ \cos(30^o) \end{bmatrix} \qquad \vec{u}_n = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\vec{u}_t \cdot \vec{u}_n = 0 = x\sin(30^o) + y\cos(30^o) \iff x\sin(30^o) = -y\cos(30^o)$$

$$\implies x = \cos(30^o) \text{ and } y = -\sin(30^o)$$

$$\therefore \vec{u}_n = \begin{bmatrix} \cos(30^o) \\ -\sin(30^o) \end{bmatrix} = \cos(30^o)\vec{i} - \sin(30^o)\vec{j}$$

And so, using equation (2.18) we obtain the following,

$$\vec{a}_B = (100)\big(\sin(30^o)\vec{i} + \cos(30^o)\vec{j}\big) + (3571.43)\big(\cos(30^o)\vec{i} - \sin(30^o)\vec{j}\big)$$

$$\therefore \vec{a}_B = [3142.95\vec{i} - 1699.11\vec{j}]\text{ms}^{-2}$$

Given that $\vec{a}_A = 0$ because car $A$ has rectilinear motion with constant speed, using equation (2.25) from above, we obtain the following result,

$$\vec{a}_{B|A} = \big[3142.95\vec{i} - 1699.11\vec{j}\big]\text{ms}^{-2} - 0$$

# Chapter 3

# Kinetics of a Particle

In this chapter we will state Newton's Second Law of Motion and define mass and weight. We will also analyse equations of motion with different coordinate systems.

## 3.1 Newton's Second Law of Motion

The motion of a particle is governed by Newton's three laws of motion. These laws are as follows,

**First Law** A particle originally at rest, or moving in a straight line at a constant velocity will remain in this state if the resultant force acting on the particle is zero.

**Second Law** If the resultant force on the particle is not zero, the particle experiences an acceleration in the same direction as the resultant force. This acceleration has a magnitude proportional to the resultant force.

**Third Law** Mutual forces of action and reaction between two particles are equal, opposite and collinear.

The first and third laws were used in developing the concepts of statics. Newton's second law forms the basis of the study of dynamics.

Mathematically, Newton's second law of motion can be written as,

$$\vec{F} = m\vec{a} \tag{3.1}$$

Where $\vec{F}$ is the resultant unbalanced force acting on the particle,
$\vec{a}$ is the acceleration of the particle,
$m$ is the mass of the particle and a positive scalar.

**Note** Newton's second law cannot be used when the particle's speed approaches the speed of light ir if the size of the particle is extremely small.

### 3.1.1 Newton's Law of Gravitational Attraction

Newton's Law of Gravitational Attraction states that any two particles or bodies have a mutually attractive gravitational force acting between them. Newton postulated the law governing this gravitational force using the following equation,

$$F = G\frac{m_1 m_2}{r^2} \tag{3.2}$$

| | | |
|---|---|---|
| Where | $F$ | is the force of attraction between the two bodies, |
| | $G$ | is the universal constant of gravitation; $G = 66.73 \cdot 10^{-12} \frac{\mathrm{m}^3}{\mathrm{kg \cdot s^2}}$, |
| | $m_1, m_2$ | are the masses of each body, |
| | $r$ | is the distance between the centres of the two bodies. |

When at or near the surface of the earth, the only gravitational force having any appreciable magnitude is that between the earth and the body. This force is called the *weight* of that body.

### 3.1.2 Mass and Weight

It is important to understand the difference between the mass and the weight of a body.

Mass is an absolute property of a body. It is independent of the gravitational field in which it is measured. The mass provides a measure of the resistance of a body to a change in velocity, as defined by Newton's second law of motion such that,

$$m = \frac{\vec{F}}{\vec{a}}$$

The weight of a body is not absolute since it depends on the gravitational field in which it is measured. Weight is defined as,

$$W = mg \tag{3.3}$$

Where $g$ is the magnitude of the acceleration due to gravity.

### 3.1.3 Relevant Units of Measure

**SI System**

In the SI System of units, mass is a base unit and weight is a derived unit.

Typically, mass is specified in kilograms (kg) and weight is calculated from $W = mg$

If the gravitational acceleration $g$ is specified in units of $\text{ms}^{-2}$, then the weight $W$ is expressed in newtons N such that,

$$W(\text{N}) = m(\text{kg}) \cdot g(\frac{\text{m}}{\text{s}^2}) \implies \text{N} = \text{kg} \cdot \frac{\text{m}}{\text{s}^2}$$

**Note** The acceleration due to gravity $g$ in the surface of Earth can be taken as,

$$g = 9.81\text{ms}^{-2}$$

**FPS System**

In the FPS System of unit, weight is a base unit and mass is a derived unit (opposite of SI System).

Weight is typically specified in pounds (lb), and mass is calculated from $m = \frac{W}{g}$.

If $g$ is specified in units of $\frac{\text{ft}}{\text{s}^2}$, then the mass is expressed in units of *slugs* such that,

$$m(\text{slugs}) = \frac{W(\text{lb})}{g(\frac{\text{ft}}{\text{s}^2})} \implies \text{slug} = \frac{\text{lb} \cdot \text{s}^2}{\text{ft}}$$

On the Earth's surface, the acceleration due to gravity $g$ is approximately,

$$g = 32.2\frac{\text{ft}}{\text{s}^2}$$

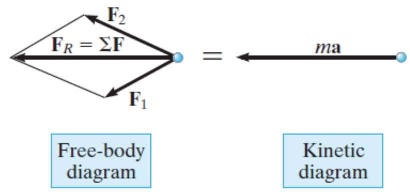
## 3.2 The Equation of Motion

The motion of a particle is governed by Newton's second law relating the unbalanced forces on a particle to its acceleration. If more than one force acts on the particle, the equation of motion can be written as,

$$\Sigma\vec{F} = \vec{F}_R = m\vec{a} \tag{3.4}$$

Where $\vec{F}_R$ is the resultant force, which is a vector summation of all the forces.

To illustrate this equation, consider a particle acted on by two forces.




First draw the particle's *free body diagram* showing all the forces acting on the particle. Next, draw the *kinetic diagram* showing the internal force $m\vec{a}$ acting in the same direction as the resultant force $\vec{F}_R$.

**Note**

$$\text{If} \quad \Sigma\vec{F} = \Sigma\vec{F}_R = 0,$$

then the acceleration is also zero so that the particle will either remain at rest or continue travelling along a rectilinear path with constant velocity.

These are the conditions of static equilibrium as defined by Newton's first law of motion.

### 3.2.1 Inertial Frame of Reference

The equation of motion discussed in the previous subsection is valid only if the acceleration is measured in a *Newtonian* of inertial frame of reference.

An inertial frame of reference has a constant velocity. That is, it is moving at a constant speed in a straight line, or it is standing still.

Understand that when something is standing still, it has a constant velocity. Its velocity is constantly zero meters per seconds.

To say that the velocity of a frame of reference is constant is the same as saying that the frame os not acceleration. So, we could define an inertial frame of reference to be a coordinate system which is not accelerating.

Such a constant velocity frame of reference is called an inertial frame because Newton's First Law holds in it.
That is, an object whose position is judged from this frame will tend to resist changes in its velocity, thus obeying Newton's First Law.

An object within this frame will not spontaneously change its velocity. An object within this frame will only change its velocity if an actual non-zero net force is applied to it.

There are several ways to describe an inertial frame. Here are few descriptions:

- An inertial frame of reference is a frame of reference with constant velocity.
- An inertial frame of reference is a non-accelerating frame of reference.
- An inertial frame of reference is a frame of reference in which the law of inertia holds.
- An inertial frame of reference is a frame of reference in which Newton's Laws of Motion Hold.
- In an inertial frame of reference, no fictitious forces arise.

Consider the act of juggling. It is just as easy (or just as difficult) to juggle balls in a room which is standing still as it is to juggle in a bus which is travelling smoothly down a straight road at constant speed.

In fact, the juggler on the bus could not determine that the bus was moving based on any clues gathered from the motion of the balls. They would move through the air within the moving bus exactly as if they were begin tossed about within the still room - as long as the bus moved with a constant velocity.

The physics of typical mechanics is always the same when it is done within a constant velocity frame of reference.

Without visual aids, such as viewing the scenery going by and without sound clues, such as the noise of the engine and drive train, the juggler physicist on the constant velocity bus, could perform no experiment to determine if the bus was moving or was parked.

Such frames of reference as our constant velocity bus are called inertial frames of reference.

The bus would however, cease to be an inertial frame of reference while it changed its velocity. That would happen if it slowed down, sped up or turned around a corner.

Each of these changes in velocity would constitute an acceleration, and, while the bus was acceleration, the act of juggling could get quite difficult.

For example, if the bus drier slammed on the brakes while some of the balls were in flight, those balls would seem to fly forward from the jugglers perspective, assuming that the juggler was facing the front of the bus.

From the viewpoint of the juggler, it would seem as if some unknown force had pushed the balls away from her, making them fly towards the front of the bus. But remember that in this situation the bus is no longer considered an inertial frame of reference. Its velocity is changing thus it is now an accelerating of reference and **Newton's Laws of Motion no longer hold**.

This accelerating frame is called a non-inertial frame of reference.

For problems concerned with motions at or near the Earth's surface, we typically assume our inertial frame to be fixed to the Earth and we neglect any acceleration effects from the Earth's rotation.

`Remark | The centripetal acceleration of a frame fixed to the surface of the Earth due only to the Earth's rotation about its axis is around 0.034 m/(s^2).`

For problems involving satellites or rockets, the inertial frame of reference is often fixed to the stars.

### 3.2.2 Summary

#### Key Points

- Newton's Second Law is a "law of nature" — meaning it is experimentally proven but not the result of an analytical proof.
- Mass is a property of an object and is a measure of the resistance to a change in velocity of the object.
- Weight is a force which depends on the local gravitational field. Calculating the weight of an object is an application of $\vec{F} = m\vec{a}$ such that $W = mg \downarrow +$.
- Unbalanced forces cause the acceleration of objects. This condition is fundamental to all dynamics problems.

#### Procedure of Application

- Select a convenient inertial coordinate system. Rectangular, normal/tangential, or cylindrical coordinates may be used.
- Draw a free body diagram showing all external forces applied to the particle. Resolve forces into their appropriate components.

- Draw the kinetic diagram showing the particle's inertial force $m\vec{a}$. Resolve this vector into its appropriate components.
- Apply the equation of motion in their scalar component form and solve these equations for their unknowns.
- It may be necessary to apply the proper kinematic relations to generate additional equations.

## 3.3 Equation of Motion — System of Particles

Consider a collection of particles in space. Each $i$-th particle has some mass $m_i$ and a position described by the vector $\vec{r}_i$. Each of these particles may be subjected to both external forces with a resultant $\vec{F}_i$ and system of internal forces from other particles in the system. These internal forces result from elastic interactions between particles and other electric or magnetic effects. Let $\vec{f}_{ij}$ be the force exerted on some $i$-th particle some other $j$-th particle such that the $i$-th particle has the resultant,

$$\vec{F}_i + \sum_{j=1}^{n} \vec{f}_{ij} = m_i \vec{a}_i$$

By Newton's 3rd Law, for every internal force $\vec{f}_{ij}$ there exists an equal opposing force $-\vec{f}_{ji}$. Thus, an equation of motion for the system of $n$ particles is obtained by summing the above equation for each particle such that,

$$\sum_{i=1}^{n} \vec{F}_i + \underbrace{\sum_{i=1}^{n} \left( \sum_{j=1}^{n} \vec{f}_j \right)}_{=0} = \sum_{i=1}^{n} m_1 \vec{a}_i$$

We can rewrite the above equation using the idea of the mass centre of a system of particles.

Now, for a system of $n$ particles, the mass centre of the system of $n$ particles, the mass centre of the system is the point $G$ defined by the position vector $\vec{r}_G$ which satisfies the relation,

$$m\vec{r}_a = \sum_{i=1}^{n} m_i \vec{r}_i$$

where $m = \sum_{i=1}^{n} m_i$ is the total mass of all the particles. Differentiating twice with respect to time assumes no mass is entering or leaving the system,

$$m\vec{a}_G = \sum m_i \vec{a}_i$$

Thus we obtain the following resultant force,

$$\sum \vec{F} = m\vec{a}_G$$

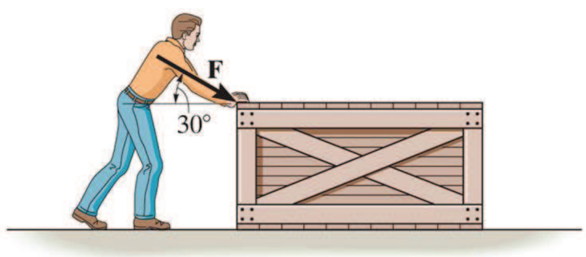
is the sum of all external forces acting on a system of particles equal to the total mass of the particles multiplied by the acceleration of it centre of mass.

Since, in reality, all particles must have a finite size to possess mass, this justifies the application of the equation of motion to a body that is represented as a single particle.

## 3.4 Equation of Motion — Rectangular Coordinates

In this section we will apply Newton's Second Law to determine forces and accelerations for particles in rectilinear motion.

Consider a man trying to move a 100lb crate. How large a force $\vec{F}$ must he exert to start moving the crate? What factors influence how large this force must be to start moving the crate?




If the crate starts moving, is there acceleration present?

What would you have to know before you could find these answers?

The equation of motion, $\vec{F} = m\vec{a}$ is best used when the problem requires finding forces (especially forces perpendicular to the path), accelerations, velocities or mass. Remember that unbalanced forces cause acceleration.

