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Chapter 1

Functions and Their Properties



Gottfried Wilhelm Leibniz
1646 - 1716

The Big Picture

This chapter deals with the definition and properties of things we call *functions*. They are used all the time in the world around us although we don't recognize them right away. Functions are a mathematical device for describing an inter-dependence between things or objects, whether real or imaginary. With this notion, the original creators of Calculus, namely, the English mathematician and physicist, Sir Isaac Newton, and the German philosopher and mathematician, Gottfried Wilhelm Leibniz (see inset), were able to quantify and express relationships between real things in a mathematical way. Most of you will have seen the famous Einstein equation

$$E = mc^2.$$

This expression defines a dependence of the quantity, E , called the *energy* on m , called the *mass*. The number c is the *speed of light* in a vacuum, some 300,000 kilometers per second. In this simple example, E is a *function* of m . Almost all naturally occurring phenomena in the universe may be quantified in terms of functions and their relationships to each other. A complete understanding of the material in this chapter will enable you to gain a foothold into the fundamental vocabulary of Calculus.

Review

You'll need to remember or learn the following material before you get a thorough understanding of this chapter. Look over your notes on functions and be familiar with all the **basic algebra** and **geometry** you learned and also don't forget to review your **basic trigonometry**. Although this seems like a lot, it is necessary as mathematics is a sort of language, and before you learn any language you need to be familiar with its vocabulary and its grammar and so it is with mathematics. Okay, let's start ...



1.1 The Meaning of a Function

You realize how important it is for you to remember your social security/insurance number when you want to get a real job! That's because the employer will associate

You can think of a function f as an I/O device, much like a computer CPU; it takes input, x , works on x , and produces only one output, which we call $f(x)$.

you with this number on the payroll. This is what a **function** does ... a function is a rule that associates to each element in some set that we like to call the **domain** (in our case, the name of anyone eligible to work) only one element of another set, called the **range** (in our case, the set of all social security/insurance numbers of these people eligible to work). In other words the function here associates to each person his/her social security/insurance number. Each person can have only one such number and this lies at the heart of the definition of a function.

Example 1.

In general everybody as an *age* counted historically from the moment he/she is born. Consider the rule that associates to each person, that person's age. You can see this depicted graphically in Figure 1. Here A has age a , person B has age c while persons C, D both have the same age, that is, b . So, by definition, this rule is a function. On the other hand, consider the rule that associates to each automobile driver the car he/she owns. In Figure 2, both persons B and C share the automobile c and this is alright, however note that person A owns two automobiles. Thus, by definition, this rule cannot be a function.

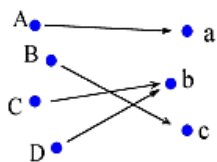


Figure 1.

NOTATION

$\text{Dom}(f)$ = Domain of f

$\text{Ran}(f)$ = Range of f

Objects in the domain of f are referred to as independent variables, while objects in the range are dependent variables.

Example 2.

Let's say that Jennifer Black has social security number 124124124. The arrow would start at a point which we label "Jennifer Black" (in the domain) and end at a point labeled "124124124" (in the range).

Okay, so here's the formal definition of a function ...

Definition 1. A function f is a rule which associates with each object (say, x) from one set named the domain, a single object (say, $f(x)$) from a second set called the range. (See Figure 1)

Rather than replace every person by their photograph, the objects of the domain of a function are replaced by **symbols** and mathematicians like to use the symbol " x " to mark some unknown quantity in the domain (this symbol is also called an **independent variable**), because it can be *any* object in the domain. If you don't like this symbol, you can use any other symbol and this won't change the function. The symbol, $f(x)$, is also called a **dependent variable** because its value generally *depends* on the value of x . Below, we'll use the "box" symbol, \square , in many cases instead of the more standard symbol, x .

Example 3.

Let f be the (name of the) function which associates a person with their height. Using a little shorthand we can write this rule as $h = f(p)$ where p is a particular person, (p is the *independent variable*) and h is that person's height (h is the *dependent variable*). The domain of this function f is the set of all persons, right? Moreover, the range of this function is a set of numbers (their height, with some units of measurement attached to each one). Once again, let's notice that many people can have the same height, and this is okay for a function, but clearly there is no one having two different heights!

A rule which is NOT a function

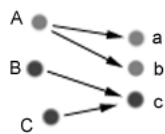


Figure 2.

LOOK OUT! When an arrow "splits" in an arrow diagram (as the arrow starting from A does in Figure 2) the resulting rule is *never* a function.

In applications of calculus to the physical and natural sciences the domain and the range of a function are both sets of real (sometimes complex) numbers. The symbols

used to represent objects within these sets may vary though ... E may denote energy, p the price of a commodity, x distance, t time, etc. The domain and the range of a function may be the same set or they may be totally unrelated sets of numbers, people, aardvarks, aliens, etc. Now, functions have to be identified somehow, so rules have been devised to *name them*. Usually, we use the lower case letters f, g, h, k, \dots to name the function itself, but you are allowed to use any other symbols too, but try not to use x as this might cause some confusion ... we already decided to name objects of the domain of the function by this symbol, x , remember?

Quick Summary Let's recapitulate. A function has a *name*, a *domain* and a *range*. It also has a *rule* which associates to every object of its domain only one object in its range. So the rule (whose name is) g which associates to a given number its square as a number, can be denoted quickly by $g(x) = x^2$ (Figure 3). You can also represent this rule by using the symbols $g(\square) = \square^2$ where \square is a "box" ... something that has nothing to do with its shape as a symbol. It's just as good a symbol as " x " and both equations represent the *same* function.

You need to think beyond the *shape* of an independent variable and just keep your mind on a generic "variable", something that has nothing to do with its shape.

Remember ... x is just a symbol for what we call an independent variable, that's all. We can read off a rule like $g(x) = x^2$ in many ways: The purist would say "The value of g at x is x^2 " while some might say, " g of x is x^2 ". What's really important though is that you *understand the rule*... in this case we would say that the function associates a symbol with its square *regardless of the shape of the symbol itself*, whether it be an x , \square , \triangle , \heartsuit , t , etc.

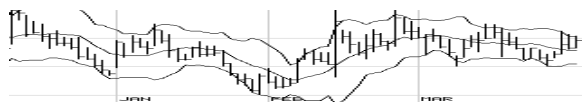


Example 4. Generally speaking,

- The association between a one-dollar bill and its serial number is a function (unless the bill is counterfeit!). Its domain is the collection of all one-dollar bills while its range is a subset of the natural numbers along with some 26 letters of the alphabet.



- The association between a CD-ROM and its own serial number is also a function (unless the CD was copied!).
- Associating a fingerprint with a specific human being is another example of a function, as is ...
- Associating a human being with the person-specific DNA (although this may be a debatable issue).
- The association between the monetary value of a stock, say, x , at time t is also function but this time it is a function of *two* variables, namely, x, t . It could be denoted by $f(x, t)$ meaning that this symbol describes the value of the stock x at time t . Its graph may look like the one below.



- The correspondence between a patent number and a given (patented) invention is a function
- If the ranges of two functions are subsets of the real numbers then the difference between these two functions is also a function. For example, if $R(x) = px$

denotes the *total revenue* function, that is, the product of the number of units, x , sold at price p , and $C(x)$ denotes the total cost of producing these x units, then the difference, $P(x) = R(x) - C(x)$ is the *profit* acquired after the sale of these x units.

Composition of Functions:

There is a fundamental operation that we can perform on two functions called their “composition”.

Let’s describe this notion by way of an example. So, consider the domain of all houses in a certain neighborhood. To each house we associate its owner (we’ll assume that to each given house there is only one owner). Then the rule that associates to a given house its owner is a function and we call it “ f ”. Next, take the rule that associates to a given owner his/her annual income from all sources, and call this rule “ g ”. Then the new rule that associates with each house the annual income of its owner is called the composition of g and f and is denoted mathematically by the symbols $g(f(x))$. Think of it . . . if x denotes a house then $f(x)$ denotes its owner (some name, or social insurance number or some other unique way of identifying that person). Then $g(f(x))$ must be the annual income of the owner, $f(x)$.

Once can continue this exercise a little further so as to define compositions of more than just two functions . . . like, maybe three or more functions. Thus, if h is a (hypothetical) rule that associates to each annual income figure the total number of years of education of the corresponding person, then the composition of the three functions defined by the symbol $h(g(f(x)))$, associates to each given house in the neighborhood the total number of years of education of its owner.

In the next section we show how to calculate the values of a composition of two given functions using symbols that we can put in “boxes” . . . so we call this the “box method” for calculating compositions. Basically you should always look at what a function does to a generic “symbol”, rather than looking at what a function does to a specific symbol like “ x ”.

NOTES:

1.2 Function Values and the Box Method

Now look at the function g defined on the domain of real numbers by the rule $g(x) = x^2$. Let's say we want to know the value of the mysterious looking symbols, $g(3x+4)$, which is really the same as asking for the composition $g(f(x))$ where $f(x) = 3x + 4$. How do we get this?

The Box Method

To find the value of $g(3x+4)$ when $g(x) = x^2$: We place all the symbols "3x+4" (i.e., all the stuff between the parentheses) in the symbol " $g(\dots)$ " inside a box, say, \square , and let the function g take the box \square to \square^2 (because this is what a function *does* to a symbol, regardless of what it looks like, right?). Then we "remove the box", replace its sides by parentheses, and there you are ... what's left is the value of $g(3x+4)$.

We call this procedure the **Box Method**.

Example 5. So, if $g(x) = x^2$, then $g(\square) = \square^2$. So, according to our rule, $g(3x+4) = g(\boxed{3x+4}) = \boxed{3x+4}^2 = (3x+4)^2$. This last quantity, when simplified, gives us $9x^2 + 24x + 16$. We have found that $g(3x+4) = 9x^2 + 24x + 16$.