Three scalar equations can be written from this vector equation. The equation of motion, being a vector equation, may be expressed in terms of its three components in the Cartesian (rectangular) coordinate system as,

$$\Sigma\vec{F} = m\vec{a} \quad \text{or} \quad \Sigma F_x\vec{i} + \Sigma F_y\vec{j} + \Sigma F_z\vec{k} = m(a_x\vec{i} + a_y\vec{j} + a_z\vec{k})$$

or, as scalar equations,

$$\Sigma F_x = ma_x, \quad \Sigma F_y = ma_y, \quad \Sigma F_z = ma_z \tag{3.5}$$

### 3.4.1 Procedure for Analysis

- Free Body Diagram
  - Establish you inertial coordinate system and draw the particle's free body diagram showing only external forces.
    These external forces usually include weight, normal forces, friction forces and applied forces.
  - Show the '$m\vec{a}$' vector, sometimes called the inertial force, on a separate diagram.
  - Make sure any friction forces act opposite to the direction of motion. If the particle is connected an elastic linear spring, a spring force equal to $ks$ should be included on the free body diagram where $k$ is the spring constant and $s$ is the displacement from equilibrium of the spring.
- Equations of Motion
  - If the forces can be resolved directly from the free-body diagram, which is often the case in two-dimensional problems, use the scalar form of the equation of motion. In more complex cases such as three-dimensional problems, write a Cartesian vector for every force and use vector analysis.
  - A Cartesian vector formulation of Newton's Second Law is,

    $$\Sigma\vec{F} = m\vec{a} \quad \text{or} \quad \Sigma F_x\vec{i} + \Sigma F_y\vec{j} + \Sigma F_z\vec{k} = m(a_x\vec{i} + a_y\vec{j} + a_z\vec{k})$$

    Three scalar equations can be written from this vector equations, although you may only need two equations if the motion is only two-dimensional.
- Kinematics
  - The second law only provides solution for forces and accelerations. If velocity or position have to be found, kinematic equations are used once the acceleration is found using the equations of motion.
  - Any of the kinematics tools learned in Chapter 2 may be needed to solve a problem.
  - Make sure to use consistent positive coordinate directions as used in the equation of motion part of the problem.

## 3.5 Equation of Motion — Normal and Tangential Components

Our objective in this section is to apply the equation of motion using normal and tangential coordinates.

Consider a centrifuge that is used to subject a passenger to a very large normal acceleration caused by rapid rotation. This acceleration is caused by the unbalanced normal force exerted on the passenger by the seat of the centrifuge.

Race tracks are often banked in the turns to reduce the frictional forces required to keep the cars from sliding up to the outer rail at high speeds.

If that car's maximum velocity and a minimum coefficient of friction between the car's tires and the track are specified, we can determine the minimum banking angle $\theta$ required to prevent the car from sliding up the track.

Satellites are held in orbit around the Earth's gravitational pull as the centripetal force—that is, the force acting to change the direction of the satellite's velocity.

Knowing the radius of orbit of the satellite, we need to determine the required speed of the satellite to maintain this orbit. What equation governs this situation?

### 3.5.1 Coordinate System

When a particle moves along a curved path, it may be more convenient to write the equation of motion in terms of normal and tangential coordinates.

**Normal Direction** The normal direction, denoted $n$, always points towards the path's centre of curvature.

**Tangential Direction** The tangential direction, denoted $t$ is tangent to the path, usually set as positive in the direction of motion of the particle.

### 3.5.2 Acceleration

The tangential acceleration, given by the following equation,

$$a_t = \frac{d}{dt}(v),$$

represents the time rate of change in the magnitude of the velocity. Depending on the tangential direction, that is negative or positive given by $\Sigma F_t$, the particle's speed will either be increasing or decreasing.

The normal acceleration, given by the following equation,

$$a_n = \frac{v^2}{\rho},$$

represents the time rate of change in the direction of the velocity vector. Remember that $a_n$ always acts toward the path's centre of curvature. Thus, the sum of forces in the normal direction, $\Sigma F_n$, will always be directed towards the centre of the path.

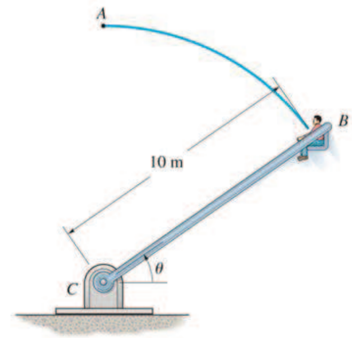
Recall that, if the path of motion is defined as $s_y = f(s_x)$, then the radius of curvature at any given point can be found using equation (2.22) such that,

$$\rho = \frac{\left(1 + \left(\frac{d}{ds_x}(s_y)\right)^2\right)^{\frac{3}{2}}}{\left|\frac{d^2}{d(s_x)^2}(s_y)\right|}$$

- Use $n$-$t$ coordinate system when a particle is moving along a known, curved path.
- Establish the $n$-$t$ coordinate system on the particle.
- Draw free-body and kinetic diagrams of the particle. The normal acceleration $a_n$ always acts "inwards" in the positive $n$-direction. The tangential acceleration $a_t$ may act in either the positive or negative $t$ direction
- Apply the equations of motion in scalar form and solve
- It may be necessary to use kinematic relations,

$$a_t = \frac{d}{dt}(v) = v\frac{d}{ds}(v) \qquad a_n = \frac{v^2}{\rho}$$

### Example (3.1)

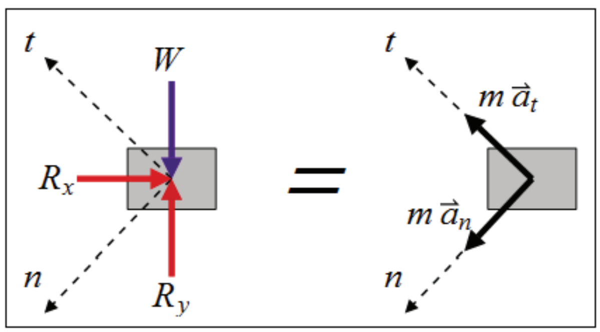



At the instant where $\theta = 45^o$, the boy with a mass of 75kg moves with a speed of $6\text{ms}^{-1}$, which is increasing at $0.5\text{ms}^{-2}$. Neglect the size and the mass of the seat and cords. The seat is pin connected to the frame.

Calculate the horizontal and vertical reactions of the seat on the boy.

**Solution** Since the problem involves a curved path, use $n$-$t$ coordinates. Draw the boy's free body and kinetic diagrams. Apply the equation of motion in the $n$-$t$ directions.

The $n$-$t$ coordinate system can be establish on the boy at an angle of $45^o$. Approximating the boy and seat together as a particle, the free body and kinetic diagrams can be drawn,




Applying the equation of motion in the $n$-$t$ direction, we find that,

$$\Sigma F_n = m\vec{a}_n; \quad -R_x \cos 45^o - R_y \sin 45^o + W \sin 45^o = m\vec{a}_n$$

$$\vec{a}_n \overset{(2.20)}{=} \frac{v^2}{\rho} = \frac{(6)^2}{(10)}\text{ms}^{-2} \qquad W = (75)(9.81)\text{N} \qquad m = 75\text{kg}$$

$$\therefore -R_x \cos 45^o + -R_y \sin 45^o + 520.3 = (75)\left(\frac{6^2}{10}\right)$$

$$\Sigma F_t = m\vec{a}_t; \quad -R_x \sin 45^o + R_y \cos 45^o - W \cos 45^o = m\vec{a}_t$$

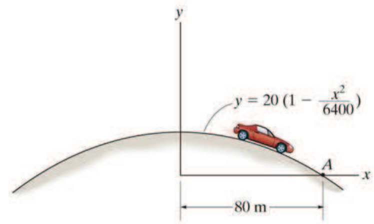
$$\therefore -R_x \sin 45^o + R_y \cos 45^o - 520.3 = 75(0.5)$$

Hence, we have two equations for two unknowns so we can solve for $R_x$ and $R_y$ such that,

$$R_x = -217\text{N} \qquad R_y = 572\text{N}$$

### Example (3.2)

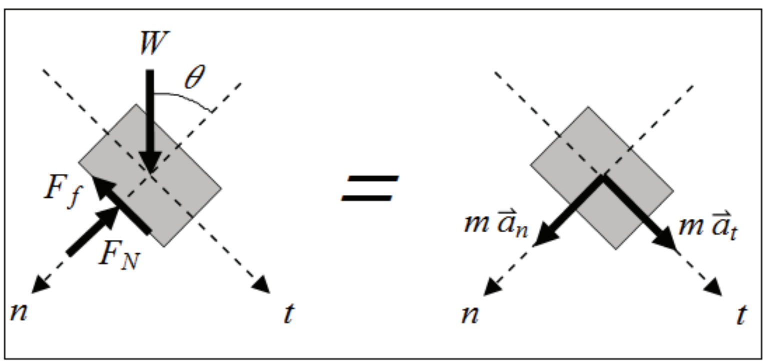
A 800kg car is travelling over the hill having the shape of a parabola. When it is at the point $A$, it is travelling at $9\text{ms}^{-1}$ and increasing its speed at $3\text{ms}^{-2}$.




Find the resultant normal force and the resultant frictional force exerted on the road at point $A$.

**Solution** We will treat the car as a particle and draw the free-body and kinetic diagrams. We will apply the equations of motion in the $n$-$t$ directions and use calculus to determine the slope and radius of the curvature of the path at point $A$.

The $n$-$t$ coordinate system can be established on the car at point $A$. Treat the car as a particle and draw the free-body and kinetic diagrams as follows,




Applying the equation of motion in the $n$-$t$ direction, we find that,

$$\Sigma F_n = m\vec{a}_n; \quad W\cos\theta - F_N = m\vec{a}_n$$

$$W = mg \qquad \vec{a}_n \overset{(2.22)}{=} \frac{v^2}{\rho} = \frac{(9)^2}{\rho}$$

$$\therefore (800)(9.81)\cos\theta - F_N = (800)\left(\frac{(81)}{\rho}\right) \iff F_N = 7848\cos\theta - \frac{64800}{\rho}$$

$$\Sigma F_t = m\vec{a}_t; \quad W\sin\theta - F_f = m\vec{a}_t$$

$$W = mg \qquad a_t = 3\text{ms}^{-2}$$

$$\therefore (800)(9.81)\sin\theta - F_f = (800)(3) \implies F_f = 7848\sin\theta - 2400$$

Determine $\rho$ by differentiating $s_y = f(s_x)$ at $s_x = 80\text{m}$ such that,

$$s_y = 20\left(1 - \frac{x^2}{6400}\right) \implies \frac{d}{ds_x}(s_y) = (-40)\frac{s_x}{6400} \implies \frac{d^2}{d(s_x)^2}(s_y) = (-40)\frac{-40}{6400}$$

$$\rho\Big|_{s_x=80\text{m}} \stackrel{(2.22)}{=} \frac{\left(1 + \left(\frac{d}{ds_x}(s_y)\right)^2\right)^{\frac{3}{2}}}{\left|\frac{d^2}{d(s_x)^2}(s_y)\right|} = \frac{\left(1 + (-0.5)^2\right)^{\frac{3}{2}}}{|0.00625|} = 223.6\text{m}$$

We can determine $\theta$ from the slope of the curve at $A$ such that,

$$\tan\theta = \frac{d}{ds_x}(s_y)\Big|_{s_x=80\text{m}} \implies \theta = \left|\arctan\left(\frac{d}{ds_x}(s_y)\right)\right| = |\arctan(0.5)| = 26.6^o$$

$$\text{Finally,} \quad F_N = 7848\cos\theta - \frac{64800}{\rho} = 7848\cos 26.6^o - \frac{64800}{223.6} = 6728\text{N}$$

$$\text{and} \quad F_f = 7848\sin\theta - 2400 = 7848\sin\left(26.6^o\right) - 2400 = 1114\text{N}$$

# Chapter 4

# Work and Energy

The objective in this chapter is to develop the principle of work and energy and its applications to problem involving force, velocity and displacement.

Another equation for working with kinetics problems involving particles can be derived by integrating the equation of motion with respect to displacement.

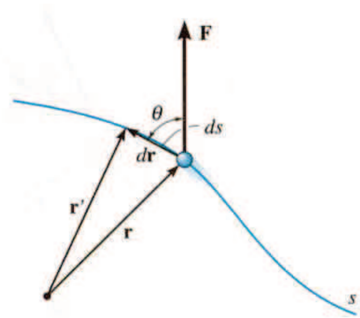
By substituting $\vec{a}_t = \vec{v}\left(\frac{d}{ds}(v)\right)$ into $F_t = m\vec{a}_t$, the result is integrated to yield an equation known as the principle of work and energy.

This principle is useful for solving problems that involve force, velocity and displacement. It can also be used to explore the concept of power.

To use this principle, we must first understand how to calculate the work of a force.

## 4.1 Work of a Force

A force does *work* on a particle when the particle undergoes a displacement along the line of action of the force.




Work is defined as the product of force and displacement components acting in the same direction. So, if the angle between the force $F$ and the displacement $s$ vector is $\theta$, then the increment of work $dU$ done by the force is given by,

$$dU = F \cdot ds \cos\theta \tag{4.1}$$

The total work can be written as the definite integral of the dot product between the force vector $\vec{F}$ and the increment in the position vector $d\vec{r}$ over an interval of displacement.

$$U\Big|_a^b = \int_{r_a}^{r_b} \vec{F} \bullet d\vec{r} \tag{4.2}$$

### 4.1.1 Work of a Variable Force

If a force vector $\vec{F}$ is a function of position $s$, we can express Equation (4-2) equivalently as,

$$U\Big|_a^b = \int_{s_a}^{s_b} \vec{F}\cos\theta \cdot ds \tag{4.3}$$

### 4.1.2 Work of a Constant Force

If both $F$ and $\theta$ are constant, we can simplify Equation (4-3) such that,

$$U\Big|_a^b = F_c \cos\left(s_2 - s_1\right) \tag{4.4}$$

Note that work is **positive** if the force and the movement are in the same direction. Work is **negative** it they are opposing. If the force and the displacement directions are orthogonal or perpendicular, then the work is zero.

### 4.1.3 Work of a Weight

The work done by the gravitational force acting on a particle, called the *weight* of an object, can be calculated by using the following equation,

$$U\Big|_a^b = \int \vec{F} \bullet d\vec{r} = -\int (W\hat{\boldsymbol{\jmath}}) \bullet (ds_x\hat{\boldsymbol{\imath}} + ds_y\hat{\boldsymbol{\jmath}} + ds_z\hat{\boldsymbol{k}})$$

$$U_G\Big|_a^b = -\int_{(s_y)_a}^{(s_y)_b} W \cdot ds_y = -W\big((s_y)_2 - (s_y)_1\big) = -W\Delta s_y \tag{4.5}$$

The work done by gravity or the weight of a particle is the product of the magnitude of the particle's weight and vertical displacement.

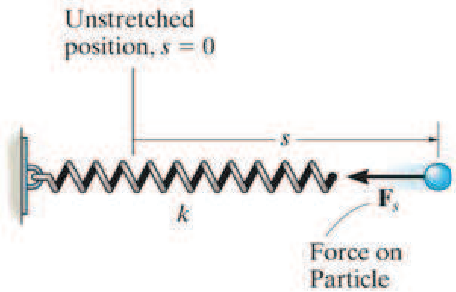
Note that if $\Delta s_y$ is positive such that the particle is moving upward, then the work done is negative because the force of weight is acting downwards.

### 4.1.4 Work of a Spring

When a linear elastic spring is stretched, it develops a force with a magnitude such that,

$$F_s = ks$$

Where $k$ is the spring constant, indicating the "stiffness" of the spring, and $s$ is the displacement from the unstretched position of the spring.




The work of a spring force moving from position $s_a$ to position $s_b$ is,

$$U\Big|_a^b = \int_{s_a}^{s_b} F_s \cdot ds = \int_{s_a}^{s_b} ks \cdot ds = \frac{1}{2}k(s_b)^2 - \frac{1}{2}k(s_a)^2$$

If a particle is attached to the spring, then the force $F$ exerted on the particle is opposing that exerted by the spring such that the work done on the particle by the spring force will be negative.

$$U_s\Big|_a^b = -\left(\frac{1}{2}k(s_b)^2 - \frac{1}{2}k(s_a)^2\right) \tag{4.6}$$

It is important to note the following about springs,

- The equations above are for **linear** springs only! Recall that a linear spring develops a force according to $F_s = ks$ which is a linear equation.
- The work of a spring is not just spring force times distance at some point.
- Always double check the sign convention of the spring work after calculating it. It is positive work if the force put on the object by the spring and the movement of the particle attached to the spring are in the same directions.

## 4.2 Principle of Work and Energy

By integrating the equation of motion,

$$\int F_t \cdot ds = \int m\vec{a}_t \cdot ds = \int m\vec{v}\left(\frac{d(\vec{v})}{ds}\right) \cdot ds = \int m\vec{v} \cdot d\vec{v} = \frac{1}{2}m(\vec{v})^2 + C$$

then the principle of work and energy can be written as,

$$\Sigma U\Big|_a^b = \frac{1}{2}m(\vec{v_b})^2 - \frac{1}{2}m(\vec{v}_a)^2 \tag{4.7}$$

$\Sigma U\Big|_a^b$ is the work done by all the forces acting on the particle as it moves from point $a$ to point $b$. Work can be either a positive or negative scalar.

We define the *kinetic energy* to be denoted $T$ such that it is a positive scalar value given by,

$$T = \frac{1}{2}m(\vec{v})^2 \tag{4.8}$$

so that Equation (4.7) can be rewritten such that,

$$\Sigma U\Bigg|_a^b = T_b - T_a \tag{4.9}$$

So the particle's initial kinetic energy plus the work done by all the forces acting on the particle as it moves from its initial to final position is equal to the particle's final kinetic energy.

Note that the principle of work and energy is not a vector equation. Each term in the equation results in a scalar value.

Both kinetic energy and work have the same units which is that of energy. Using the SI System of units, the unit for energy is called a *joule* denoted 'J' equivalent to 'N · m'. In the FPS System of units, the unit for energy is ft · lb.

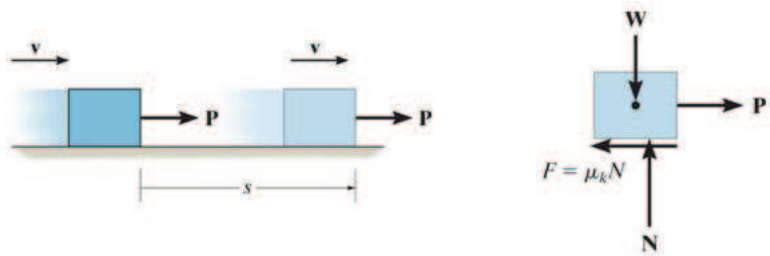
The principle of work and energy cannot be used, in general, to determine forces directed normal to the path since these orthogonal forces do no work.

## 4.3 Work and Energy — System of Particles

The principle of work and energy can also be applied to a system of particles by summing the kinetic energies of all particles in the system and the work due to all forces acting on the system such that,

$$\Sigma U\Bigg|_a^b = \Sigma T_b - \Sigma T_a \tag{4.10}$$

### 4.3.1 Work of Friction




The case of a body sliding over a rough surface merits special consideration.

Consider a block which is moving over a rough surface. If the applied force $P$ just balances the resultant frictional force $F = \mu_k F_N$, then a constant velocity $v$ would be maintained.

The principle of work and energy would be applied as,

$$\frac{1}{2}m(\vec{v})^2 + Ps - (\mu_k F_N)s = \frac{1}{2}m(\vec{v})^2$$

This equation is satisfied if $P = \mu_k F_N$. However, we know from experience that friction generates heat, a form of energy that does not seem to be accounted for in this equation. It can be shown that the work term $(\mu_k F_N)s$ represents both the external work of the friction force and the internal work that is converted into heat.

The principle of work and energy can be applied to problems involving sliding friction, but it should be understood that the work of the resultant frictional force is not represents by $(\mu_k F_N)s$. Instead, this term represents both the external work of friction,

$$(\mu_k F_N)s', \text{ where } s' < s$$

and internal work,

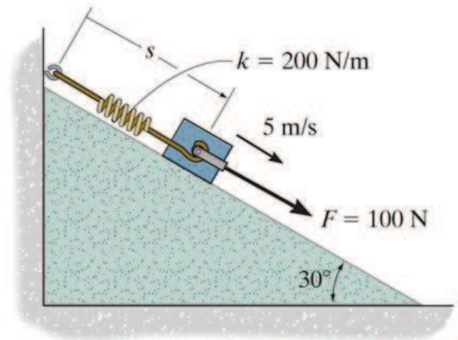
$$(\mu_k F_N)(s - s')$$

which is converted into various forms of internal energy such as heat.

### Example (4.1)

When $s = 0.6$m, the spring is not stretched or compressed and the 10kg block, which is subjected to a force of $F = 100$N, has a speed of 5ms$^{-1}$ down the smooth plane.

Find the distance $s$ when the block stops.




### Solution

Since this problem involves forces, velocity and displacement, we can apply the principle of work and energy to determine $s$.

Apply the principle of work and energy between the initial position $s_a = 0.6$m and the final position $s_b$. Note that the normal force $F_N$ does no work since it is always perpendicular to the displacement.

There is work done by three different forces,

- Work of force $F = 100$N,

$$U_F = 100(s_b - s_a) = 100(s_b - 0.6)\text{N}$$

- Work of the block's weight,

$$U_W = 10(9.81)(s_b - s_a)\sin 30^o = 49.05(s_b - 0.6)$$

- Work of the spring's force,

$$U_s \stackrel{(4.6)}{=} -\left(\frac{1}{2}k((s_s)_b)^2 - \frac{1}{2}k((s_s)_a)^2\right)$$

  Where $(s_s)_a$ and $(s_s)_b$ are the displacement from equilibrium of the spring at positions $a$ and $b$ respectively.

Given $(s_s)_a = 0$m such that the spring is initially unstretched, and $(s_s)_b = (s_b - 0.6)$m, we find,

$$U_s \overset{(4.6)}{=} -\frac{1}{2}(200)(s_b - 0.6)^2 = -100(s_b - 0.6)^2$$

Since the block is at rest at $s_b$, we can assume $T_b = 0$. Applying the principle of work and energy we find,

$$\Sigma U \overset{(4.9)}{=} T_b - T_a$$

$$\underbrace{100(s_b - 0.6)}_{U_F} + \underbrace{49.05(s_b - 0.6)}_{U_W} - \underbrace{100(s_b - 0.6)^2}_{U_s} = \underbrace{0}_{T_b} - \underbrace{\frac{1}{2}(10)(5)^2}_{T_a}$$

$$100(s_b - 0.6)^2 - 149.05(s_b - 0.6) - 125 = 0$$

Solving with the quadratic equation for $(s_b - 0.6)$,

$$(s_b - 0.6) = \frac{-(-149.05) \pm \sqrt{(-149.05)^2 - 4(100)(-125)}}{2(100)}$$

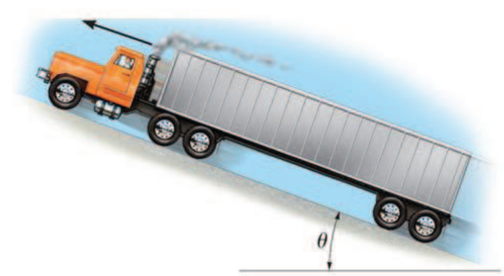
Since the spring is **extending**, we will select the positive root such that,

$$(s_b - 0.6) = 2.09\text{m} \implies s_b = 2.69\text{m}$$

## 4.4 Power and Efficiency

In this section, we will determine the power generated by a machine, engine or motor and calculate the mechanical efficiency of a machine.

Consider the speed at which a truck can climb a hill. It depends, in part, on the power output of the engine and the angle inclination of the hill.




For a given angle, how can we determine the speed of this truck knowing the power transmitted by the engine to the wheels? Can we find the speed if we know the power?

If we know the engine's power output and the speed of the truck, can we determine the maximum angle that the truck can climb?

### 4.4.1 Power

We define *power* as the amount of work performed per unit of time.

If a machine or engine performs a certain amount of work $dU$ during some given time interval $dt$, then the power generated can be calculated as,

$$P = \frac{d}{dt}U \tag{4.11}$$

Since the work can be expressed as $dU = \vec{F} \bullet d\vec{r}$, then the power can be written such that,

$$P = \frac{d}{dt}U = \frac{(\vec{F} \bullet d\vec{r})}{dt} = \vec{F} \bullet \left(\frac{d}{dt}\vec{r}\right) = \vec{F} \bullet \vec{r} \tag{4.12}$$

Note that power is a scalar value. Using scalar notation power can be written such that,

$$P = \vec{F} \bullet \vec{v} = Fv\cos(\theta) \tag{4.13}$$

where $\theta$ is the angle between the force and velocity vectors.

So, if the velocity of a body acted on by some force $\vec{F}$ is known, then the power $P$ can be determined by calculating the dot product of the vectors $\vec{F}$ and $\vec{v}$.

The unit for power is the SI System of units is the *Watt*, denoted W such that,

$$1\text{W} = 1\frac{\text{J}}{\text{s}} = 1\frac{(\text{N} \cdot \text{m})}{\text{s}}$$

In the FPS System of units, power is usually expressed in units of *horsepower* denoted hp such that,

$$1\text{hp} = 550\frac{(\text{ft} \cdot \text{lb})}{s} = 746\text{W}$$

### 4.4.2 Efficiency

The *mechanical efficiency* of a machine is defined to be the ratio of the useful power produced, or the *output power* over the power supplied to the machine called the *input power* such that if $\varepsilon$ denotes the mechanical efficiency of a machine,

$$\varepsilon \equiv \frac{\text{power ouput}}{\text{power input}} \tag{4.14}$$

If the input and removal of energy occur at the same time, then the efficiency may also be expressed in terms of the ratio of the output energy over the input energy.

Machines will always have frictional forces. Since frictional forces dissipate energy, then additional power will be required to overcome these forces. Consequently, the efficiency of a machine is always less than 1.

### 4.4.3 Procedure for Analysis

- Find the resultant external force acting on the body causing its motion. It may be necessary to draw a free-body diagram.
- Determine the velocity of the point on the body at which the force is applied. Energy or the equation of motion and appropriate kinematic relations may be necessary to consider.
- Multiply the magnitude of the external force by the component of velocity acting in the direction of $\vec{F}$ to determine the power supplied to the body or particle. That is,

$$P = |F| v \cos(\theta)$$

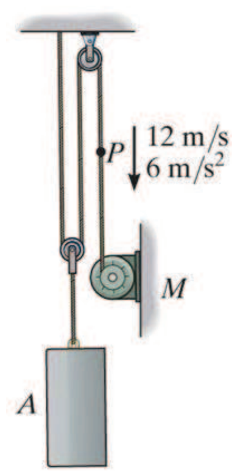
- In some cases, power may be determined by calculating the work done per unit of time such that,

$$P = \frac{d}{dt}(U)$$

- If the mechanical efficiency of the machine is known, then either the power input or output can be determined.

### Example (4.2)

Suppose a 50kg block $A$ is hoisted by the pulley system and motor $M$. The motor has an efficiency of 0.8. At the instant shown, point $P$ on the cable has a downwards velocity of $12\text{ms}^{-1}$ which is increasing at a rate of $6\text{ms}^{-2}$. Neglect the mass of the pulleys and cable system.




Find the power supplied to the motor at the instant shown.

#### Solution

We will relate the velocities of the cable and the block by defining position coordinates relative to a fixed datum. Draw a free body diagram of the block.

Using the equation of motion, determine the tension in the cable.

Calculate the power supplied by the motor and then that supplied to the motor using the efficiency ratio.

First, define position coordinate to relate velocities. Here, $s_P$ is defined to a point on the cable. Also, $s_A$ is defined only to the lower pulley since the block moves with the pulley. Using kinetics we find,

$$s_P + 2s_A = 1 \implies a_P + 2a_A = 0 \implies a_A = \frac{-a_P}{2} = -3\text{ms}^{-2}(+\uparrow)$$

Draw the free body diagram of block $A$ showing a downward weight force vector and upward force vector twice that of the tension in the cable. A kinetic diagram of the block shows an overall upwards acceleration.

The tension of the cable can be obtained by applying the equation of motion to the block such that,

$$(+\uparrow)\sum F_y = m_A a_A \implies 2F_T - 490.5 = 50(3) \implies F_T = 320.2\text{N}$$

The power supplied by the motor is the dot product of the force applied to the cable and the velocity of the cable.

$$P_o = \vec{F} \bullet \vec{v} = (320.3)(12) = 3844\text{W}$$

The power supplied to the motor is determined using the motor's efficiency and the basic efficiency equation such that,

$$P_i = \frac{P_o}{\varepsilon} = \frac{3844}{0.8} = 4804\text{W} \simeq 4.80\text{kW}$$

## 4.5 Conservative Forces and Potential Energy

Our objective in this section is to understand the concept of conservative forces and determine the potential energy of such forces.

The principle of work and energy os now simplified and becomes *conservation of energy*. Know the difference between the two and when and how to apply them.

Image you do some work against some force. Can you get that work back later? Are there forces that "store" the work for re-use later?

For example, consider lifting an object some distance $h$ against the force of gravity. The work done is the product of the mass of the object $m$, the distance $h$, and the force of gravity $g$. If you then let the object fall, you essentially get that work back, either in the form of the object's kinetic energy or, if you attach that object to some kind of string or pulley system, you can transfer that returned work. This idea is referred to as *conservative force.*

On the other hand, a block of mass $m$ projected onto a rough surface will be brought to rest by the kinetic frictional force. There is no way to get back the original kinetic energy of the block after the frictional force has been brought to rest. The directed motion of the block has been transformed into kinetic energy of the randomly directed moving atoms that make up the block and the plane — hence, a frictional cannot "store" energy.