Example 6. If f is a new function defined by the rule $f(x) = x^3 - 4$ then $f(\square) = \square^3 - 4$ (regardless of *what's in the box!*), and

$$f(a+h) = f(\boxed{a+h}) = \boxed{a+h}^3 - 4 = (a+h)^3 - 4 = a^3 + 3a^2h + 3ah^2 + h^3 - 4.$$

Also,

$$f(2) = 2^3 - 4 = 4,$$

and

$$\begin{aligned} f(-1) &= (-1)^3 - 4 = -5, \\ f(a) &= a^3 - 4, \end{aligned}$$

where a is another symbol for any object in the domain of f .

Example 7. Let $f(x) = \frac{1.24x^2}{\sqrt{2.63x-1}}$. Find the value of $f(n+6)$ where n is a positive integer.

Solution The Box Method gives

$$\begin{aligned} f(x) &= \frac{1.24x^2}{\sqrt{2.63x-1}} \\ f(\square) &= \frac{1.24\square^2}{\sqrt{2.63\square-1}} \\ f(\boxed{n+6}) &= \frac{1.24\boxed{n+6}^2}{\sqrt{2.63\boxed{n+6}-1}} \\ f(n+6) &= \frac{1.24(n+6)^2}{\sqrt{2.63(n+6)-1}} \\ &= \frac{1.24(n^2+12n+36)}{\sqrt{2.63n+15.78-1}} \\ &= \frac{1.24n^2+14.88n+44.64}{\sqrt{2.63n+14.78}}. \end{aligned}$$

The function $g(x) = x^2$ and some of its values.

x	$g(x)$
-2	4
-1	1
0.5	0.25
1.5	2.25
3	9
0.1	0.01
-2.5	6.25
5	25
10	100
-3	9

Figure 3.

NOTATION for Intervals.

$(a, b) = \{x : a < x < b\}$, and this is called an **open interval**. $[a, b] = \{x : a \leq x \leq b\}$, is called a **closed interval**. $(a, b], [a, b)$ each denote the sets $\{x : a < x \leq b\}$ and $\{x : a \leq x < b\}$, respectively (either one of these is called a **semi-open interval**).

“Formerly, when one invented a new function, it was to further some practical purpose; today one invents them in order to make incorrect the reasoning of our fathers, and nothing more will ever be accomplished by these inventions.”

Henri Poincaré, 1854 - 1912
French mathematician

Example 8. On the other hand, if $f(x) = 2x^2 - x + 1$, and $h \neq 0$ is some real number, how do we find the value of the quotient

$$\frac{f(x+h) - f(x)}{h} ?$$

Solution Well, we know that $f(\square) = 2\square^2 - \square + 1$. So, the idea is to put the symbols $x+h$ inside the box, use the rule for f on the *box* symbol, then expand the whole thing and subtract the quantity $f(x)$ (and, finally, divide this result by h). Now, the value of f evaluated at $x+h$, that is, $f(x+h)$, is given by

$$\begin{aligned} f(\boxed{x+h}) &= 2\boxed{x+h}^2 - \boxed{x+h} + 1, \\ &= 2(x+h)^2 - (x+h) + 1 \\ &= 2(x^2 + 2xh + h^2) - x - h + 1 \\ &= 2x^2 + 4xh + 2h^2 - x - h + 1. \end{aligned}$$

From this, provided $h \neq 0$, we get

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{2x^2 + 4xh + 2h^2 - x - h + 1 - (2x^2 - x + 1)}{h} \\ &= \frac{4xh + 2h^2 - h}{h} \\ &= \frac{h(4x + 2h - 1)}{h} \\ &= 4x + 2h - 1. \end{aligned}$$

Example 9. Let $f(x) = 6x^2 - 0.5x$. Write the values of f at an integer by $f(n)$, where the symbol “ n ” is used to denote an integer. Thus $f(1) = 5.5$. Now write $f(n) = a_n$. Calculate the quantity $\frac{a_{n+1}}{a_n}$.

Solution The Box method tells us that since $f(n) = a_n$, we must have $f(\square) = a_\square$. Thus, $a_{n+1} = f(n+1)$. Furthermore, another application of the Box Method gives $a_{n+1} = f(n+1) = f(\boxed{n+1}) = 6\boxed{n+1}^2 - 0.5\boxed{n+1}$. So,

$$\frac{a_{n+1}}{a_n} = \frac{6(n+1)^2 - 0.5(n+1)}{6n^2 - 0.5n} = \frac{6n^2 + 11.5n + 5.5}{6n^2 - 0.5n}.$$

Example 10. Given that

$$f(x) = \frac{3x+2}{3x-2}$$

determine $f(x-2)$.

Solution Here, $f(\square) = \frac{3\square+2}{3\square-2}$. Placing the symbol “ $x-2$ ” into the box, collecting terms and simplifying, we get,

$$f(x-2) = f(\boxed{x-2}) = \frac{3\boxed{x-2} + 2}{3\boxed{x-2} - 2} = \frac{3(x-2) + 2}{3(x-2) - 2} = \frac{3x-4}{3x-8}.$$

EXAMPLES



Example 11. If $g(x) = x^2 + 1$ find the value of $g(\sqrt{x-1})$.

Solution Since $g(\square) = \square^2 + 1$ it follows that

$$g(\sqrt{x-1}) = [\sqrt{x-1}]^2 + 1 = [x-1] + 1 = x$$

on account of the fact that $\sqrt{\square}^2 = \square$, regardless of “what’s in the box”.

Example 12. If $f(x) = 3x^2 - 2x + 1$ and $h \neq 0$, find the value of

$$\frac{f(x+h) - f(x-h)}{2h}.$$

Solution We know that since $f(x) = 3x^2 - 2x + 1$ then $f(\square) = 3\square^2 - 2\square + 1$. It follows that

$$\begin{aligned} f(x+h) - f(x-h) &= \{3[\square{x+h}]^2 - 2[\square{x+h}] + 1\} - \{3[\square{x-h}]^2 - 2[\square{x-h}] + 1\} \\ &= 3\{(x+h)^2 - (x-h)^2\} - 2\{(x+h) - (x-h)\} \\ &= 3(4xh) - 2(2h) = 12xh - 4h. \end{aligned}$$

It follows that for $h \neq 0$,

$$\frac{f(x+h) - f(x-h)}{2h} = \frac{12xh - 4h}{2h} = 6x - 2.$$

Example 13. Let f be defined by

$$f(x) = \begin{cases} x+1, & \text{if } -1 \leq x < 0, \\ x^2, & \text{if } 0 < x \leq 3, \end{cases}$$

This type of function is said to be “**defined in pieces**”, because it takes on different values depending on where the “ x ” is...

- What is $f(-1)$?
- Evaluate $f(0.70714)$.
- Given that $0 < x < 1$ evaluate $f(2x+1)$.

Solution a) Since $f(x) = x+1$ for any x in the interval $-1 \leq x < 0$ and $x = -1$ is in this interval, it follows that $f(-1) = (-1) + 1 = 0$.

b) Since $f(x) = x^2$ for any x in the interval $0 < x \leq 3$ and $x = 0.70714$ is in this interval, it follows that $f(0.70714) = (0.70714)^2 = 0.50005$

c) First we need to know what f does to the symbol $2x+1$, that is, what is the value of $f(2x+1)$? But this means that we have to know where the values of $2x+1$ are when $0 < x < 1$, right? So, for $0 < x < 1$ we know that $0 < 2x < 2$ and so once we add 1 to each of the terms in the inequality we see that $1 = 0+1 < 2x+1 < 2+1 = 3$. In other words, whenever $0 < x < 1$, the values of the expression $2x+1$ must lie in the interval $1 < 2x+1 < 3$. We now use the Box method: Since f takes a symbol to its square whenever the symbol is in the interval $(0, 3]$, we can write by definition $f(\square) = \square^2$ whenever $0 < \square \leq 3$. Putting $2x+1$ in the box, (and using the fact that $1 < \square{2x+1} < 3$) we find that $f(\square{2x+1}) = \square{2x+1}^2$ from which we deduce $f(2x+1) = (2x+1)^2$ for $0 < x < 1$.

Some useful angles expressed in radians

<i>degrees</i>	<i>radians</i>
0	0
30°	$\pi/6$
45°	$\pi/4$
60°	$\pi/3$
90°	$\pi/2$
180°	π
270°	$3\pi/2$
360°	2π

We'll need to recall some notions from geometry in the next section.

Don't forget that, in Calculus, we always assume that angles are described in **radians** and not degrees. The conversion is given by

$$\text{Radians} = \frac{(\text{Degrees}) \times (\pi)}{180}$$

For example, 45 degrees = $45 \pi/180 = \pi/4 \approx 0.7853981633974$ radians.

NOTES:

Figure 4.

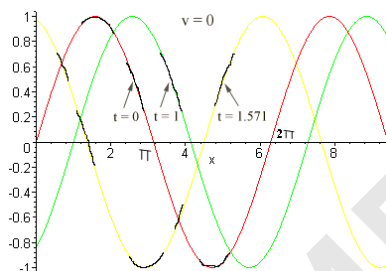
SAMPLE

SNAPSHOTS

Example 14. This example requires knowledge of trigonometry. Given that $h(t) = t^2 \cos(t)$ and $Dom(h) = (-\infty, \infty)$. Determine the value of $h(\sin x)$.

Solution We know that $h(\square) = \square^2 \cos(\square)$. So, $h(\boxed{\sin x}) = \boxed{\sin x}^2 \cos(\boxed{\sin x})$. Removing the box we get, $h(\sin x) = (\sin x)^2 \cos(\sin x)$, or, equivalently, $h(\sin x) = (\sin x)^2 \cos(\sin x) = \sin^2(x) \cos(\sin x)$.

Example 15. Let f be defined by the rule $f(x) = \sin x$. Then the function whose values are defined by $f(x - vt) = \sin(x - vt)$ can be thought of as representing a travelling wave moving to the right with velocity $v \geq 0$. Here t represents *time* and we take it that $t \geq 0$. You can get a feel for this motion from the graph below where we assume that $v = 0$ and use three increasing times to simulate the motion of the wave to the right.



The function $f(x) = \sin x$ and some of its values.

x (in radians)	$\sin x$
0	0
$\pi/6$	$1/2 = 0.5$
$-\pi/6$	$-1/2 = -0.5$
$\pi/3$	$\sqrt{3}/2 \approx 0.8660$
$-\pi/3$	$-\sqrt{3}/2 \approx -0.8660$
$\pi/2$	1
$-\pi/2$	-1
$\pi/4$	$\sqrt{2}/2 \approx 0.7071$
$-\pi/4$	$-\sqrt{2}/2 \approx -0.7071$
$3\pi/2$	-1
$-3\pi/2$	1
2π	0

Figure 5.

Example 16. On the surface of our moon, an object P falling from rest will fall a distance, $f(t)$, of approximately $5.3t^2$ feet in t seconds. Let's take it for granted that its, so-called, **instantaneous velocity**, denoted by the symbol $f'(t)$, at time $t = t_0 \geq 0$ is given by the expression

$$\text{Instantaneous velocity at time } t = f'(t) = 10.6 t.$$

Determine its (instantaneous) velocity after 1 second (at $t = 1$) and after 2.6 seconds ($t = 2.6$).

Solution We calculate its instantaneous velocity, at $t = t_0 = 1$ second. Since, in this case, $f(t) = 5.3t^2$, it follows that its instantaneous velocity at $t = 1$ second is given by $10.6(1) = 10.6$ feet per second, obtained by setting $t = 1$ in the formula for $f'(t)$. Similarly, $f'(2.6) = 10.6(2.6) = 27.56$ feet per second. The observation here is that one can conclude that an object falling from rest on the surface of the moon will fall at approximately *one-third* the rate it does on earth (neglecting air resistance, here).

Example 17. Now let's say that f is defined by

$$f(x) = \begin{cases} x^2 + 1, & \text{if } -1 \leq x \leq 0, \\ \cos x, & \text{if } 0 < x \leq \pi, \\ x - \pi, & \text{if } \pi < x \leq 2\pi. \end{cases}$$

Find an expression for $f(x + 1)$.

Solution Note that this function is defined *in pieces* (see Example 13) ... Its domain is obtained by taking the union of all the intervals on the right side making up one big interval, which, in our case is the interval $-1 \leq x \leq 2\pi$. So we see that $f(2) = \cos(2)$



because the number 2 is in the interval $0 < x \leq \pi$. On the other hand, $f(8)$ is not defined because 8 is not within the domain of definition of our function, since $2\pi \approx 6.28$.

Now, the value of $f(x+1)$, say, will be different depending on where the symbol “ $x+1$ ” is. We can still use the “box” method to write down the values $f(x+1)$. In fact, we replace every occurrence of the symbol x by our standard “box”, insert the symbols “ $x+1$ ” inside the box, and then remove the boxes... We’ll find

$$f(\boxed{x+1}) = \begin{cases} \boxed{x+1}^2 + 1, & \text{if } -1 \leq \boxed{x+1} \leq 0, \\ \cos \boxed{x+1}, & \text{if } 0 < \boxed{x+1} \leq \pi, \\ \boxed{x+1} - \pi, & \text{if } \pi < \boxed{x+1} \leq 2\pi. \end{cases}$$

or

$$f(x+1) = \begin{cases} (x+1)^2 + 1, & \text{if } -1 \leq x+1 \leq 0, \\ \cos(x+1), & \text{if } 0 < x+1 \leq \pi, \\ (x+1) - \pi, & \text{if } \pi < x+1 \leq 2\pi. \end{cases}$$

We now solve the inequalities on the right for the symbol x (by subtracting 1 from each side of the inequality). This gives us the values

$$f(x+1) = \begin{cases} x^2 + 2x + 2, & \text{if } -2 \leq x \leq -1, \\ \cos(x+1), & \text{if } -1 < x \leq \pi - 1, \\ x + 1 - \pi, & \text{if } \pi - 1 < x \leq 2\pi - 1. \end{cases}$$

Note that the graph of the function $f(x+1)$ is really the graph of the function $f(x)$ shifted to the left by 1 unit. We call this a “translate” of f .



Exercise Set 1.

Use the method of this section to evaluate the following functions at the indicated point(s) or symbol.

- $f(x) = x^2 + 2x - 1$. What is $f(-1)$? $f(0)$? $f(+1)$? $f(1/2)$?
- $g(t) = t^3 \sin t$. Evaluate $g(x+1)$.
- $h(z) = z + 2 \sin z - \cos(z+2)$. Evaluate $h(z-2)$.
- $k(x) = -2 \cos(x-ct)$. Evaluate $k(x+2ct)$.
- $f(x) = \sin(\cos x)$. Find the value of $f(\pi/2)$. [Hint: $\cos(\pi/2) = 0$]
- $f(x) = x^2 + 1$. Find the value of

$$\frac{f(x+h) - f(x)}{h}$$

whenever $h \neq 0$. Simplify this expression as much as you can!

- $g(t) = \sin(t+3)$. Evaluate

$$\frac{g(t+h) - g(t)}{h}$$

whenever $h \neq 0$ and simplify this as much as possible.

Hint: Use the trigonometric identity

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

valid for any two angles A, B where we can set $A = t+3$ and $B = h$ (just two more **symbols**, right?)

8. Let x_0, x_1 be two symbols which denote real numbers. In addition, for any real number x let $f(x) = 2x^2 \cos x$.

a) If $x_0 = 0$ and $x_1 = \pi$, evaluate the expression

$$\frac{f(x_0) + f(x_1)}{2}.$$

Hint: $\cos(\pi) = -1$ b) What is the value of the expression

$$f(x_0) + 2 f(x_1) + f(x_2)$$

if we are given that $x_0 = 0$, $x_1 = \pi$, and $x_2 = 2\pi$?

Hint: $\cos(2\pi) = 1$

9. Let f be defined by

$$f(x) = \begin{cases} x + 1, & \text{if } -1 \leq x \leq 0, \\ -x + 1, & \text{if } 0 < x \leq 2, \\ x^2, & \text{if } 2 < x \leq 6. \end{cases}$$

- a) What is $f(0)$?
 b) Evaluate $f(0.142857)$.
 c) Given that $0 < x < 1$ evaluate $f(3x + 2)$.

Hint Use the ideas in Example 17.

10. Let $f(x) = 2x^2 - 2$ and $F(x) = \sqrt{\frac{x}{2} + 1}$. Calculate the values $f(F(x))$ and $F(f(x))$ using the *box method* of this section. Don't forget to expand completely and simplify your answers as much as possible.
11. $g(x) = x^2 - 2x + 1$. Show that $g(x + 1) = x^2$ for every value of x .
12. $h(x) = \frac{2x + 1}{1 + x}$. Show that $h\left(\frac{x - 1}{2 - x}\right) = x$, for $x \neq 2$.
13. Let $f(x) = 4x^2 - 5x + 1$, and $h \neq 0$ a real number. Evaluate the expression

$$\frac{f(x + h) - 2f(x) + f(x - h)}{h^2}.$$

14. Let f be defined by

$$f(x) = \begin{cases} x - 1, & \text{if } 0 \leq x \leq 2, \\ 2x, & \text{if } 2 < x \leq 4, \end{cases}$$

Find an expression for $f(x + 1)$ when $1 < x \leq 2$.

Suggested Homework Set 1. Go into a library or the World Wide Web and come up with five (5) functions that appear in the literature. Maybe they have names associated with them? Are they useful in science, engineering or in commerce? Once you have your functions, identify the dependent and independent variables and try to evaluate those functions at various points in their domain.

NOTES

If you want to use your calculator for this question (you don't have to...) don't forget to change your angle settings to **radians!**

The **Binomial Theorem** states that, in particular,

$$(\square + \triangle)^2 = \square^2 + 2\square\triangle + \triangle^2$$

for any two symbols \square, \triangle representing real numbers, functions, etc. **Don't forget the middle terms**, namely, " $2\square\triangle$ " in this formula.



1.3 The Absolute Value of a Function

Definition of the absolute value function

One of the most important functions in the study of calculus is the absolute value function.

Definition 2. The function whose rule is defined by setting

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

is called the **absolute value function**.

For example, $|-5| = -(-5) = +5$, and $|6.1| = 6.1$. You see from this Definition that the absolute value of a number is either that same number (if it is positive) or the original *unsigned* number (dropping the minus sign completely). Thus, $|-5| = -(-5) = 5$, since $-5 < 0$ while $|3.45| = 3.45$ since $3.45 > 0$. Now the inequality

$$(0 \leq) |\square| \leq \triangle \quad (1.1)$$

between the symbols \square and \triangle is equivalent to (i.e., exactly the same as) the inequality

$$-\triangle \leq \square \leq \triangle. \quad (1.2)$$

where \square and \triangle are any two symbols whatsoever (x , t , or any function of x , etc). Why is this true? Well, there are only two cases. That is, $\square \geq 0$ and $\square \leq 0$, right? Let's say $\square \geq 0$. In this case $\square = |\square|$ and so (1.1) implies (1.2) immediately since the left side of (1.2) is already negative. On the other hand if $\square \leq 0$ then, by (1.1), $|\square| = -\square \leq \triangle$ which implies $\square \geq -\triangle$. Furthermore, since $\square \leq 0$ we have that $\square \leq -\square \leq \triangle$ and this gives (1.2). For example, **the inequality** $|x - a| < 1$ **means that** the distance from x to a is at most 1 and, in terms of an inequality, this can be written as

$$|x - a| < 1 \quad \text{is equivalent to} \quad -1 < x - a < +1.$$

Why? Well, put $x - a$ in the box of (1.1) and the number 1 in the triangle. Move these symbols to (1.2) and remove the box and triangle, then what's left is what you want. That's all. Now, adding a to both ends and the middle term of this latest inequality we find the equivalent statement

$$|x - a| < 1 \quad \text{is also equivalent to} \quad a - 1 < x < a + 1.$$

This business of passing from (1.1) to (1.2) is **really important in Calculus** and you should be able to do this without thinking (after lots of practice you will, don't worry).