Generally speaking, the laws of thermodynamics forbid energy which has been converted into heat from being converted back to its original form — hence, we say that friction is a *non-conservative force* because it dissipates energy rather than conserving it.

This suggests that conservative forces can be associated with some form of potential energy.

A force $\vec{F}$ is said to be a conservative force if the work done by the force in moving some particle between two points $A$ and $B$ is independent of the path taken by the particle as it moves from $A$ to $B$. This also means that the work done by the force $\vec{F}$ in closed path from $A$ to $B$ and then back to $A$ is zero.

$$\oint \vec{F} \cdot d\vec{r} = 0$$

Thus, we say that the work is conserved.

The work done by a conservative force depends only on the positions of the particle and is independent of its velocity and acceleration.

The conservative potential energy of a particle or system is typically written as a function $V$ of potential energy. There are two major components to $V$ commonly encountered in mechanical systems — the potential energy from gravity, and the potential energy from linear elastic springs or other elastic elements. That is,

$$V_{\text{total}} = V_{\text{gravity}} + V_{\text{elastic}} \tag{4.15}$$

Recall that potential energy is a measure of the amount of work a conservative force will do when a body changes position.

In general, for any conservative force system, we define the potential energy function $V$ a function of position. The work done by conservative forces as the particle moves is equal to the change in the value of the potential energy function or the sum of gravitational and elastic potential.

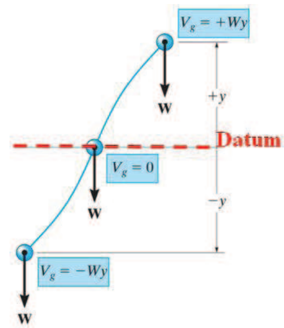
It is important to become familiar with these two types of potential energy and know how to calculate their magnitudes.

### 4.5.1 Gravitational Potential Energy

The potential energy function for a gravitational force or *weight* is the force multiplied by its displacement from some fixed datum defined at any convenient location. Then,

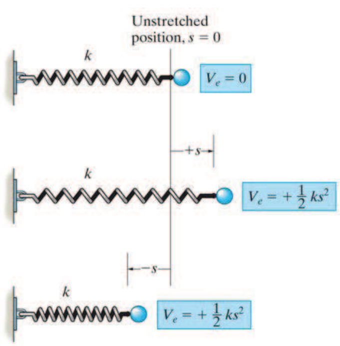
$$V_g = \pm W y \tag{4.16}$$

such that $V_g$ is a **positive** scalar when displacement is above the datum and **negative** when the displacement is below the datum.



Gravitational potential energy

### 4.5.2 Elastic Potential Energy



Elastic potential energy

Recall that the force of an elastic spring is $F_s = ks$ or the product of the spring constant $k$ and the springs displacement $s$ from equilibrium.

Then, the potential energy function $V_e$ of an elastic spring is given by,

$$V_e = \frac{1}{2} k s^2 \tag{4.17}$$

Note that the potential energy function of an elastic system is always positive.

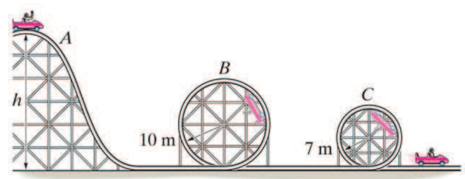
## 4.6 Conservation of Energy

Consider a particle acted upon by a system of conservative forces. The work done by these forces is conserved and the sum of the kinetic energy and the potential energy remains constant. In other words, as the particle moves, kinetic energy is converted to potential energy and vice versa. This principle is called the principle of conservation of energy and is expressed,

$$T_a + V_a = T_b + V_b \equiv \text{constant} \tag{4.18}$$

where the sum of kinetic energy $T$ and the potential energy function $V$ are equal at some points $a$ and $b$. Recall the kinetic energy is defined as $T = \frac{1}{2}mv^2$.

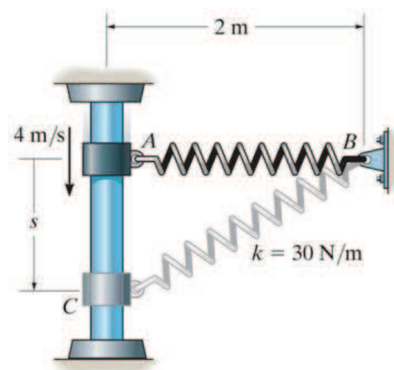
Consider a roller coaster released from rest at the top of the hill. As the roller coaster moves down the hill, the potential energy is transformed into kinetic energy.




## Example (4.2)

The 2kg collar is moving down with a velocity of $4\text{ms}^{-1}$ at point $A$. The spring constant is $30\text{Nm}^{-1}$. The unstretched length of the spring is $1m$.

Find the velocity of the collar when $s = 1\text{m}$




### Solution

Apply the conservation of energy equation between points $A$ and $C$. Set the gravitational potential energy datum at point $A$.

Note that the potential energy at point $C$ has two parts,

$$V_C = (V_C)_e + (V_C)_g$$

$$V_C = \frac{1}{2}(30)(\sqrt{5} - 1)^2 - (2)(9.81)(1)$$

The kinetic energy at point $C$ is,

$$T_C = \frac{1}{2}(2)v^2$$

Similarly, at point $A$, potential and kinetic energies are respectively,

$$V_A = \frac{1}{2}(30)(2 - 1)^2 \qquad T_A = \frac{1}{2}(2)(4)^2$$

Hence, the energy conservation equation becomes,

$$T_A + V_A = T_C T_C$$

$$\left(\frac{1}{2}(30)(\sqrt{5} - 1)^2 - (2)(9.81)(1)\right) + \frac{1}{2}(2)v^2 = \left(\frac{1}{2}(30)(2 - 1)^2\right) + \frac{1}{2}(2)(4)^2$$

$$\implies v = 5.26\text{ms}^{-1}$$

# Chapter 5

# Impulse and Momentum

In this chapter we will develop the principle of linear impulse and momentum for a particle and apply it to solve problems that involve force, velocity and time. We will study the conservation of linear momentum for particles and analyse the mechanics of impact.

## 5.1 Principle of Linear Impulse and Momentum

The next method we will consider for solving particle kinetics problems is obtained by integrating the equation of motion with respect to time. The result is referred to as the principle of impulse and momentum. It can be applied to problems involving both linear and angular motion.

This principle us useful for solving problems that involve force, velocity and time. It can also be used to analyse the mechanics if impact.

The principle of linear impulse and momentum is obtained by integrating the equation of motion with respect to time. The equation of motion can be written as,

$$\sum \vec{F} = m\vec{a} = m\frac{d}{dt}(\vec{v})$$

Separating variables and integrating between the points $a$ and $b$ result in,

$$\sum \int_{t_a}^{t_b} \vec{F} \cdot dt = \int_{\vec{v}_a}^{\vec{v}_b} m \cdot d\vec{v} = m\vec{v}_b - m\vec{v}_a$$

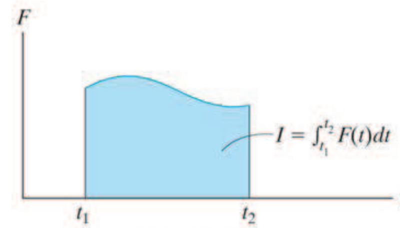
This equation represents the principle of linear impulse and momentum. It relates the final and initial velocity to the forces acting on the particle as a function of time.

### 5.1.1 Linear Momentum

We define linear momentum as the vector $m\vec{v}$ and denote it $\vec{L}$. This vector has the same direction as $\vec{v}$. The linear momentum vector has units of $\frac{\text{kg}\cdot\text{m}}{\text{s}}$ or $\frac{\text{slug}\cdot\text{ft}}{\text{s}}$

### 5.1.2 Linear Impulse

The integral $\int \vec{F} \cdot dt$ is defined as the linear impulse and denoted $I$. It is a vector quantity measuring the effect of a force during its time interval of action. Impulse acts in the same direction as $\vec{F}$ and has units of $\text{N} \cdot \text{s}$ or $\text{lb} \cdot \text{s}$.



Variable Force

The impulse may be determined by direct integration. Graphically, it can be represented by the area under the force versus time curve. If $\vec{F}$ is constant then,

$$\vec{I} = \vec{F}(t_b - t_a) \tag{5.1}$$

The principle of linear impulse and momentum in vector form is written as,

$$m\vec{v}_a + \sum \int_{t_a}^{t_b} \cdot dt = m\vec{v}_b \tag{5.2}$$

The particle's initial momentum plus the sum of all the impulses applied from $t_a$ to $t_b$ is equal to the particle's final momentum.

The two momentum diagrams indicate direction and magnitude of the particle's initial and final momentum, $m\vec{v}_a$ and $m\vec{v}_b$. The impulse diagram is similar to a free body diagram, but includes the time duration of the forces acting on the particle.

### 5.1.3 Scalar Equations of Impulse and Momentum

Since the principle of linear impulse and momentum is a vector equation, it can be resolved into its $x$, $y$ and $z$ scalar components such that,

$$m(v_a)_x + \sum \int_{t_a}^{t_b} F_x \cdot dt = m(v_2)_x \tag{5.3}$$

$$m(v_a)_y + \sum \int_{t_a}^{t_b} F_y \cdot dt = m(v_2)_y \tag{5.4}$$

$$m(v_a)_z + \sum \int_{t_a}^{t_b} F_z \cdot dt = m(v_2)_z \tag{5.5}$$

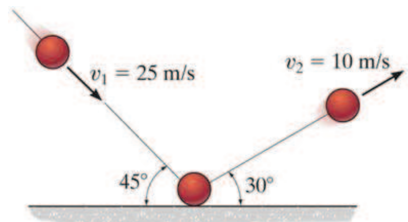
The scalar equations provide a convenient means for applying the principle of linear impulse and momentum once the velocity and force vectors have been resolved into $x$, $y$ and $z$ components.

### 5.1.4 Problem Solving

- Establish the $x$, $y$ and $z$ coordinate system.
- Draw the particle's free body diagram and establish the direction of the particle's initial and final velocities. Draw the impulse and momentum diagrams for the particle. Show the linear momenta and force impulse vectors.
- Resolve the force and velocity or impulse and momentum vectors into their $x$, $y$ and $z$ components and apply the principle of linear impulse and momentum using its scalar form.
- Forces as functions of time must be integrated to obtain impulses. If a force is constant, its impulse is the product of the force's magnitude and the time interval over which it acts

### Example (5.1)

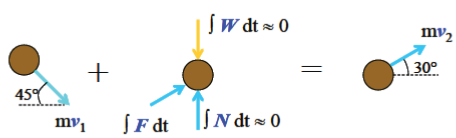
A 0.5kg ball strikes the rough ground and rebounds with the velocities shown. Neglect the ball's weight during the time it impacts the ground.




Find the magnitude of impulsive force exerted on the ball.

#### Solution

Draw the momentum and impulse diagrams of the ball as it hits the surface. Apply the principle of impulse and momentum to determine the impulsive force. The impulse caused




by the ball's weight and the normal force $N$ can be neglected because their magnitudes are very small as compared to the impulse from the ground.

The principle of impulse and momentum can be applied along the direction of motion such that,

$$m\vec{v}_a + \sum \int_{t_a}^{t_b} \vec{F} \cdot dt = m\vec{v}_2$$

$$0.5(25\cos{(45^o)}\hat{\boldsymbol{\imath}} - 25\sin{(45^o)}\hat{\boldsymbol{\jmath}}) + \int_{t_a}^{t_b} \sum \vec{F} \cdot dt = 0.5(10\cos{(30^o)}\hat{\boldsymbol{\imath}} + 10\sin{(30^o)}\hat{\boldsymbol{\jmath}})$$

The impulsive force vector is,

$$\vec{I} = \int_{t_a}^{t_b} \sum \vec{F} \cdot dt = (4.509\hat{\imath} + 11.34\hat{\jmath})\text{N} \cdot \text{s}$$

Finally, the magnitude of impulse is $I = \sqrt{(4.509)^2 + (11.34)^2} = 12.2\text{N} \cdot \text{s}$

## 5.2 Impulse and Momentum for a System of Particles

The principle of linear impulse and momentum for a system of particles moving relative to an inertial frame, is obtained from applying the equation of motion to each particle in the system. That is,

$$\sum \vec{F_i} = \sum m_i \frac{d}{dt}(\vec{v_i}) \tag{5.6}$$

Consider a system of particles. The internal forces $f_i$ between particles occur in pairs with equal magnitudes and opposite directions resulting in a total internal impulse of zero.

Thus, the equation for linear impulse and momentum of this system includes only the external forces. That is,

$$\sum m_i(\vec{v_i})_a + \sum \int_{t_a}^{t_b} \vec{F_i} \cdot dt = \sum m - i(\vec{v_i})_b \tag{5.7}$$

### 5.2.1 Motion of the Centroid

Given a system of particles, we can define a fictitious centre of mass as an aggregate particle $G$ who's mass is the sum $\sum m_i$ of all the particles. This particle then has an aggregate velocity such that,

$$\vec{v}_G = \frac{\sum m_i \vec{v_i}}{\sum m_i} \tag{5.8}$$

We can describe the motion of the centroid or centre of mass by the position vector $\vec{r}$ such that,

$$\vec{r}_G = \frac{\sum m_i \vec{r_i}}{\sum m_i} \tag{5.9}$$

Thus, we can write the principle of linear impulse and momentum for a system of particles in terms of the centroid $G$ of the system who's mass is the total mass of the system such that,

$$m(\vec{v}_G)_a + \sum \int_{t_a}^{t_b} \vec{F_i} \cdot dt = m(\vec{v}_G)_b \tag{5.10}$$

## 5.3 Conservation of Linear Momentum

When the sum of external impulses acting on a system of objects is zero, as in an inertial frame, then the linear impulse and momentum equation is simplified such that,

$$\sum m_i(\vec{v}_i)_a = \sum m_i(\vec{v}_i)_2 \tag{5.11}$$

This equation is referred to as the conservation of linear momentum. It states that the total linear momentum for a system of particles remains constant during the time period. Then, given a constant mass,

$$(\vec{v}_G)_a = (\vec{v}_G)_b$$

indicating that the velocity of the centroid of a system of particles does not change if no external impulses are applied to the system.