Example 18.

Write down the values of the function f defined by the rule $f(x) = |1 - x^2|$ as a function defined in pieces. That is "remove the absolute value" around the $1 - x^2$.

Solution Looks tough? Those absolute values can be very frustrating sometimes! Just use the Box Method of Section 1.2. That is, use Definition 2 and replace all the symbols between the vertical bars by a \square . This really makes life easy, let's try it out. Put the symbols between the vertical bars, namely, the " $1 - x^2$ " inside a box, \square . Since, by definition,

$$|\square| = \begin{cases} \square, & \text{if } \square \geq 0, \\ -\square, & \text{if } \square < 0, \end{cases} \quad (1.3)$$



we also have

$$|1 - x^2| = \boxed{1 - x^2} = \begin{cases} 1 - x^2, & \text{if } \boxed{1 - x^2} \geq 0, \\ -\boxed{1 - x^2}, & \text{if } \boxed{1 - x^2} < 0. \end{cases}$$

Removing the boxes and replacing them by parentheses we find

$$|1 - x^2| = |(1 - x^2)| = \begin{cases} (1 - x^2), & \text{if } (1 - x^2) \geq 0, \\ -(1 - x^2), & \text{if } (1 - x^2) < 0. \end{cases}$$

Adding x^2 to both sides of the inequality on the right we see that the above display is equivalent to the display

$$|1 - x^2| = \begin{cases} (1 - x^2), & \text{if } 1 \geq x^2, \\ -(1 - x^2), & \text{if } 1 < x^2. \end{cases}$$

Almost done! We just need to solve for x on the right, above. To do this, we're going to use the results in Figure 6 with $A = 1$. So, if $x^2 \leq 1$ then $|x| \leq 1$ too. Similarly, if $1 < x^2$ then $1 < |x|$, too. Finally, we find

$$|1 - x^2| = \begin{cases} (1 - x^2), & \text{if } 1 \geq |x|, \\ -(1 - x^2), & \text{if } 1 < |x|. \end{cases}$$

Now by (1.1)-(1.2) the inequality $|x| \leq 1$ is equivalent to the inequality $-1 \leq x \leq 1$. In addition, $1 < |x|$ is equivalent to the double statement "either $x > 1$ or $x < -1$ ". Hence the last display for $|1 - x^2|$ may be rewritten as

$$|1 - x^2| = \begin{cases} 1 - x^2, & \text{if } -1 \leq x \leq +1, \\ x^2 - 1, & \text{if } x > 1 \text{ or } x < -1. \end{cases}$$

A glance at this latest result shows that the natural domain of f (see the Appendix) is the set of all real numbers.

NOTE The procedure described in Example 18 will be referred to as the process of **removing the absolute value**. You just can't leave out those vertical bars because you feel like it! Other functions defined by absolute values are handled in the same way.

Example 19. Remove the absolute value in the expression $f(x) = |x^2 + 2x|$.

Solution We note that since $x^2 + 2x$ is a polynomial it is defined for every value of x , that is, its natural domain is the set of all real numbers, $(-\infty, +\infty)$. Let's use the Box Method. Since for any symbol say, \square , we have by definition,

$$|\square| = \begin{cases} \square, & \text{if } \square \geq 0, \\ -\square, & \text{if } \square < 0. \end{cases}$$

we see that, upon inserting the symbols $x^2 + 2x$ inside the box and then removing

Solving a square root inequality!

If for some real numbers A and x , we have

$$x^2 < A,$$

then, it follows that

$$|x| < \sqrt{A}$$

More generally, this result is true if x is replaced by any other symbol (including functions!), say, \square . That is, if for some real numbers A and \square , we have

$$\square^2 < A,$$

then, it follows that

$$|\square| < \sqrt{A}$$

These results are still true if we replace " $<$ " by " \leq " or if we reverse the inequality and $A > 0$.

Figure 6.

Steps in removing absolute values in a function f

- Look at that part of f with the absolute values,
- Put all the stuff between the vertical bars in a box,
- Use the definition of the absolute value, equation 1.3.
- Remove the boxes, and replace them by parentheses,
- Solve the inequalities involving x 's for the symbol x .
- Rewrite f in pieces

(See Examples 18 and 20)

Figure 7.

its sides, we get

$$\left| x^2 + 2x \right| = \begin{cases} x^2 + 2x, & \text{if } x^2 + 2x \geq 0, \\ -(x^2 + 2x), & \text{if } x^2 + 2x < 0. \end{cases}$$

So the required function *defined in pieces* is given by

$$\left| x^2 + 2x \right| = \begin{cases} x^2 + 2x, & \text{if } x^2 + 2x \geq 0, \\ -(x^2 + 2x), & \text{if } x^2 + 2x < 0. \end{cases}$$



Sir Isaac Newton
1642 - 1727

where we need to solve the inequalities $x^2 + 2x \geq 0$ and $x^2 + 2x < 0$, for x . But $x^2 + 2x = x(x + 2)$. Since we want $x(x + 2) \geq 0$, there are now *two cases*. Either *both* quantities $x, x + 2$ must be greater than or equal to zero, OR *both* quantities $x, x + 2$ must be less than or equal to zero (so that $x(x + 2) \geq 0$ once again). The other case, the one where $x(x + 2) < 0$, will be considered separately.

Solving $x^2 + 2x \geq 0$:

Case 1: $x \geq 0, (x + 2) \geq 0$. In this case, it is clear that $x \geq 0$ (since if $x \geq 0$ then $x + 2 \geq 0$ too). This means that the polynomial inequality $x^2 + 2x \geq 0$ has among its solutions the set of real numbers $\{x : x \geq 0\}$.

Case 2: $x \leq 0, (x + 2) \leq 0$. In this case, we see that $x + 2 \leq 0$ implies that $x \leq -2$. On the other hand, for such x we also have $x \leq 0$, (since if $x \leq -2$ then $x \leq 0$ too). This means that the polynomial inequality $x^2 + 2x \geq 0$ has for its *solution* the set of real numbers $\{x : x \leq -2 \text{ or } x \geq 0\}$.

A similar argument applies to the case where we need to solve $x(x + 2) < 0$. Once again there are two cases, namely, the case where $x > 0$ and $x + 2 < 0$ and the separate case where $x < 0$ and $x + 2 > 0$. Hence,

Solving $x^2 + 2x < 0$:

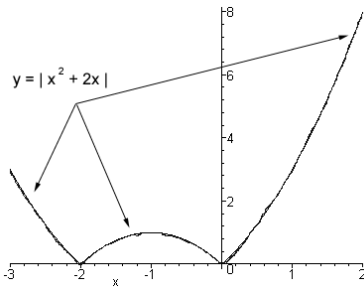
Case 1: $x > 0$ and $(x + 2) < 0$. This case is impossible since, if $x > 0$ then $x + 2 > 2$ and so $x + 2 < 0$ is impossible. This means that there are no x such that $x > 0$ and $x + 2 < 0$.

Case 2: $x < 0$ and $(x + 2) > 0$. This implies that $x < 0$ and $x > -2$, which gives the inequality $x^2 + 2x < 0$. So, the the solution set is $\{x : -2 < x < 0\}$.

Combining the conclusions of each of these cases, our function takes the form,

$$\left| x^2 + 2x \right| = \begin{cases} x^2 + 2x, & \text{if } x \leq -2 \text{ or } x \geq 0, \\ -(x^2 + 2x), & \text{if } -2 < x < 0. \end{cases}$$

Its graph appears in the margin.



The graph of $f(x) = |x^2 + 2x|$.

Example 20. Rewrite the function f defined by

$$f(x) = |x - 1| + |x + 1|$$

for $-\infty < x < +\infty$, as a function defined in pieces.

Solution How do we start this? Basically we need to understand the definition of the absolute value and apply it here. In other words, there are really 4 cases to consider when we want to remove these absolute values, the cases in question being:

1. $x - 1 \geq 0$ and $x + 1 \geq 0$. These two together imply that $x \geq 1$ and $x \geq -1$, that is, $x \geq 1$.
2. $x - 1 \geq 0$ and $x + 1 \leq 0$. These two together imply that $x \geq 1$ and $x \leq -1$ which is impossible.
3. $x - 1 \leq 0$ and $x + 1 \geq 0$. These two together imply that $x \leq 1$ and $x \geq -1$, or $-1 \leq x \leq 1$.
4. $x - 1 \leq 0$ and $x + 1 \leq 0$. These two together imply that $x \leq 1$ and $x \leq -1$, or $x \leq -1$.

Combining these four cases we see that we only need to consider the three cases where $x \leq -1$, $x \geq 1$ and $-1 \leq x \leq 1$ separately.

In the first instance, if $x \leq -1$, then $x + 1 \leq 0$ and so $|x + 1| = -(x + 1)$. What about $|x - 1|$? Well, since $x \leq -1$ it follows that $x - 1 \leq -2 < 0$. Hence $|x - 1| = -(x - 1)$ for such x . Combining these two results about the absolute values we get that

$$f(x) = |x + 1| + |x - 1| = -(x + 1) - (x - 1) = -2x,$$

for $x \leq -1$.

In the second instance, if $x \geq 1$, then $x - 1 \geq 0$ so that $|x - 1| = x - 1$. In addition, since $x + 1 \geq 2$ we see that $|x + 1| = x + 1$. Combining these two we get

$$f(x) = |x + 1| + |x - 1| = (x + 1) + (x - 1) = 2x,$$

for $x \geq 1$.

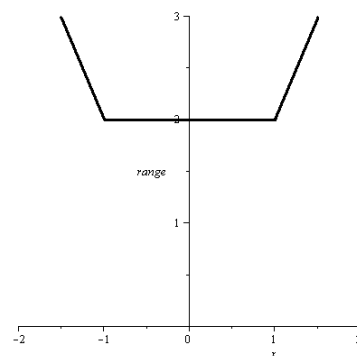
In the third and final instance, if $-1 \leq x \leq 1$ then $x + 1 \geq 0$ and so $|x + 1| = x + 1$. Furthermore, $x - 1 \leq 0$ implies $|x - 1| = -(x - 1)$. Hence we conclude that

$$f(x) = |x + 1| + |x - 1| = (x + 1) - (x - 1) = 2,$$

for $-1 \leq x \leq 1$. Combining these three displays for the pieces that make up f we can write f as follows:

$$|x + 1| + |x - 1| = \begin{cases} -2x, & \text{if } x \leq -1, \\ 2, & \text{if } -1 \leq x \leq 1. \\ 2x, & \text{if } x \geq 1. \end{cases}$$

and this completes the description of the function f as required. Its graph consists of the darkened lines in the adjoining Figure.



The graph of $f(x) = |x + 1| + |x - 1|$.

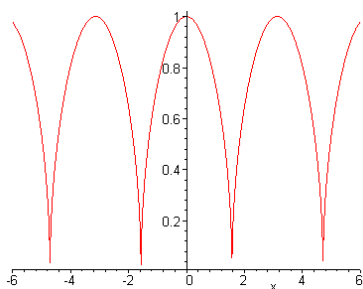
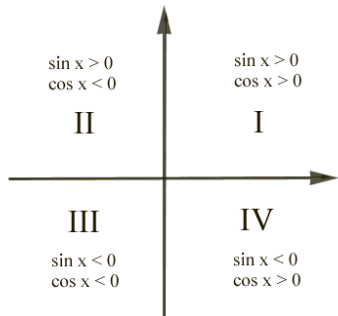
Example 21. Remove the absolute value in the function f defined by $f(x) = \sqrt{|\cos x|}$ for $-\infty < x < +\infty$.

Solution First, we notice that $|\cos x| \geq 0$ regardless of the value of x , right? So, the square root of this absolute value is defined for every value of x too, and this explains the fact that its natural domain is the open interval $-\infty < x < +\infty$. We use the method in Figure 7.

- Let's look at that part of f which has the absolute value signs in it...
In this case, it's the part with the $|\cos x|$ term in it.
- Then, take all the stuff between the vertical bars of the absolute value and stick them in a box ...

Using Definition 2 in disguise namely, Equation 1.3, we see that

$$|\cos x| = \boxed{\cos x} = \begin{cases} \boxed{\cos x}, & \text{if } \boxed{\cos x} \geq 0, \\ -\boxed{\cos x}, & \text{if } \boxed{\cos x} < 0. \end{cases}$$



The graph of $y = \sqrt{|\cos x|}$

Figure 8.

- Now, remove the boxes, and replace them by parentheses if need be ...

$$|\cos x| = \begin{cases} \cos x, & \text{if } \cos x \geq 0, \\ -\cos x, & \text{if } \cos x < 0. \end{cases}$$

- Next, solve the inequalities on the right of the last display above for x .

In this case, this means that we have to figure out when $\cos x \geq 0$ and when $\cos x < 0$, okay? There's a few ways of doing this... One way is to look at the graphs of each one of these functions and just find those intervals where the graph lies above the x -axis.

Another way involves remembering the trigonometric fact that the cosine function is positive in Quadrants I and IV (see the margin for a quick recall). Turning this last statement into symbols means that if x is between $-\pi/2$ and $\pi/2$, then $\cos x \geq 0$. Putting it another way, if x is in the interval $[-\pi/2, +\pi/2]$ then $\cos x \geq 0$.

But we can always add positive and negative multiples of 2π to this and get more and more intervals where the cosine function is positive ... why?. Either way, we get that $\cos x \geq 0$ whenever x is in the closed intervals $[-\pi/2, +\pi/2]$, $[3\pi/2, +5\pi/2]$, $[7\pi/2, +9\pi/2]$, ... or if x is in the closed intervals $[-5\pi/2, -3\pi/2]$, $[-9\pi/2, -7\pi/2]$, ... (Each one of these intervals is obtained by adding multiples of 2π to the endpoints of the basic interval $[-\pi/2, +\pi/2]$ and rearranging the numbers in increasing order).

Combining these results we can write

$$|\cos x| = \begin{cases} \cos x, & \text{if } x \text{ is in } [-\pi/2, +\pi/2], [3\pi/2, +5\pi/2], \\ & [7\pi/2, +9\pi/2], \dots, \text{ or if } x \text{ is in} \\ & [-5\pi/2, -3\pi/2], [-9\pi/2, -7\pi/2], \dots, \\ -\cos x, & \text{if } x \text{ is NOT IN ANY ONE} \\ & \text{of the above intervals.} \end{cases}$$

- Feed all this information back into the original function to get it "in pieces"
Taking the square root of all the cosine terms above we get

$$\sqrt{|\cos x|} = \begin{cases} \sqrt{\cos x}, & \text{if } x \text{ is in } [-\pi/2, +\pi/2], [3\pi/2, +5\pi/2], \\ & [7\pi/2, +9\pi/2], \dots, \text{ or if } x \text{ is in} \\ & [-5\pi/2, -3\pi/2], [-9\pi/2, -7\pi/2], \dots, \\ \sqrt{-\cos x}, & \text{if } x \text{ is NOT IN ANY ONE} \\ & \text{of the above intervals.} \end{cases}$$

Phew, that's it! Look at Fig. 8 to see what this function looks like.

You shouldn't worry about the minus sign appearing inside the square root sign above because, inside those intervals, the cosine is negative, so the negative of the cosine is positive, and so we can take its square root without any problem! Try to understand this example completely; then you'll be on your way to mastering one of the most useful concepts in Calculus, handling absolute values!

EXAMPLES



SNAPSHOTS

Example 22. The natural domain of the function h defined by $h(x) = \sqrt{x^2 - 9}$ is the set of all real numbers x such that $x^2 - 9 \geq 0$ or, equivalently, $|x| \geq 3$ (See Table 15.1 and Figure 6).

Example 23. The function f defined by $f(x) = x^{2/3} - 9$ has its natural domain given by the set of all real numbers, $(-\infty, \infty)$! No exceptions! All of them...why?

Solution Look at Table 15.1 and notice that, by algebra, $x^{2/3} = (\sqrt[3]{x})^2$. Since the natural domain of the “cube root” function is $(-\infty, \infty)$, the same is true of its “square”. Subtracting “9” doesn’t change the domain, that’s all!

Example 24. Find the natural domain of the the function f defined by

$$f(x) = \frac{x}{\sin x \cos x}.$$

Solution The natural domain of f is given by the set of all real numbers with the property that $\sin x \cos x \neq 0$, (cf., Table 15.1), that is, the set of all real numbers x with $x \neq \pm \pi/2, \pm 3\pi/2, \pm 5\pi/2, \pm 7\pi/2, \dots, 0, \pm \pi, \pm 2\pi, \pm 3\pi, \pm 4\pi, \dots$ (as these are the values where the denominator is zero).

Example 25. The natural domain of the function f given by

$$f(x) = \frac{|\sin x|}{\sqrt{1-x^2}}$$

is given by the set of all real numbers x with the property that $1 - x^2 > 0$ or, equivalently, the open interval $|x| < 1$, or, $-1 < x < +1$ (See Equations 1.1 and 1.2).

Example 26. Find the natural domain of the function f defined by $f(x) = \sqrt{|\sin x|}$

Solution The natural domain is given by the set of all real numbers x in the interval $(-\infty, \infty)$, that’s right, *all* real numbers! Looks weird right, because of the square root business! But the absolute value will turn any negative number inside the root into a positive one (or zero), so the square root is always defined, and, as a consequence, f is defined everywhere too.

List of Important Trigonometric Identities

Recall that an **identity** is an equation which is true for any value of the variable for which the expressions are defined. So, this means that the identities are true regardless of whether or not the variable looks like an $x, y, \square, \diamond, f(x)$, etc.

YOU’VE GOT TO KNOW THESE!

Odd-even identities

$$\begin{aligned}\sin(-x) &= -\sin x, \\ \cos(-x) &= \cos x.\end{aligned}$$

Pythagorean identities

$$\begin{aligned}\sin^2 x + \cos^2 x &= 1, \\ \sec^2 x - \tan^2 x &= 1, \\ \csc^2 x - \cot^2 x &= 1,\end{aligned}$$

Sofya Kovalevskaya (1850 - 1891)

Celebrated immortal mathematician, writer, and revolutionary, she was appointed Professor of Mathematics at the University of Stockholm, in Sweden, in 1889, the first woman ever to be so honored in her time and the second such in Europe (Sophie Germain, being the other one). At one point, she was apparently courted by Alfred Nobel and brothers (of Nobel Prize fame). Author of slightly more than 10 mathematical papers, she proved what is now known as the **Cauchy-Kovalevski Theorem**, one of the first deep results in a field called *Partial Differential Equations*, an area used in the study of airplane wings, satellite motion, wavefronts, fluid flow, among many other applications. A. L. Cauchy’s name appears because he had proved a more basic version of this result earlier.