When particles collide, only *impulsive forces* cause a change of linear momentum. Consider a sledgehammer applying an impulsive force to a post. The weight of the post is considered *non-impulsive* since it is negligible as compared to the force of the sledgehammer. Similarly, we neglect the impulse of the ground of the post.

When we consider the hammer and post as a system, then the impulsive force from the hammer to the post is **internal**. That is, we can apply the conservation of linear momentum.

### Example (5.2)

An explosion has broken a mass $M$ into three smaller particles $a$, $b$ and $c$ such that,

$$\begin{array}{r|rrrrrl} M = 100\text{kg} & \vec{v}_o = & 0\hat{\boldsymbol{\imath}} & + & 20\hat{\boldsymbol{\jmath}} & + & 0\hat{\boldsymbol{k}} \ \text{ms}^{-1} \\ m_a = 20\text{kg} & \vec{v}_a = & 50\hat{\boldsymbol{\imath}} & + & 50\hat{\boldsymbol{\jmath}} & + & 0\hat{\boldsymbol{k}} \ \text{ms}^{-1} \\ m_b = 30\text{kg} & \vec{v}_b = & -30\hat{\boldsymbol{\imath}} & + & 0\hat{\boldsymbol{\jmath}} & - & 50\hat{\boldsymbol{k}} \ \text{ms}^{-1} \end{array}$$

Find the velocity of particle $c$ after the explosion.

**Solution**

Since the internal forces of the explosion sum to zero, we can apply the conservation of linear momentum to the system such that,

$$\sum m_i(\vec{v}_i)_a = \sum m_i(\vec{v}_i)_b$$

$$M\vec{v}_i = m_a\vec{v}_a + m_b\vec{v}_b + m_c\vec{v}_c$$

$$100(20\hat{\boldsymbol{\jmath}}) = 20(50\hat{\boldsymbol{\imath}} + 50\hat{\boldsymbol{\jmath}}) + 30(-30\hat{\boldsymbol{\imath}} - 50\hat{\boldsymbol{k}}) + 50\left[(v_c)_x\hat{\boldsymbol{\imath}} + (v_c)_y\hat{\boldsymbol{\jmath}} + (v_c)_z\hat{\boldsymbol{k}}\right]$$

$$\begin{aligned} 1000 - 900 + 50\left((v_c)_x\right) &\implies ((v_c)_x) = -1\text{ms}^{-1} \\ 1000 + 50 + 50\left((v_c)_y\right) &\implies ((v_c)_y) = 20\text{ms}^{-1} \\ -1500 + 50\left((v_c)_z\right) &\implies ((v_c)_z) = 30\text{ms}^{-1} \end{aligned}$$

$$\text{So,} \qquad \vec{v}_c = (-2\hat{\boldsymbol{\imath}} + 20\hat{\boldsymbol{\jmath}} + 30\hat{\boldsymbol{k}})\text{ms}^{-1}$$

## Example (5.3)

Consider two rail cars with masses of $m_A = 20\text{Mg}$ and $m_B = 15\text{Mg}$ and velocities as shown.


Find the speed of the car $A$ after the collision if the cars collide and rebound such that $B$ moves to the right with a speed of $2\text{ms}^{-1}$. Also find the average impulsive force between the cars if the collision lasts 0.5s.

**Solution**

Use conservation of linear momentum to find the velocity of the car $A$ after the collision. Note that all impulsive forces cancel out. Use the principle of impulse and momentum to find the impulsive force by looking at only one car.

$$m_A(v_A)_a + m_B(v_B)_a = m_A(v_A)_b + m_B(v_B)_b$$

$$20000(3) + 15000(-1.5) = (20000)(v_A)_b + 15000(2)$$

$$\implies (v_A)_b = 0.375\text{ms}^-1$$

$$m_A(v_A)_a + \int F \cdot dt = m_A(v_A)_b$$

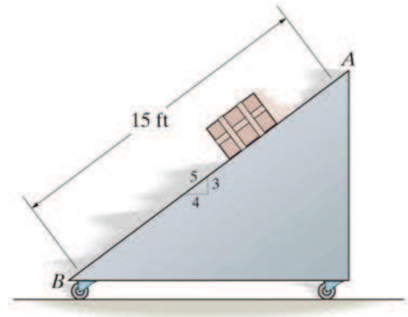
$$20000(3) - \int F \cdot dt = 20000(0.375)$$

$$\int F \cdot dt = 52500\text{N} \cdot \text{s}$$

$$\int F \cdot dt = 52500\text{N} \cdot \text{s} = F_{avg}(0.5\text{sec}) \implies F_{avg} = 105\text{kN}$$

### Example (5.4)

The free rolling ramp has a weight of 120lb. The 80lb crate slides from rest at point $A$ down the ramp to point $B$. Assume that the ramp is smooth and neglect the mass of the wheels.




Find the ramp's speed when the crate reaches point $B$.

### Solution

Use the energy conservation equation as well as the conservation of linear momentum and the relative velocity equation to find the velocity of the ramp.

Using the conservation of energy equation,

$$0 + 80(\frac{3}{5})(15) = \frac{1}{2}(\frac{120}{32.2})(v_{ramp})^2 + \frac{1}{2}(\frac{80}{32.2})(v_{crate})^2$$

Relate $\vec{v}_{crate}$ and $\vec{v}_{ramp}$ using the conservation of linear momentum,

$$(\overset{+}{\rightarrow}) \quad 0 = (\frac{120}{32.2})(v_{ramp}) - (\frac{80}{32.2})(v_{crate})_x \implies (v_{crate})_x = (1.5)v_{ramp}$$

$$\vec{v}_{crate} = \vec{v}_{ramp} + \vec{v}_{crate|ramp}$$

$$-(v_{crate})_x\hat{\boldsymbol{\imath}} - (v_{crate})_y\hat{\boldsymbol{\jmath}} = v_{ramp}\hat{\boldsymbol{\imath}} + v_{crate|ramp}\left(-\frac{4}{5}\hat{\boldsymbol{\imath}} - \frac{3}{5}\hat{\boldsymbol{\jmath}}\right)$$

$$-(v_{crate})_x = v_{ramp} - (\frac{4}{5})v_{crate|ramp} \qquad -(v_{crate})_y = -(\frac{3}{5})v_{crate|ramp}$$

Hence, we find $(v_{crate})_y = (1.875)v_{ramp}$

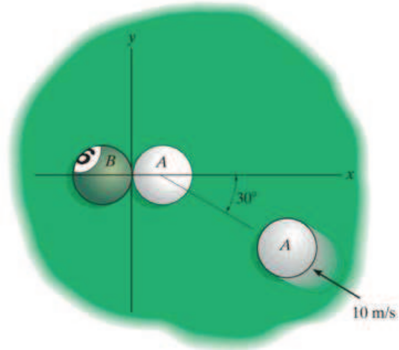
Then, using the conservation of energy equation again,

$$0 + 80(\frac{3}{5})(15) = \frac{1}{2}(\frac{80}{32.2})(v_{crate})^2 + \frac{1}{2}(\frac{120}{32.2})(v_{ramp})^2$$

$$0 + 80(\frac{3}{5})(15) = \frac{1}{2}(\frac{80}{32.2})\left[\left((1.5)v_{ramp}\right)^2 + \left((1.875)v_{ramp}\right)^2\right] + \frac{1}{2}(\frac{120}{32.2})(v_{ramp})^2$$

$$\implies v_{ramp} = 8.93\frac{\text{ft}}{\text{s}}$$

## 5.4 Impact



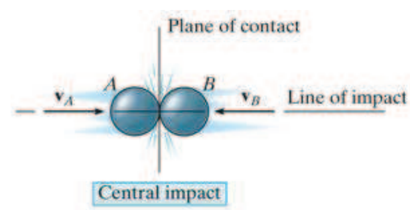

Consider a tennis ball. The quality of that tennis ball is measured by the height if its bounce. This can be quantified by the *coefficient of restitution* of the ball.

I a game of billiards, it is important to be able to predict the trajectory and speed of a ball after it is struck by another ball.

If we know the velocity of a ball $A$ before the impact, how can we determine the magnitude and direction of the velocity of ball $B$ after the impact?

Impact occurs when two bodies collide causing impulsive forces to be exerted between the bodies. The *line of impact* is a line through the mass centres of the colliding particles. In general, there are two types of impact.

### 5.4.1 Central Impact




*Central impact* occurs when the directions of motion of the two colliding particles are along the line of impact.

Once the particles collide, they may *deform* if they are non-rigid bodies. In any case, energy is transferred between the two particles.

For impulsive force to occur, we will assume that the particles are non-rigid. The particles will undergo a *period of deformation* such that they exert an equal but opposite impulse on each other, call it $\int \vec{P} \cdot dt$ on each other.

At the instant of *maximum deformation*, the relative motion of the particles zero and so they will have a common velocity.

After this instant, a *period of restitution* occurs and equal but opposite *impulse of restitution* $\int \vec{R} \cdot dt$ push the particles apart.

In most problems, the initial velocities of the particles will be known and it will be necessary to determine their final velocities. Recall that momentum is conserved throughout the system since the internal impulses cancel. Hence,

$$(\overset{+}{\rightarrow}) \qquad m_A(v_A)_a + m_B(v_B)_a = m_A(v_A)_b + m_B(v_B)_b \tag{5.12}$$

By analysing the impulse and momentum of each particle, we can obtain a second equation. During the deformation phase, we find,

$$m_A(v_A)_a - \int P \cdot dt = m_A v$$

and in the restitution phase,

$$m_A v - \int R \cdot dt = m_A(v_A)b$$

The ratio of the restitution impulse over the deformation impulse is called the *coefficient of restitution*, denoted $e$. Then, applying this to both particles,

$$e = \frac{\int R \cdot dt}{\int P \cdot dt} = \frac{v - (v_A)_b}{(v_A)_a - v} \qquad e = \frac{\int R \cdot dt}{\int P \cdot dt} = \frac{(v_B)_b - v}{v - (v_B)_a}$$

If the unknown velocity $v$ is eliminated from the above equation, then the coefficient of restitution can be expressed in terms of the particle's initial and final velocities such that,

$$(\overset{+}{\rightarrow}) \qquad e = \frac{(v_B)_b - (v_A)_b}{(v_A)_a - (v_B)_a} \tag{5.13}$$

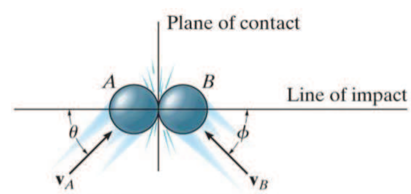
### Coefficient of Restitution

Note that when the coefficient of restitution $e = 1$, it indicates a *perfectly elastic* collision and that the impulse of deformation and restitution are equal and opposite.

If the coefficient of restitution is zero, the collision is said to be *inelastic* or *plastic* such that the resulting restitution impulse is zero. After the collision, both particles will **stick** and move with a common velocity.

## 5.4.2 Oblique Impact

*Oblique impact* occurs when the direction of motion of one or both of the particles is at an angle to the line of impact. Typically, there will be four unknowns that are the magnitudes $(v_A)_b$, $(v_B)_b$ and directions $\theta_b$, $\phi_b$ of the final velocities.



Oblique impact

Establish a coordinate system such that the $x$-axis is along the line of impact and the $y$-axis is the plane of contact. The impulsive forces of deformation and restitution act only in the $x$ direction. By resolving the velocity or momentum vectors into components along the $x$ and $y$ axis, it is then possible to write four independent scalar equations to determine our unknowns.

Momentum of the system is conserved along the line of impact or $x$-axis so that,

$$\sum m(v_x)_a = \sum m(v_x)_b$$

The coefficient of restitution, given by Equation (5.13) relates the relative velocity components of the particles along the line of impact.

If these two equations are solved simultaneously, then we obtain $(v_A)_b$ and $(v_B)_b$ in the $x$ direction.

The momentum of the particles is conserved along the plane of contact, or the $y$-axis, perpendicular to the line of impact. Because no impulse acts on the particles in this direction. So then,

$$((v_A)_y)_a = ((v_A)_y)_b \qquad ((v_B)_y)_a = ((v_B)_y)_b$$

**Procedure for Analysis**

- In most impact problems, the initial velocities of the particles and the coefficient of restitution are given, with the final velocities to be determined.
- Define the coordinate system such that the $x$-axis is along the line of impact and the $y$-axis is along the plane of contact perpendicular to the $x$-axis.
- For both central and oblique impact problems, the following equations apply along the line of impact in the $x$-direction,

$$\sum m(v_x)_a = \sum m(v_x)_b \qquad e = \frac{((v_B)_x)_b - ((v_A)_x)_b}{((v_A)_x)_a - ((v_B)_x)_a}$$

- For oblique impact problems, the following equations are also required, applied perpendicular to the line of impact in the $y$-direction,

$$m_A((v_A)_y)_a = m_A((v_A)_y)_b \qquad m_B((v_B)_y)_a = m_B((v_B)_y)_b$$

## Example (5.5)

The ball strikes the smooth wall with a velocity $(v_b)_1 = 20\text{ms}^{-1}$. The coefficient of restitution between the ball and the wall is $e = 0.75//$

Find the velocity of the ball just after the impact

**Solution**

The collision is an oblique impact with the line of impact perpendicular to the plane. Thus, the coefficient of restitution applies perpendicular to the wall and the momentum of the ball is conserved along the wall.

Using the $x$,$y$-axes defined along and perpendicular to the line of impact respectively,

$$m(v_b)_1 \sin 30^o = m(v_b)_2 \sin\theta$$

$$(v_b)_2 \sin\theta = 10\text{ms}^{-1}$$

Applying the coefficient of restitution in the $x$-direction,

$$e = \frac{0 - ((v_b)_x)_2}{((v_b)_x)_1 - 0} \implies 0.75 = \frac{0 - (-v_b)_2 \cos\theta}{20\cos 30^o - 0}$$

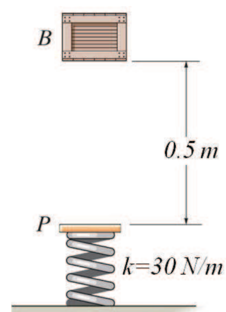
$$\implies (v_b)_2 \cos\theta = 12.99\text{ms}^{-1}$$

$$(v_b)_2 = \sqrt{12.99^2 + 10^2} = 16.4\text{ms}^{-1} \qquad \arctan(\frac{10}{12.99}) = 37.6^o$$

## Example (5.6)

A 2kg crate $B$ is released from rest, falls a distance $h = 0.5$m and strikes the plate $P$ is $e = 0.6$ and the spring stiffness is $k = 30\text{Nm}^{-1}$

Find the velocity of crate $B$ just after the collision.




**Solution**

Determine the speed of the crate just before the collision using projectile motion or energy analysis.

Analyse the collision as a central impact problem.

Determine the speed of block $B$ just before impact by using conservation of energy. Define the gravitational datum at the initial position of the block and note the block is released from rest.

$$T_1 + V_1 = T_2 + V_2$$

$$\frac{1}{2}m(v_1)^2 + mgh_1 = \frac{1}{2}m(v_2)^2 + mgh_2$$

$$0 + 0 = \frac{1}{2}(2)(v_2)^2 + (2)(9.81)(-0.5)$$

$$\implies v_2 = 3.132$$

This is the speed of the block just before the collision with the plate $P$ at rest.

Analyse the collision as a central impact problem.

Applying the conservation of momentum to the system in the vertical direction,

$$(+\uparrow) m_B(v_B)_1 - m_P(v_P)_1 = m_B(v_B)_2 = m_P(v_P)_2$$

$$(2)(-3.132) + 0 = (2)(v_B)_2 + (3)(v_P)_2$$

Using the coefficient of restitution,

$$(+\uparrow) \qquad e = \frac{(v_P)_2 - (v_B)_2}{(v_B)_1 - (v_P)_1} \implies 0.6 = \frac{(v_P)_2 - (v_B)_2}{-3.132 - 0} \implies (v_P)_2 - (v_B)_2 = -1.879$$

$$(v_B)_2 = -0.125\text{ms}^{-1} \downarrow \qquad (v_P)_2 = -2.00\text{ms}^{-1} \downarrow$$

# Chapter 6

# Planar Kinematics of a Rigid Body

Our objective in this chapter is to analyse the kinematics of a rigid body undergoing planar translation or rotation about a fixed axis.

## 6.1 Planar Rigid Body Motion

There are cases where an object cannot be treated as a particle. In these cases, the size or shape of the body must be considered. Rotation of the body about its centre of mass requires a different approach.

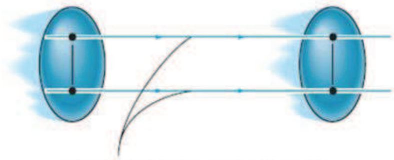
For example, in the design of gears, cams and links in machinery or mechanisms, rotation of the body is an important aspect in the analysis of motion.

We will now start to study rigid body motion. The analysis will be limited to planar motion.

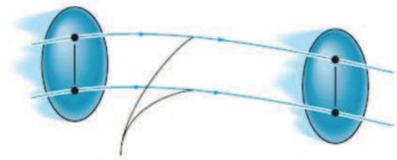
A body is said to undergo planar motion when all parts of the body move along paths equidistant from a fixed plane.

## 6.2 Translation

Translation occurs if every line segment on the body remains parallel to its original direction during the motion. When all point move along straight lines, the motion is called rectilinear translation. When the paths of motion are curved lines, the motion is called curvilinear translation.



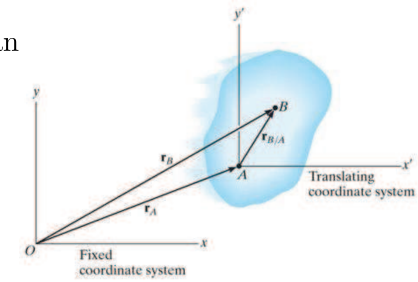
Path of rectilinear translation



Path of curvilinear translation

Recall, the positions of two points $A$ and $B$ on a tran

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B|A}$$




where $\vec{r}_A$ and $\vec{r}_B$ are the absolute position vectors defined from the fixed $x$,$y$ coordinate system and $\vec{r}_{B|A}$ is the relative position vector between $B$ and $A$.

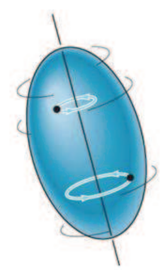
The velocity at $B$ is, $\quad \vec{v}_B = \vec{v}_A + \frac{d}{dt}\vec{r}_{B|A}$

Now $\frac{d}{dt}\vec{r}_{B|A} = 0$ since $\vec{r}_{B|A}$ is constant. So, $\vec{v}_B = \vec{v}_A$ and by similar logic, $\vec{a}_B = \vec{a}_A$.

Note that all point in a rigid body subjected to translation move with the same velocity and acceleration.

## 6.3 Rotation

In rotation about a fixed axis, all the particles of the body except those on the axis of rotation, move along circular paths in planes perpendicular to the axis of rotation.

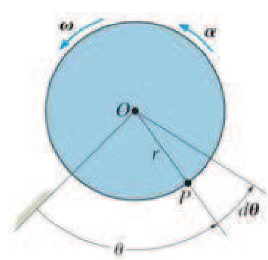


Rotation about a fixed axis

In the case of *general plane motion*, the body undergoes both translation and rotation. Translation occurs within a plan and rotation occurs about an axis perpendicular to this plane.

When a body rotates about a fixed axis, any point $P$ in the body travels along a circular path. the angular position of $P$ is defined by $\theta$.

The change in angular position, $d\theta$, is called the *angular displacement* with the units of either radians or revolutions.



*Angular velocity* $\omega$, is obtained by taking the time derivative of angular displacement such that,

$$\left(\underset{+}{\curvearrowleft}\right) \quad \omega = \frac{d}{dt}\theta \tag{6.1}$$

Similarly, the *angular acceleration* is obtained such that,

$$\left(\underset{+}{\curvearrowleft}\right) \quad \alpha = \frac{d^2}{dt^2}\theta = \frac{d}{dt}\omega = \omega \cdot \frac{d}{d\theta}\omega \tag{6.2}$$

If the angular acceleration of the body is constant, that is $\alpha = \alpha_c$, the equations for angular velocity and acceleration can integrated to yield the set of algebraic equations below,

$$\omega = \omega_o + \alpha_c t \tag{6.3}$$

$$\theta = \theta_o + \omega_o t + \frac{1}{2}\alpha_c t^2 \tag{6.4}$$

$$\omega^2 = (\omega_o)^2 + 2\alpha_c(\theta - \theta_o) \tag{6.5}$$

Where $\theta_o$ and $\omega_o$ are the initial values of the body's angular position and angular velocity respectively. Note these equations are very similar to the constant acceleration relations developed for the rectilinear motion of a particle.

### 6.3.1 Velocity

The magnitude of the velocity of point $P$ can be found by dividing $ds = rd\theta$ by $dt$ so that,

$$v = \omega r \tag{6.6}$$

The velocity's direction is tangent to the circular path of $P$.

In the vector formulation, the magnitude and direction of $\vec{v}$ can be determined from the cross product of $\vec{\omega}$ and $\vec{r}_p$. Here $\vec{r}_p$ is a position vector from any point on the axis of rotation to a point $P$.

$$\vec{v} = \vec{\omega} \times \vec{r}_p \tag{6.7}$$

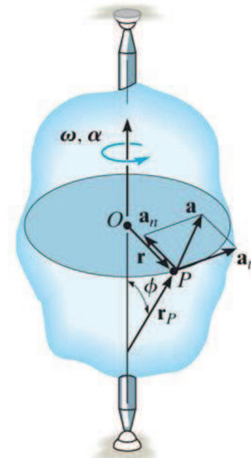
Note that this equation is non-commutative.

### 6.3.2 Acceleration

The acceleration of a point $P$ can be expressed in terms of its normal and tangential components.

Since $a_t = \frac{d}{dt}v$ and $a_n = \frac{v^2}{\rho}$ where $\rho = r$, $v = \omega r$, $\alpha = \frac{d}{dt}\omega$ we have

$$a_t = \alpha r \qquad\qquad a_n = \omega^2 r \tag{6.8}$$




The tangential component of acceleration represents the time rate of change in the velocity's magnitude. If the speed of $P$ is increasing, then $\vec{a}_t$ acts in the same direction as $\vec{v}$ and vice versa where speed is decreasing.

The normal component of acceleration represents the time rate of change in the velocity's direction. The direction of $\vec{a}_n$ is always towards the centroid $O$ of the circular path.

Like the velocity, the acceleration of point $P$ can be expressed in terms of the vector cross product. Taking the time derivative of Equation (6.7) we have,

$$\vec{a} = \vec{\alpha} \times \vec{r}_p + \vec{\omega} \times \frac{d}{dt}\vec{r}_p$$

Recall that $\vec{\alpha} = \frac{d}{dt}\vec{\omega}$ such that,

$$\vec{a} = \vec{\alpha} \times \vec{r}_p + \vec{\omega} \times (\vec{\omega} \times \vec{r}_p) \tag{6.9}$$

From the definition of the cross product, the first term $\vec{\alpha} \times \vec{r}_p$ has a magnitude $a_t = \alpha r_p \sin\phi = \alpha r$ and the direction of $\vec{a}_t$. Similarly the magnitude of the second term is $a_n = \omega^2 r_p \sin\phi = \omega^2 r$, and its direction is that of $\vec{a}_n$. Since this is also the direction $-\vec{r}$ we can express $\vec{a}_n$ such that $\vec{a}_n = -\omega^2\vec{r}$ such that,

$$\vec{a} = \vec{a}_t + \vec{a}_n = \vec{\alpha} \times \vec{r}_p - \omega^2\vec{r} \tag{6.10}$$

Then, since the vectors are orthogonal, the magnitude is given by,

$$a = \sqrt{(a_t)^2 + (a_n)^2}$$

### 6.3.3 Summary

- Establish a sign convention along the axis of rotation
- If a relationship is known between any two of the variables $\alpha$, $\omega$, $\theta$ or $t$, the variable can be determined from the the equations,

$$\omega = \frac{d}{dt}\theta \qquad \alpha = \frac{d}{dt}\omega \qquad \alpha \cdot d\theta = \omega \cdot d\omega$$

- If $\alpha$is constant, use the equations for constant angular acceleration,

$$\omega = \omega_o + \alpha_c t \qquad \theta = \theta_o = \omega_o t + \frac{1}{2}\alpha_c t^2 \qquad \omega^2 = (\omega_o)^2 + 2\alpha_c(\theta - \theta_o)$$

- To determine the motion of a point, the following scalar equations can be used,

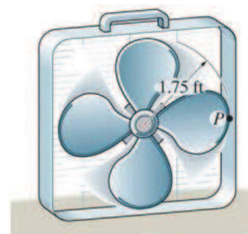
$$v = \omega r \qquad a_t = \alpha \qquad a_n = \omega^2 r \qquad a = \sqrt{(a_t)^2 + (a_n)^2}$$

- Alternatively, the vector form of these equations can be used,

$$\vec{v} = \vec{\omega} \times \vec{r}_p = \vec{\omega} \times \vec{r}$$

$$\vec{a} = \vec{a}_t + \vec{a}_n = \vec{\alpha} \times \vec{r}_p + \vec{\omega} \times (\vec{\omega} \times \vec{r}_p) = \vec{\alpha} \times \vec{r} - \omega^2 \vec{r}$$

## Example (6.1)




The motor gives the blade an angular acceleration $\alpha = 20e^{-0.6t}\frac{\text{rad}}{\text{s}^2}$ where $t$ is in seconds. The blade is initially at rest.

Find the velocity and acceleration of the tip $P$ of one the blades when $t = 3\text{s}$. How many revolutions has the blade turned in 3s?

### Solution

Determine the angular velocity and displacement of the blade using kinetics of angular motion.

The magnitude of the velocity and acceleration of point $P$ can be determined from the scalar equations of motion for a point on a rotating body.

Since the angular acceleration is given as a function of time, $\alpha = 20e^{-0.6t}\frac{\text{rad}}{\text{s}^2}$, the angular velocity and displacement can be found by integration.

$$\omega = \int \alpha \cdot dt = 20 \int e^{-0.6t} \cdot dt \frac{20}{-0.6} e^{-0.6t} \implies \omega\Big|_{t=3\text{s}} = -5.510\frac{\text{rad}}{\text{s}}$$

$$\theta = \int \omega \cdot dt = \frac{20}{-(0.6)} \int e^{-0.6t} \cdot dt = \frac{20}{(-0.6)^2} e^{-0.6t} \implies \theta\Big|_{t=3\text{s}} = 9.183\text{rad} = 1.46\text{rev}$$

$$\alpha\Big|_{t=3\text{s}} = 20e^{-0.6(3)} = 3.306\frac{\text{rad}}{\text{s}^2}$$

The velocity of point $P$ on the fan at a radius of 1.75ft is determined as,

$$v_P = \omega r = (5.510)(1.75) = 9.64\frac{\text{ft}}{\text{s}}$$

The normal and tangential components of acceleration of point $P$ are calculated as,

$$a_n = \omega^2 r = (5.510)^2(1.75) = 53.12\frac{\text{ft}}{\text{s}^2}$$

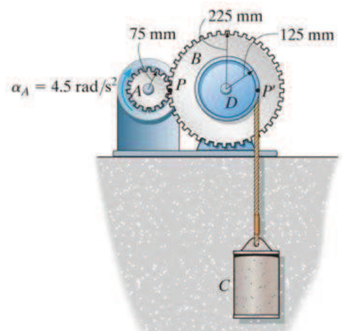
$$a_t = \alpha r = (3.306)(1.75) = 5.786 \frac{\text{ft}}{\text{s}^2}$$

Then the magnitude of the acceleration of $P$ is determined by,

$$a_P = \sqrt{(a_n)^2 + (a_t)^2} = \sqrt{(53.12)^2 + (5.786)^2} = 53.4 \frac{\text{ft}}{\text{s}^2}$$

## Example (6.2)

Starting from rest, when a gear $A$ is given a constant angular acceleration $\alpha_A = 4.5 \frac{\text{rad}}{\text{s}^2}$. The cord is wrapped around pulley $D$ which is rigidly attached to gear $B$.




Find the velocity of cylinder $C$ and the distance it travels in 3 seconds.

### Solution

The angular acceleration of gear $B$ and pulley $D$ can be related to $\alpha_A$

The acceleration of cylinder $C$ can be determined by using the equations for motion of a point on a rotating body since $(a_t)_D$ at point $P$ is the same as $a_C$.

The velocity and distance of $C$ can be found by using the constant acceleration equations.