Addition identities

$$\sin(x + y) = \sin x \cos y + \cos x \sin y,$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y.$$

Double-angle identities

$$\sin(2x) = 2 \sin x \cos x,$$

$$\cos(2x) = \cos^2 x - \sin^2 x,$$

$$\sin^2 x = \frac{1 - \cos(2x)}{2},$$

$$\cos^2 x = \frac{1 + \cos(2x)}{2}.$$

You can derive the identities below from the ones above, or ... you'll have to memorize them! Well, it's best if you know how to get to them from the ones above using some basic algebra.

$$\tan(-x) = -\tan(x)$$

$$\sin x + \sin y = 2 \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$$

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$

$$\cos x + \cos y = 2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$

$$\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$$

$$\tan\left(\frac{\pi}{2} - x\right) = \cot x$$

$$\sin x \sin y = -\frac{1}{2}[\cos(x+y) - \cos(x-y)]$$

$$\cos(2x) = 1 - 2 \sin^2 x$$

$$\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$$

$$\cos(2x) = 2 \cos^2 x - 1$$

$$\sin \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{2}}$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\cos \frac{x}{2} = \pm \sqrt{\frac{1 + \cos x}{2}}$$



NOTES:

Exercise Set 2.

Use the method of Example 18, Example 20, Example 21 and the discussion following Definition 2 to **remove the absolute value** appearing in the values of the functions defined below. Note that, once the absolute value is removed, the function will be defined *in pieces*.

1. $f(x) = |x^2 - 1|$, for $-\infty < x < +\infty$.

2. $g(x) = |3x + 4|$, for $-\infty < x < +\infty$.

Hint: Put the symbols $3x + 4$ in a box and use the idea in Example 18

3. $h(x) = x|x|$, for $-\infty < x < +\infty$.

4. $f(t) = 1 - |t|$, for $-\infty < t < +\infty$.

5. $g(w) = |\sin w|$, for $-\infty < w < +\infty$.

Hint: $\sin w \geq 0$ when w is in Quadrants I and II, or, equivalently, when w is between 0 and π radians, 2π and 3π radians, etc.

6.

$$f(x) = \frac{1}{|x|\sqrt{x^2 - 1}}$$

for $|x| > 1$.

7. The *signum* function, whose name is simply *sgn* (and pronounced *the sign of x*) where

$$\text{sgn}(x) = \frac{x}{|x|}$$

for $x \neq 0$. The motivation for the name comes from the fact that the values of this function correspond to the *sign* of x (whether it is positive or negative).

8. $f(x) = x + |x|$, for $-\infty < x < +\infty$.

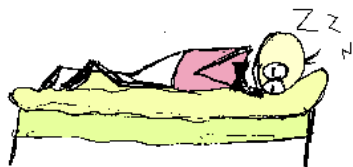
9. $f(x) = x - \sqrt{x^2}$, for $-\infty < x < +\infty$.



Suggested Homework Set 2. Do problems 2, 4, 6, 8, 10, 17, 23, above.

1. $\sin(A + B) = \sin A \cos(B) + \cos A \sin B$
2. $\cos(A + B) = \cos A \cos(B) - \sin A \sin B$
3. $\sin^2 x + \cos^2 x = 1$
4. $\sec^2 x - \tan^2 x = 1$
5. $\csc^2 x - \cot^2 x = 1$
6. $\cos 2x = \cos^2 x - \sin^2 x$
7. $\sin 2x = 2 \sin x \cos x$
8. $\cos^2 x = \frac{1 + \cos 2x}{2}$
9. $\sin^2 x = \frac{1 - \cos 2x}{2}$

Table 1.1: Useful Trigonometric Identities



WHAT'S WRONG WITH THIS ??

$$\begin{aligned}
 -1 &= -1 \\
 \sqrt{\frac{1}{-1}} &= \sqrt{\frac{-1}{1}} \\
 \frac{\sqrt{1}}{\sqrt{-1}} &= \frac{\sqrt{-1}}{\sqrt{1}} \\
 \sqrt{1} \sqrt{1} &= \sqrt{-1} \sqrt{-1} \\
 (\sqrt{1})^2 &= (\sqrt{-1})^2 \\
 1 &= -1
 \end{aligned}$$

Crazy, right?

A Basic Inequality	
If	$0 < \square \leq \triangle,$
then	$\frac{1}{\square} \geq \frac{1}{\triangle},$
regardless of the meaning of the box or the triangle or what's in them!	
OR	
You reverse the inequality when you take reciprocals !	

Table 1.2: Reciprocal Inequalities Among Positive Quantities

Inequalities among reciprocals	
If	$0 < \frac{1}{\square} \leq \frac{1}{\triangle},$
then	$\square \geq \triangle,$
regardless of the meaning of the box or the triangle or what's in them!	
OR	
You <u>still</u> reverse the inequality when you take reciprocals !	

Table 1.3: Another Reciprocal Inequality Among Positive Quantities

1.4 A Quick Review of Inequalities

In this section we will review basic inequalities because they are really important in Calculus. Knowing how to manipulate basic inequalities will come in handy when we look at how graphs of functions are sketched, in our examination of the monotonicity of functions, their convexity and many other areas. We leave the subject of reviewing the solution of basic and polynomial inequalities to Chapter 5. So, this is one section you should know well!

In this section, as in previous ones, we make heavy use of the generic symbols \square and \triangle , that is, our box and triangle. Just remember that variables *don't have to be* called x , and any other symbol will do as well.

Recall that the **reciprocal** of a number is simply the number 1 divided by the number. Table 1.2 shows what happens when you **take the reciprocal** of each term in an inequality involving two **positive** terms. You see, you need to **reverse the sign!** The same result is true had we started out with an inequality among reciprocals of positive quantities, see Table 1.3.

The results mentioned in these tables are really useful! For example,

Example 27.Show that given any number $x \neq 0$,

$$\frac{1}{x^2} > \frac{1}{x^2 + 1}.$$

Solution We know that $0 < x^2 < x^2 + 1$, and this is true regardless of the value of x , so long as $x \neq 0$, which we have assumed anyhow. So, if we put x^2 in the box in Table 1.2 and (make the triangle big enough so that we can) put $x^2 + 1$ in the triangle, then we'll find, as a conclusion, that

$$\frac{1}{x^2} > \frac{1}{x^2 + 1},$$

and this is true for any value of $x \neq 0$, whether x be positive or negative.

Example 28.Solve the inequality $|2x - 1| < 3$ for x .

Solution Recall that $|\square| < \triangle$ is equivalent to $-\triangle < \square < \triangle$ for any two symbols (which we denote by \square and \triangle). In this case, putting $2x - 1$ in the box and 3 in the triangle, we see that we are looking for x 's such that $-3 < 2x - 1 < 3$. Adding 1 to all the terms gives $-2 < 2x < 4$. Finally, dividing by 2 right across the inequality we get $-1 < x < 2$ and this is our answer.

Example 29.

Solve the inequality

$$\left| \frac{x+1}{2x+3} \right| < 2$$

for x .

Solution Once again, by definition of the absolute value, this means we are looking for x 's such that

$$-2 < \frac{x+1}{2x+3} < 2.$$

There now two main cases: **Case 1** where $2x + 3 > 0$ OR **Case 2** where $2x + 3 < 0$. Of course, when $2x + 3 = 0$, the fraction is undefined (actually infinite) and so this is not a solution of our inequality. We consider the cases in turn:

Case 1: $2x + 3 > 0$ In this case we multiply the last display throughout by $2x + 3$ and keep the inequalities as they are (by the rules in Figure 9). In other words, we must now have

$$-2(2x + 3) < x + 1 < 2(2x + 3).$$

Grouping all the x 's in the "middle" and all the constants "on the ends" we find the two inequalities $-4x - 6 < x + 1$ and $x + 1 < 4x + 6$. Solving for x in both instances we get $5x > -7$ or $x > -\frac{7}{5}$ and $3x > -5$ or $x > -\frac{5}{3}$. Now what?

Well, let's recapitulate. We have shown that if $2x + 3 > 0$ then we must have $x > -\frac{7}{5} = -1.4$ and $x > -\frac{5}{3} \approx -1.667$. On the one hand if $x > -1.4$ then $x > -1.667$ for sure and so the two inequalities together imply that $x > -1.4$, or $x > -\frac{7}{5}$. But this is not all! You see, we still need to guarantee that $2x + 3 > 0$ (by the main assumption of this case)! That is, we need to make sure that we have BOTH $2x + 3 > 0$ AND $x > -\frac{7}{5}$, i.e., we need $x > -\frac{3}{2}$ and $x > -\frac{7}{5}$. These last two inequalities together imply that

$$x > -\frac{7}{5}.$$

Case 2: $2x + 3 < 0$ In this case we still multiply the last display throughout by $2x + 3$ but now we must REVERSE the inequalities (by the rules in Figure 9, since

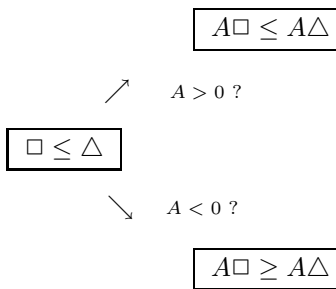


Figure 9.



now $A = 2x + 3 < 0$). In other words, we must now have

$$-2(2x + 3) > x + 1 > 2(2x + 3).$$

As before we can derive the two inequalities $-4x - 6 > x + 1$ and $x + 1 > 4x + 6$. Solving for x in both instances we get $5x < -7$ or $x < -\frac{7}{5}$ and $3x < -5$ or $x < -\frac{5}{3}$. But now, these two inequalities together imply that $x < -\frac{5}{3} \approx -1.667$. This result, combined with the basic assumption that $2x + 3 < 0$ or $x < -\frac{3}{2} = -1.5$, gives us that $x < -\frac{5}{3}$ (since $-1.667 < -1.5$). The solution in this case is therefore given by the inequality

$$x < -\frac{5}{3}.$$



Combining the two cases we get the final solution as (see the margin)

$$-1.4 = -\frac{7}{5} < x \quad \text{OR} \quad x < -\frac{5}{3} \approx -1.667.$$

In terms of intervals the answer is the set of points x such that

$$-\infty < x < -\frac{5}{3} \quad \text{OR} \quad -\frac{7}{5} < x < \infty.$$

1.4.1 The triangle inequalities

Let x, y, a, \square be any real numbers with $a \geq 0$. Recall that the statement

$$|\square| \leq a \quad \text{is equivalent to} \quad -a \leq \square \leq a \tag{1.4}$$

where the symbols \square, a may represent actual numbers, variables, function values, etc. Replacing a here by $|x|$ and \square by x we get, by (1.4),

$$-|x| \leq x \leq |x|. \tag{1.5}$$

We get a similar statement for y , that is,

$$-|y| \leq y \leq |y|. \tag{1.6}$$

Since we can add inequalities together we can combine (1.5) and (1.6) to find

$$-|x| - |y| \leq x + y \leq |x| + |y|, \tag{1.7}$$

or equivalently

$$-(|x| + |y|) \leq x + y \leq |x| + |y|. \tag{1.8}$$

Now replace $x + y$ by \square and $|x| + |y|$ by a in (1.8) and apply (1.4). Then (1.8) is equivalent to the statement

$$|x + y| \leq |x| + |y|, \tag{1.9}$$

for any two real numbers x, y . This is called the **Triangle Inequality**.