Gear $A$ and $B$ will have the same speed and tangential component of acceleration at the point where they mesh such that,

$$a_t = \alpha_A r_A = \alpha_B r_B$$

$$(4.5)(75) = \alpha_B(225) \implies \alpha_B = 1.5 \frac{\text{rad}}{\text{s}^2}$$

Since gear $B$ and pulley $D$ turn together,

$$\alpha_D = \alpha_B = 1.5 \frac{\text{rad}}{\text{s}^2}$$

Assuming the cord attached to pulley $D$ is inextensible and does not slip, the velocity and acceleration of cylinder $C$ will be the same as the velocity and acceleration of cylinder $D$ will be the same as the velocity and tangential component of acceleration along the pulley,

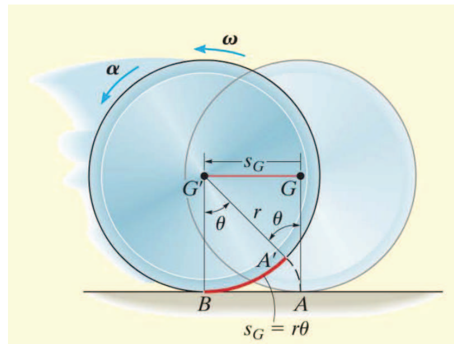
$$a_C = (a_t)_D = \alpha_D r_D = (1.5)(0.125) = 0.1875 \frac{\text{m}}{\text{s}^2} \quad \uparrow$$

Since $\alpha_A$ is constant, $\alpha_D$ and $a_C$ will be constant. The constant acceleration equation for rectilinear motion can be used to determine the velocity and displacement of cylinder $C$ when $t = 3$s

$$v_C = v_o + a_C t = 0 + 0.1875(3) = 0.563\frac{\text{m}}{\text{s}} \quad \uparrow$$

$$s_C = s_o + v_o t + \frac{1}{2}a_C t^2 = 0 + 0 + \frac{1}{2}0.1875(3)^2 = 0.844\text{m} \quad \uparrow$$

## Example (6.3)




At a given instant, the cylinder of radius $r$ shown has an angular velocity $\omega$ and angular acceleration $\alpha$.

Determine the velocity and acceleration of its centre $G$ if the cylinder rolls without slipping.

### Solution

The cylinder undergoes general plane motion since it simultaneously translates and rotates. By inspection, point $G$ moves in a straight line to the left from $G$ to $G'$ as the cylinder rolls. Consequently, its new position $G'$ will be specified by the horizontal position coordinate $s_G$, which is measured from $G$ to $G'$. Also as the cylinder rolls without slipping, the arc length $A'B$ on the rim which was in contact with the ground from $A$ to $B$ is equivalent to $s_G$. Consequently, the motion requires the radial line $GA$ to rotate $\theta$ to the position $G'A'$. Since the arc $A'B = r\theta$, then $G$ travels a distance $s_G = \vec{r}\theta$

Taking successive time derivatives of this equation, realizing that $r$ is constant, $\omega = \frac{d}{dt}\theta$ and $\alpha = \frac{d}{dt}\omega$ gives the necessary relationships,

$$s_G = r\theta \qquad v_G = r\omega \qquad a_G = r\alpha$$

# Chapter 7

# Moment of Inertia

Our objective in this chapter is to determine the mass moment of inertia of a rigid body or a system of rigid bodies.

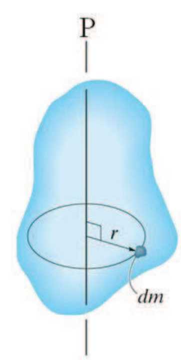
Consider a rigid body with a centre of mass at a point $G$. This body is free to rotate about the $x$-axis which basses through $G$. Now, suppose we apply a torque $T$ about the $z$-axis to the body such that the body begins to rotate with an angular acceleration of $\alpha$.

## 7.1 Mass Moment of Inertia

$T$ and $\alpha$ are related by the equation $T = I\alpha$. In this equation, $I$ is called the *mass moment of inertia* about the $z$-axis.

The mass moment of inertia of a body is a property that measures the resistance of the body to angular acceleration. It is often used when analysing rotational motion.

Again, consider a rigid body and some arbitrary $P$-axis. The mass moment of inertia about the $P$-axis is defined as,

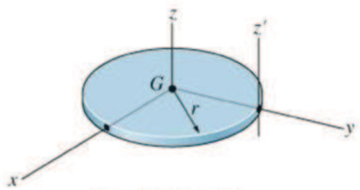



$$I = \int_m r^2 \cdot dm \tag{7.1}$$

where $r$ is the perpendicular distance from the axis to the arbitrary element $dm = \rho \cdot dV$.

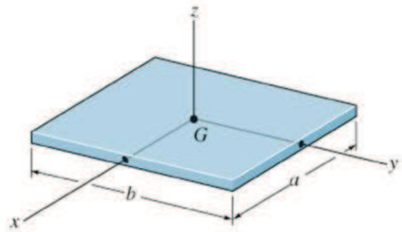
The mass moment of inertia is always a positive quantity and has units of $\text{kg} \cdot \text{m}^2$ or $\text{slug} \cdot \text{ft}^2$.

The figures show the mass moment of inertia formulations for two flat plate shapes commonly used when working with three dimensional bodies. The shapes are often used as the differential element being integrated over the entire body.




Thin Circular disk

$I_{xx} = I_{yy} = \frac{1}{4}mr^2 \quad I_{zz} = \frac{1}{2}mr^2 \quad I_{z'z'} = \frac{3}{2}mr^2$

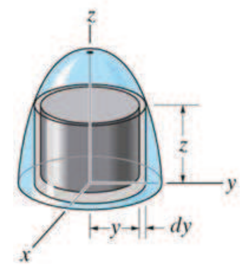



Thin plate

$I_{xx} = \frac{1}{12}mb^2 \quad I_{yy} = \frac{1}{12}ma^2 \quad I_{zz} = \frac{1}{12}m(a^2+b^2)$

When using direct integration, only symmetric bodies having surfaces generated by revolving a curve about an axis will be considered.

### 7.1.1 Shell Element

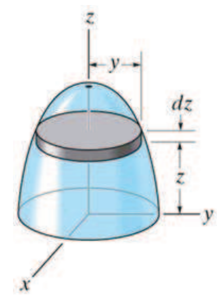



If a shell element having a height $z$ and radius $r$ along the $y$-axis and a thickness $dy$ is to be integrated, then the volume element is,

$$dV = (2\pi y)(z) \cdot dy$$

This element may be used to find the moment of inertia $I_z$ since the entire element lies equal perpendicular distance from the $z$-axis.

### 7.1.2 Disk Element

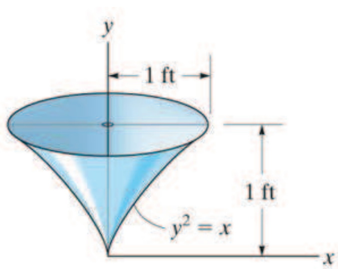



If a disk element having a radius $y$ and a thickness $dz$ is chosen for integration, then the volume element is,

$$dV = (\pi y^2) \cdot dz$$

Using the moment of inertia of the disk element, we can integrate to determine the moment of inertia of the entire body.

## Example (7.1)



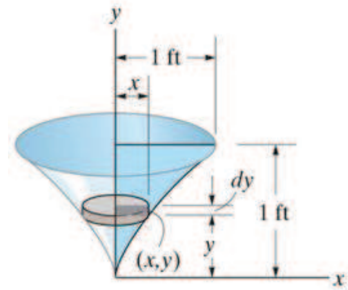

If the shape shown has a density $\rho = 5\frac{\text{slug}}{\text{ft}^3}$, find the mass moment of inertia of this body about the $y$-axis.

### Solution

Find the mass moment of inertia $dI_y$ of a disk element about the $y$-axis and integrate.

The moment of inertia of a disk about an axis perpendicular to its plane is,

$$I = \frac{1}{2}mr^2$$




Thus, for the disk element, we have,

$$dI_y = \frac{1}{2}(dm)x^2$$

where the differential mass is,

$$dm = \rho \cdot dV = \rho\pi x^2 \cdot dy$$

$$I_y = \int_0^1 \frac{\rho\pi x^4}{2} \cdot dy = \frac{\rho\pi}{2}\int_0^1 y^8 \cdot dy = \frac{\pi(5)}{18} = 0.873\text{slug} \cdot \text{ft}^2$$

### 7.1.3 Parallel Axis Theorem

If the mass moment of inertia of a body about an axis passing through the body's mass centre is known, then the moment of inertia about any other parallel axis may be determined by using the parallel axis theorem such that,

$$I = I_G + md^2 \tag{7.2}$$

where $I_G$ is the mass moment of inertia about the body's mass centre, $m$ is the mass of the body and $d$ is the perpendicular distance between the parallel axes.

### 7.1.4 Radius of Gyration

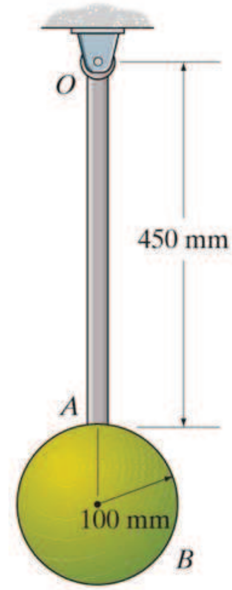
The mass moment of inertia of a body about a specific axis can be defined using the *radius of gyration k*. The radius of gyration has units of length and is a measure of the distribution of the body's mass about the axis at which the moment of inertia is defined.

$$I = mk^2 \tag{7.3}$$

### 7.1.5 Composite Bodies

If a body is constructed of a number of simple shapes, such as disks, spheres or rods, then the mass moment of inertia of the body about any axis can be determined by algebraically adding together all the mass moments of inertia found about the same axis of the different shapes.

### Example (7.2)




The pendulum consists of a slender rod with a mass of 10kg and sphere with a mass of 15kg.

Find the pendulum's mass moment of inertia about an axis perpendicular to the page and passing through point $O$.

#### Solution

Follow steps similar to finding the mass moment of inertia for a composite area. The pendulums can be divided into a slender rod and sphere.

The centre of mass of the rod is 0.225m from point $O$. The centre of mass of the sphere is 0.55m from the point $O$.

The mass moment of inertia data for a slender rod and sphere are given on the insider back cover of the textbook. Using those data, and the parallel axis theorem, calculate the following,

$$I_O = I_G + (m)(d)^2$$

$$(I_O)_{rod} = \frac{1}{12}(10)(0.45)^2 + 10(0.225)^2 = 0.675\text{kg}\cdot\text{m}^2$$

$$(I_O)_{sphere} = \frac{2}{5}(15)(0.1)^2 + 15(0.55)^2 = 4.598\text{kg}\cdot\text{m}^2$$

Now, adding the two moments of inertia about point $O$, we get,

$$I_O = (I_O)_{rod} + (I_O)_{sphere} = 5.27\text{kg}\cdot\text{m}^2$$

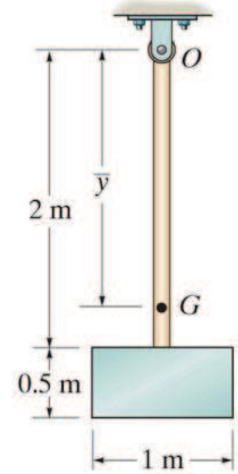
### 7.1.6 Centre of Mass — Composite Body

Recall that the coordinates $(\overline{x}, \overline{y}, \overline{z})$ of the centre of mass of a composite body is given by,

$$\overline{x} = \frac{\sum \tilde{x}W}{\sum W} \qquad \overline{y} = \frac{\sum \tilde{y}W}{\sum W} \qquad \overline{z} = \frac{\sum \tilde{z}W}{\sum W}$$

Where tilde denotes the mass centre of each body.

### Example (7.3)




The pendulum consists of a 5kg plat and a 3kg slender rod.

Find the radius of gyration of the pendulum about an axis perpendicular to the screen and passing through point $G$.

### Solution

Determine the mass moment of inertia of the pendulum using the method for composite bodies. Then determine the radius of gyration using the moments of inertia and mass values.

If we put the $y$-axis along the rod in the plane of the page, then clearly $\tilde{x}$ and $\tilde{x}$ and zero for the rod and plate. This reflects the fact that if an object is symmetrical about an axis, then the centre of mass lies on that axis.

Separate the pendulum into a square plate $P$ and a slender rod $R$.

The centre of mass of the plate and rod are 2.25m and 1m from point $O$ respectively.

The centre of mass of the pendulum lies on the axis of symmetry,

$$\overline{y} = \frac{\sum \tilde{y}m}{\sum m} = \frac{(1)(3) + (2.25)(5)}{(3+5)} = 1.781\text{m}$$

The moment of inertia data on plates and slender rods are given on the inside cover of the textbook. Using those data and the parallel axis theorem,

$$I_P = \frac{1}{12}(5)(0.5^2 + 1^2) + 5(2.25 - 1.781)^2 = 1.621\text{kg} \cdot \text{m}^2$$

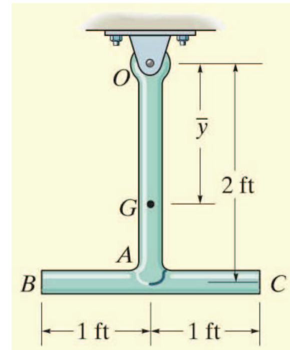
$$I_R = \frac{1}{12}(3)(2)^2 = 3(1.781 - 1)^2 = 2.830\text{kg} \cdot \text{m}^2$$

$$I_G = I_P + I_R = 1.621 + 2.830 = 4.45\text{kg} \cdot \text{m}^2$$

The total mass $m$ is 8kg. Then, the radius of gyration $k$ is,

$$k = \sqrt{\frac{I_G}{m}} = 0.746\text{m}$$

## Example (7.4)




The pendulum is suspended from the pin at $O$ and consists of two thin rods, each having a weight of 10lb.

Determine the moment of inertia of the pendulum about an axis passing through point $O$ and the mass centre $G$ of the pendulum.

### Solution

Using the table on the inside back cover, the moment of inertia of rod $OA$ about an axis perpendicular to the page and passing through point $O$ of the rod is,

$$(I_{OA})_O = \frac{1}{3}ml^2 = \frac{1}{3}\left(\frac{10}{32.2}\right)(2)^2 = 0.414\text{slug}\cdot\text{ft}^2$$

This same value can be obtained using the parallel-axis theorem,

$$(I_{OA})_O = \frac{1}{12}ml^2 + md^2 = \frac{1}{12}\left(\frac{10}{32.2}\right)(2)^2 + \left(\frac{10}{32.2}\right)(1)^2 = 0.414\text{slug}\cdot\text{ft}^2$$

For rod $BC$ we have,

$$(I_{BC})_O = \frac{1}{12}ml^2 + md^2 = \frac{1}{12}\left(\frac{10}{32.2}\right)(2)^2 + \left(\frac{10}{32.2}\right)(2)^2 = 1.346\text{slug}\cdot\text{ft}^2$$

The moment of inertia of the pendulum about $O$ is therefore,

$$I_O = 0.414 + 1.346 = 1.76\text{slug}\cdot\text{ft}^2$$

The mass centre $G$ will be located some distance $\overline{y}$ from point $O$. Using the formula for determining the mass centre, we have,

$$\overline{y} = \frac{\sum \tilde{y}m}{\sum m} = \frac{1\left(\frac{10}{32.2}\right) + 2\left(\frac{10}{32.2}\right)}{\left(\frac{10}{32.2}\right) + \left(\frac{10}{32.2}\right)} = 1.50\text{ft}$$

The moment of inertia $I_G$ may be found in the same manner as $I_O$ which requires successive applications of the parallel axis theorem to transfer the moments of inertia of rods $OA$ and $BC$ to $G$. A more direct solution however, involves using the result for $I_O$ such that,

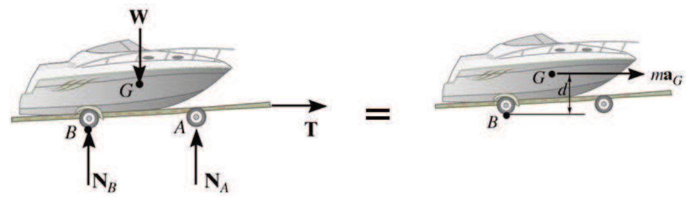
$$I_O = I_G + md^2$$

$$1.76 = I_G + \left(\frac{20}{32.2}\right)(1.50)^2 \implies I_G = 0.362\text{slug}\cdot\text{s}^2$$

## 7.2 Planar Kinetics with Equations of Motion

In this section we will apply the three equations of motion for a rigid body in planar motion. We will analyse problems involving translational motion.

Consider a boat and trailer undergoing rectilinear motion. In order to find the reactions at the trailer's wheels and the acceleration of the boat, we need to draw the free body diagram and kinetic diagram for the boat and trailer.




How can we solve this problem using equations of motion?

### 7.2.1 Equations of Translational Motion

We will limit our study of planar kinetics to rigid bodies that are symmetric with respect to a fixed reference plane.

As discussed in the previous chapter, when a body is subjected to general planar motion, it undergoes a combination of translation and rotation.

First, a coordinate system with its origin at an arbitrary point $P$ is established.

The $x$,$y$-axes should not rotate and can either be fixed or translate with constant velocity.

If a body undergoes translational motion, then the equation of motion applies where,

$$\sum \vec{F} = m\vec{a}_G$$

That is, the sum of all the external forces acting on the body is equal to the product of the body's mass and the acceleration of its mass centre.

### 7.2.2 Equations of Rotational Motion

We need to determine the effects caused by the moments of an external force system.

The moment about point $P$ can be written as,

$$\sum(\vec{r}_i \times \vec{F}_i) + \sum \vec{M}_i = \bar{r} \times m\vec{a}_G + I_G\alpha \qquad \sum M_P = \sum(K_k)_P$$

where $\bar{r} = \bar{x}\hat{\imath} + \bar{y}\hat{\jmath}$ and $\sum M_P$ is the resultant moment about $P$ due to all the external forces. The term $\sum(M_k)_P$ is called the *kinetic moment* about point $P$.

If a point $P$ coincides with the mass centre $G$, then this equation reduces to the scalar equation,

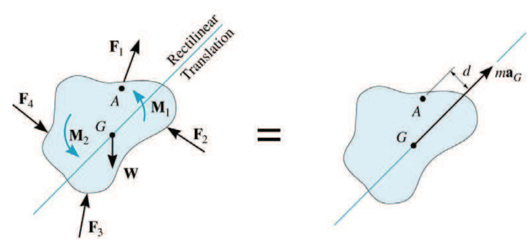
$$\sum M_G = I_G \alpha$$

that is, the resultant moment about the mass centre due to all the external forces is equal to the product of the bodies angular acceleration and the moment of inertia about the bodies mass centre.

Thus, three independent scalar equations of motion may be used to describe the general planar motion of a rigid body. These equations are,

$$\sum F_x = m(a_G)_x \qquad \sum F_y = m(a_G)_y \qquad \sum M_G = I_G \alpha \quad \text{or} \quad \sum M_P = \sum (M_k)_p$$

## 7.3 Strictly Translational Motion

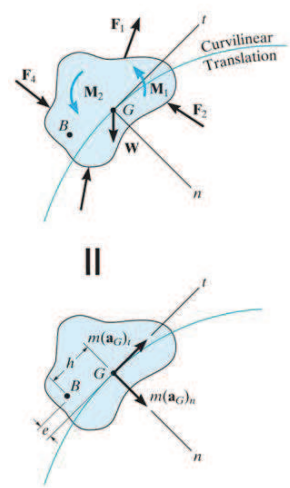



When a rigid body undergoes only translation, all the particles of the body have the same acceleration such that,

$$\vec{a}_G = \vec{a} \qquad \vec{\alpha} = 0$$

Thus, the equations of motion become simply,

$$\sum F_x = m(a_G)_x \qquad \sum F_y = m(a_G)_y \qquad \sum M_G = 0$$




Note that, if it makes the problem easier, then the moment equation can be applied about another point instead of the mass centre, for example point $A$ such that,

$$\sum M_A = (ma_G)d$$

When a rigid body is subjected to curvilinear translation, it is best to use an $n$,$t$-coordinate system. Apply the equations of motion as written below,

$$\sum F_n = m(a_G)_n \qquad \sum F_t = m(a_G)_t$$

$$\sum M_B = e\Big(m(a_G)_t\Big) - h\Big(m(a_G)_n\Big)$$

### 7.3.1 Procedure for Analysis

Problems involving only the kinetics of translation of a rigid body should be solved using the following procedure,

- Establish an $x$,$y$ or $n$,$t$ inertial coordinate system and specify the sense and direction of acceleration $\vec{a}_G$ of the mass centre.
- Draw a free body diagram and kinetic diagram showing all external and inertial forces and couple.
- Identify the unknowns.
- Apply the three equations of motion using one set or the other,

$$\sum F_x = m(a_G)_x \qquad \sum F_y = m(a_G)_y \qquad \Bigg| \qquad \sum F_n = m(a_G)n \qquad \sum F_t = m(a_G)_t$$

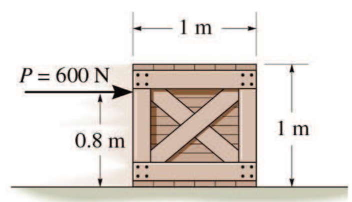
$$\sum M_G = 0 \text{ —or— } \sum M_P = \sum(M_k)_P \qquad \Bigg| \qquad \sum M_G = 0 \text{ —or— } \sum M_P = \sum(M_k)_P$$

- Remember that friction forces always act opposing the motion of the body.

## Example (7.5)

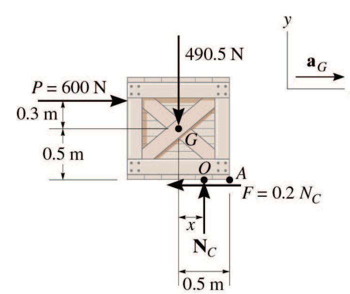
A 50kg crate rests on a horizontal surface for which the kinetic friction coefficient $\mu_k = 0.2$.

Find the acceleration of the crate if $P = 600$N.




**Solution**

Follow the procedure for analysis. Note that the load $P$ can cause the crate either to slide or to tip over. Let us assume that the crate slide, an assumption we will check later.




The coordinate system and free body diagram are shown. The weight of the crate is applied at the centre of mass and the normal force $N_c$ acts at $O$. Point $O$ is some distance $x$ from the crate's centre line. The unknowns are $N_c$, $x$ and $a_G$.

$$\begin{array}{lcccl} \sum F_x = m(a_G)_x; & 600 - 0.2N_c = 50a_G & & & N_c = 490\text{N} \\ \sum F_y = m(a_G)_y; & N_c - 490.5 = 0 & \Longrightarrow & & x = 0.467\text{m} \\ \sum M_G = 0; & -600(0.3) + N_c x - 0.2N_c(0.5) = 0 & & & a_G = 10.0\text{ms}^{-2} \end{array}$$

Since $x = 0.467\text{m} < 0.5\text{m} = x_{max}$, the crate slides as assumed.

# Chapter 8

# Planar Kinetics with Work and Energy

In this chapter, we will define the various ways a force and couple do work. We will apply the principle of work and energy to a rigid body.

## 8.1 Kinetic Energy

Consider a slab moving in an inertial plane. Take an arbitrary $i$th particle with a mass $dm$ located some distance $r$ from an arbitrary fixed point $P$. Then that particles kinetic energy is,

$$T_i = \frac{1}{2} dm (v_i)^2$$

Extending this to the summation of each point, we obtain the kinetic energy of the body. Further, we can expressed velocity in terms of the body's angular velocity $\omega$ such that,

$$\vec{v}_i = \vec{v}_P + \vec{v}_{i|P} = \Big[(v_P)_x - \omega y\Big]\hat{\boldsymbol{\imath}} + \Big[(v_P)_y + \omega x\Big]\hat{\boldsymbol{\jmath}}$$

By squaring the magnitude of this expression, we get,

$$\vec{v}_i \bullet \vec{v}_i = (v_P)^2 - 2(v_P)_x \omega y + 2(v_P)_y \omega x + \omega^2 r^2$$

Substituting this into our previous equation for kinetic energy gives,

$$T = \frac{1}{2}\left(\int_m dm\right)(v_P)^2 - (v_P)_x \omega \left(\int_m y \cdot dm\right) + (v_P)_y \omega \left(\int_m x \cdot dm\right) + \frac{1}{2}\omega^2 \left(\int_m r^2 \cdot dm\right)$$

The integrals from left to right represent the body's mass $m$, $\overline{y}m$ and $\overline{x}m$, and finally the body's moment of inertia $I_P$ about the $z$-axis through point $P$. Thus, we can simplify the expression such that,

$$T = \frac{1}{2}m(v_P)^2 - (v_P)_x\omega\overline{y}m + (v_P)_y\omega\overline{x}m + \frac{1}{2}I_P\omega^2 \tag{8.1}$$

In the special case of considering point $P$ itself such that $\overline{x} = \overline{y} = 0$,

$$T = \frac{1}{2}m(v_G)^2 + \frac{1}{2}I_G\omega^2 \tag{8.2}$$

### 8.1.1 Pure Translation

*Pure Translation* occurs when a rigid body is subjected to only curvilinear or rectilinear translation so the rotational kinetic energy is zero. Therefore,

$$T = \frac{1}{2}m(v_G)^2 \tag{8.3}$$

### 8.1.2 Pure Rotation

When a rigid body is rotating about a fixed axis passing through point $O$, the body has both translational and rotational kinetic energy. Therefore,

$$T = \frac{1}{2}m(v_G)^2 + \frac{1}{2}I_G\omega^2$$

Since $v_G = r_G\omega$, we can express the kinetic energy of the body, by the parallel axis theorem, as,

$$T = \frac{1}{2}\left(I_G + m(r_G)^2\right)\omega^2 = \frac{1}{2}I_O\omega^2$$

If the rotation occurs about the mass centre $G$, then what is the value of $v_G$?

In this case, the velocity of the mass centre is equal to zero. So the kinetic energy equation reduces to,

$$T = \frac{1}{2}I_G\omega^2 \tag{8.4}$$

## 8.2 Work of a Force

Recall that the work done by a force can be written as,

$$U_F = \int \vec{F} \bullet d\vec{r} = \int_s (F\cos\theta) \cdot ds$$

### 8.2.1 Work of a Weight

As before, the work can be expressed as,

$$U_W = -W\Delta y$$

Remember, if the force and movement are in the same direction, then the work is positive.

### 8.2.2 Work of a Spring Force

For a linear spring, the work is given by,

$$U_s = -\frac{1}{2}k\Big((s_2)^2 - (s_1)^2\Big)$$

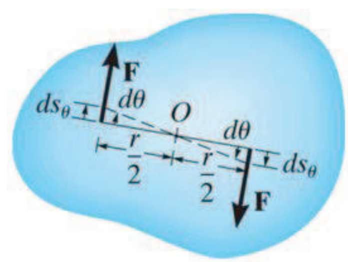
### 8.2.3 Forces that don't Work

Some external forces do no work. For instance, reactions at fixed supports do no work because there is no displacement at those points.

Normal and frictional forces acting on rolling bodies the are **not slipping** over a rough surface also do no work since there is no instantaneous displacement of the point in contact with the ground.

Internal forces do no work because they always act in equal and opposite pairs.

## 8.3 Work of a Couple




When a body subjected to a couple experiences general plane motion, the two couple forces do work only when the body undergoes rotation.

If the body rotates through an angular displacement $d\theta$, the work of the couple moment $M$ is,

$$U_M = \int_{\theta_1}^{\theta_2} M \cdot d\theta$$

If the couple moment $M$ is constant, then

$$U_M = M(\theta_2 - \theta_1)$$

Here the work is positive, provided $M$ and $(\theta_2 - \theta_1)$ are in the same direction.

## 8.4 Principle of Work and Energy

Recall the statement of the principle of work and energy used earlier,

$$T_1 + \sum U_{1|2} = T_2$$

In the case of general plane motion, this equation states that the sum of the initial kinetic energy, both translational and rotational, and the work done by all external forces and couple moments equals the body's final kinetic energy.

This equation is a scalar and can be applied to a system of rigid bodies by summing contributions from all bodies.

### Example (8.1)

The disk weighs 40lb and has a radius of gyration $k_G$ of 0.6ft. A 15ft · lb moment is applied and the spring has a spring constant of $10\frac{\text{lb}}{\text{ft}}$

Find the angular velocity of the wheel when point $G$ moves 0.5ft. The wheel starts from rest and rolls without slipping. The spring is originally unstretched.

#### Solution

Use the principle of work and energy to solve the problem since distance is the primary parameter. Draw a free body diagram of the disk and calculate the work of the external forces.