The second triangle inequality is just as important as it provides a lower bound on the absolute value of a sum of two numbers. To get this we replace x by $x - y$ in (1.9) and re-arrange terms to find

$$|x - y| \geq |x| - |y|.$$

There are 2 other really important inequalities called the **Triangle Inequalities**: If \square, \triangle are any 2 symbols representing real numbers, functions, etc. then

$$|\square + \triangle| \leq |\square| + |\triangle|,$$

and

$$|\square - \triangle| \geq ||\square| - |\triangle||.$$

Similarly, replacing y in (1.9) by $y - x$ we obtain

$$|y - x| \geq |y| - |x| = -(|x| - |y|).$$

But $|x - y| = |y - x|$. So, combining the last two displays gives us

$$|x - y| \geq \pm(|x| - |y|),$$

and this statement is equivalent to the statement

$$|x| - |y| \leq |x - y|. \quad (1.10)$$

We may call this the **second triangle inequality**.

Example 30.

Show that if x is any number, $x \geq 1$, then

$$\frac{1}{\sqrt{x}} \geq \frac{1}{|x|}.$$

Solution If $x = 1$ the result is clear. Now, everyone believes that, if $x > 1$, then $x < x^2$. OK, well, we can take the square root of both sides and use Figure 6 to get $\sqrt{x} < \sqrt{x^2} = |x|$. From this we get,

$$\text{If } x > 1, \quad \frac{1}{\sqrt{x}} > \frac{1}{|x|}.$$

On the other hand, one has to be careful with the opposite inequality $x > x^2$ if $x < 1$... This is *true*, even though it doesn't seem right!

Example 31.

Show that if x is any number, $0 < x \leq 1$, then

$$\frac{1}{\sqrt{x}} \leq \frac{1}{x}.$$

Solution Once again, if $x = 1$ the result is clear. Using Figure 6 again, we now find that if $x < 1$, then $\sqrt{x} > \sqrt{x^2} = |x|$, and so,

$$\text{If } 0 < x < 1, \quad \frac{1}{\sqrt{x}} < \frac{1}{|x|} = \frac{1}{x}.$$

These inequalities can allow us to estimate how big or how small functions can be!

Example 32.

We know that $\square < \square + 1$ for any \square representing a positive number. The box can even be a function! In other words, we can put a function of x inside the box, apply the reciprocal inequality of Table 1.2, (where we put the symbols $\square + 1$ inside the triangle) and get a new inequality, as follows. Since $\square < \square + 1$, then

$$\frac{1}{\square} > \frac{1}{\square + 1}.$$

Now, put the function f defined by $f(x) = x^2 + 3x^4 + |x| + 1$ inside the box. Note that $f(x) > 0$ (this is really important!). It follows that, for example,

$$\frac{1}{x^2 + 3x^4 + |x| + 1} > \frac{1}{x^2 + 3x^4 + |x| + 1 + 1} = \frac{1}{x^2 + 3x^4 + |x| + 2}.$$

Multiplying inequalities by an unknown quantity

- If $A > 0$, is any symbol (variable, function, number, fraction, ...) and

$$\square \leq \Delta,$$

then

$$A\square \leq A\Delta,$$

- If $A < 0$, is any symbol (variable, function, number, fraction, ...) and

$$\square \leq \Delta,$$

then

$$A\square \geq A\Delta,$$

Don't forget to **reverse the inequality sign when $A < 0$!**

Table 1.4: Multiplying Inequalities Together

Example 33. How “big” is the function f defined by $f(x) = x^2 + \cos x$ if x is in the interval $[0, 1]$?

Solution The best way to figure out how big f is, is to try and estimate each term which makes it up. Let's leave x^2 alone for the time being and concentrate on the $\cos x$ term. We know from trigonometry that $|\cos x| \leq 1$ for any value of x . OK, since $\pm \cos x \leq |\cos x|$ by definition of the absolute value, and $|\cos x| \leq 1$ it follows that

$$\pm \cos x \leq 1$$

for any value of x . Choosing the plus sign, because that's what we want, we add x^2 to both sides and this gives

$$f(x) = x^2 + \cos x \leq x^2 + 1$$

and this is true for any value of x . But we're only given that x is between 0 and 1. So, we take the right-most term, the $x^2 + 1$, and *replace it by something “larger”*. The simplest way to do this is to notice that, since $x \leq 1$ (then $x^2 \leq 1$ too) and $x^2 + 1 \leq 1^2 + 1 = 2$. Okay, now we combine the inequalities above to find that, if $0 \leq x \leq 1$,

$$f(x) = x^2 + \cos x \leq x^2 + 1 \leq 2.$$

This shows that $f(x) \leq 2$ for such x 's and yet we never had to calculate the range of f to get this ... We just used inequalities! You can see this too by plotting its graph as in Figure 10.

NOTE: We have just shown that the so-called **maximum value of the function f over the interval $[0, 1]$** denoted mathematically by the symbols

$$\max_{x \text{ in } [0,1]} f(x)$$

is not greater than 2, that is,

$$\max_{x \text{ in } [0,1]} f(x) \leq 2.$$

For a ‘flowchart interpretation’ of Table 1.4 see Figure 9 in the margin.

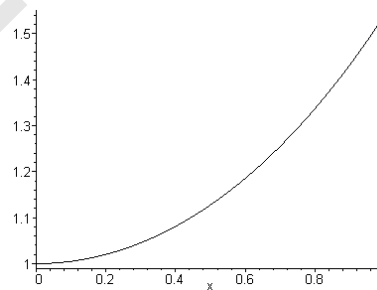
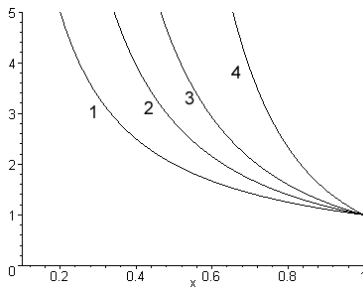


Figure 10.



1. The graph of $1/x$
2. The graph of $1/x^{1.5}$
3. The graph of $1/x^{2.1}$
4. The graph of $1/x^{3.8}$

Figure 11.

Example 34.

Show that if x is any real number, then $-x^3 \geq -x^2(x+1)$.

Solution We know that, for any value of x , $x < x+1$ so, by Table 1.4, with $A = -2$ we find that $-2x > -2(x+1)$. You see that we reversed the inequality since we multiplied the original inequality by a negative number! We could also have multiplied the original inequality by $A = -x^2 \leq 0$, in which case we find, $-x^3 \geq -x^2(x+1)$ for any value of x , as being true too.

Example 35.

Show that if $p \geq 1$, and $x \geq 1$, then

$$\frac{1}{x^p} \leq \frac{1}{x}.$$

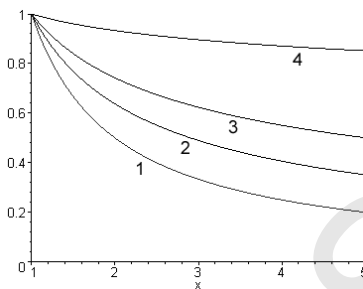
Solution Let $p \geq 1$ be any number, (e.g. $p = 1.657, p = 2, \dots$). Then you'll believe that if $p-1 \geq 0$ and if $x \geq 1$ then $x^{p-1} \geq 1$ (for example, if $x = 2$ and $p = 1.5$, then $2^{1.5-1} = 2^{0.5} = 2^{1/2} = \sqrt{2} = 1.414\dots > 1$). Since $x^{p-1} \geq 1$ we can multiply both sides of this inequality by $x > 1$, which is positive, and find, by Figure 9 with $A = x$, that $x^p \geq x$. From this and Table 1.2 we obtain the result

$$\frac{1}{x^p} \leq \frac{1}{x}, \quad \text{if } p \geq 1, \text{ and } x \geq 1$$

Example 36.

Show that if $p > 1$, and $0 < x \leq 1$, then

$$\frac{1}{x^p} \geq \frac{1}{x}.$$



1. The graph of $1/x$
2. The graph of $1/x^{0.65}$
3. The graph of $1/x^{0.43}$
4. The graph of $1/x^{0.1}$

Figure 12.

Solution In this example we change the preceding example slightly by requiring that $0 < x \leq 1$. In this case we get the opposite inequality, namely, if $p > 1$ then $x^{p-1} \leq 1$ (e.g., if $x = 1/2, p = 1.5$, then $(1/2)^{1.5-1} = (1/2)^{0.5} = (1/2)^{1/2} = 1/\sqrt{2} = 0.707\dots < 1$). Since $x^{p-1} \leq 1$ we can multiply both sides of this inequality by $x > 0$, and find, by Figure 9 with $A = x$, again, that $x^p \leq x$. From this and Table 1.2 we obtain the result (see Fig. 11)

$$\frac{1}{x^p} \geq \frac{1}{x}, \quad \text{if } p \geq 1, \text{ and } 0 < x \leq 1$$

Example 37.

Show that if $0 < p \leq 1$, and $x \geq 1$, then

$$\frac{1}{x^p} \geq \frac{1}{x}.$$

Solution In this final example of this type we look at what happens if we change the p values of the preceding two examples and keep the x -values larger than 1. Okay, let's say that $0 < p \leq 1$ and $x \geq 1$. Then you'll believe that, since $p-1 \leq 0$, $0 < x^{p-1} \leq 1$, (try $x = 2, p = 1/2$, say). Multiplying both sides by x and taking reciprocals we get the important inequality (see Fig. 12),

$$\frac{1}{x^p} \geq \frac{1}{x}, \quad \text{if } 0 < p \leq 1, \text{ and } x \geq 1 \quad (1.11)$$

NOTES:

SNAPSHOTS

Example 38. We know that $\sin x \geq 0$ in Quadrants I and II, by trigonometry. Combining this with Equation (1.11), we find that: If x is an angle expressed in radians and $1 \leq x \leq \pi$ then

$$\frac{\sin x}{x^p} \geq \frac{\sin x}{x}, \quad 0 < p \leq 1$$

Think about why we had to have some restriction like $x \leq \pi$ here.

Example 39. On the other hand, $\cos x \leq 0$ if $\pi/2 \leq x \leq \pi$ (notice that $x > 1$ automatically in this case, since $\pi/2 = 1.57... > 1$). So this, combined with Equation (1.11), gives

$$\frac{\cos x}{x^p} \leq \frac{\cos x}{x} \quad 0 < p \leq 1$$

where we had to “reverse” the inequality (1.11) as $\cos x \leq 0$.

Example 40. There is this really cool (and old) inequality called the AG-inequality, (meaning the Arithmetic-Geometric Inequality). It is an inequality between the “arithmetic mean” of two positive numbers, \square and \triangle , and their “geometric mean”. By definition, the **arithmetic mean** of \square and \triangle , is $(\square + \triangle)/2$, or more simply, their “average”. The **geometric mean** of \square and \triangle , is, by definition, $\sqrt{\square \triangle}$. The inequality states that if $\square \geq 0$, $\triangle \geq 0$, then

$$\frac{\square + \triangle}{2} \geq \sqrt{\square \triangle}$$

Do you see why this is true? Just start out with the inequality $(\sqrt{\square} - \sqrt{\triangle})^2 \geq 0$, expand the terms, rearrange them, and then divide by 2.

Example 41. For example, if we set x^2 in the box and x^4 in the triangle and apply the AG-inequality (Example 40) to these two positive numbers we get the “new” inequality

$$\frac{x^2 + x^4}{2} \geq x^3$$

valid for any value of x , something that is not easy to see if we don’t use the AG-inequality.

We recall the **general form of the Binomial Theorem**. It states that if n is any positive integer, and \square is any symbol (a function, the variable x , or a positive number, or negative, or even zero) then

$$(1 + \square)^n = 1 + n\square + \frac{n(n-1)}{2!}\square^2 + \frac{n(n-1)(n-2)}{3!}\square^3 + \dots + \frac{n(n-1)\dots(2)(1)}{n!}\square^n \tag{1.12}$$

where the symbols that look like $3!$, or $n!$, called **factorials**, mean that we multiply all the integers from 1 to n together. For example, $2! = (1)(2) = 2$, $3! = (1)(2)(3) = 6$, $4! = (1)(2)(3)(4) = 24$ and, generally, “ n factorial” is defined by

$$n! = (1)(2)(3)\dots(n-1)(n) \tag{1.13}$$

When $n = 0$ we all agree that $0! = 1$. . . Don’t worry about why this is true right now, but it has something to do with a function called the **Gamma Function**, which will be defined later when we study things called **improper integrals**. Using this we can arrive at identities like:

$$(1 - x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 \pm \dots + (-1)^n \frac{n(n-1)\dots(2)(1)}{n!}x^n,$$

Remember, if

$$\square^2 \leq \triangle^2,$$

then

$$|\square| \leq |\triangle|,$$

regardless of the values of the variables or symbols involved. If $\square > 0$, this is also true for positive powers, p , other than 2. So, if, for example, $\square > 0$, and

$$\square^p \leq \triangle^p,$$

then

$$|\square| \leq |\triangle|.$$

This result is not true if $\square < 0$ since, for example, $(-2)^3 < 1^3$ but $|-2| > |1|$.

Figure 13.



obtained by setting $\square = -x$ in (1.12) or even

$$2^n = 1 + n + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} + \cdots + \frac{n(n-1)\cdots(2)(1)}{n!}$$

(just let $\square = 1$ in the Binomial Theorem), and finally,

$$(2x-3)^n = (-1)^n 3^n \left(1 - \frac{2nx}{3} + \frac{4n(n-1)x^2}{9 \cdot 2!} + \cdots + (-1)^n \frac{2^n x^n}{3^n} \right),$$

found by noting that

$$(2x-3)^n = \left(-3 \left(1 - \frac{2x}{3} \right) \right)^n = (-3)^n \left(1 - \frac{2x}{3} \right)^n = (-1)^n 3^n \left(1 - \frac{2x}{3} \right)^n$$

and then using the boxed formula (1.12) above with $\square = -\frac{2x}{3}$. In the above formulae note that

$$\frac{n(n-1)\cdots(2)(1)}{n!} = 1$$

by definition of the factorial symbol.

If you already know something about improper integrals then the Gamma Function can be written as,

$$\Gamma(p) = \int_0^{\infty} x^{p-1} e^{-x} dx$$

where $p \geq 1$. One can actually prove that $\Gamma(n+1) = n!$ if n is a positive integer. The study of this function dates back to Euler.

Exercise Set 3.

Determine which of the following 7 statements is true, if any. If the statement is false give an example that shows this. Give reasons either way.

1. If $-A < B$ then $-\frac{1}{A} > \frac{1}{B}$
2. If $-\frac{1}{A} < B$ then $-A > -B$
3. If $A < B$ then $A^2 < B^2$
4. If $A > B$ then $1/A < 1/B$
5. If $A < B$ then $-A < -B$
6. If $A^2 < B^2$ and $A > 0$, then $A < B$
7. If $A^2 > B^2$ and $A > 0$, then $|B| < A$
8. Start with the obvious $\sin x < \sin x + 1$ and find an interval of x 's in which we can conclude that

$$\csc x > \frac{1}{\sin x + 1}$$
9. How big is the function f defined by $f(x) = x^2 + 2 \sin x$ if x is in the interval $[0, 2]$?
10. How big is the function g defined by $g(x) = 1/x$ if x is in the interval $[-1, 4]$?
11. Start with the inequality $x > x - 1$ and conclude that for $x > 1$, we have $x^2 > (x-1)^2$.
12. If x is an angle expressed in radians and $1 \leq x \leq \pi$ show that

$$\frac{\sin x}{x^{p-1}} \geq \sin x, \quad p \leq 1.$$

13. Use the AG-inequality to show that if $x \geq 0$ then $x + x^2 \geq \sqrt{x^3}$. Be careful here, you'll need to use the fact that $2 > 1$! Are you allowed to "square both sides" of this inequality to find that, if $x \geq 0$, $(x + x^2)^2 \geq x^3$? Justify your answer.

14. Can you replace the x 's in the inequality $x^2 \geq 2x - 1$ by an arbitrary symbol, like \square ? Under what conditions on the symbol?
15. Use the ideas surrounding Equation (1.11) to show that, if $p \leq 1$ and $|x| \geq 1$, then

$$\frac{1}{|x|^p} \geq \frac{1}{|x|}$$

Hint: Note that if $|x| \geq 1$ then $|x|^{1-p} \geq 1$.

16. In the theory of relativity developed by A. Einstein, H. Lorentz and others at the turn of this century, there appears the quantity γ , read as “gamma”, defined as a function of the velocity, v , of an object by

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where c is the speed of light. Determine the conditions on v which give us that the quantity γ is a real number. In other words, find the natural domain of γ .

Hint: This involves an inequality and an absolute value.

17. Show that for any integer $n \geq 1$ there holds the inequalities

$$2 \leq \left(1 + \frac{1}{n}\right)^n < 3.$$

Hint: 

This is a really hard problem! Use the Binomial Theorem, (1.5), with $\square = \frac{1}{n}$. But first of all, get a *feel* for this result by using your calculator and setting $n = 2, 3, 4, \dots, 10$ and seeing that this works!

Suggested Homework Set 3. *Work out problems 3, 6, 11, 12, 14*

Web Links

Additional information on Functions may be found at:
<http://www.coolmath.com/func1.htm>

For more on inequalities see:
<http://math.usask.ca/readin/ineq.html>

More on the AG- inequality at:
<http://www.cut-the-knot.com/pythagoras/corollary.html>

Mathematics is not always done by mathematicians. For example, **Giordano Bruno**, 1543-1600, Renaissance Philosopher, once a Dominican monk, was burned at the stake in the year 1600 for heresy. He wrote around 20 books in many of which he upheld the view that Copernicus held, i.e., that of a sun centered solar system (called *heliocentric*). A statue has been erected in his honor in the *Campo dei Fiori*, in Rome. Awesome. The rest is history...What really impresses me about Bruno is his steadfastness in the face of criticism and ultimate torture and execution. It is said that he died without uttering a groan. Few would drive on this narrow road...not even **Galileo Galilei**, 1564 - 1642, physicist, who came shortly after him. On the advice of a Franciscan, Galileo retracted his support for the heliocentric theory when called before the Inquisition. As a result, he stayed under house arrest until his death, in 1642. The theories of Copernicus and Galileo would eventually be absorbed into Newton and Leibniz's Calculus as a consequence of the basic laws of motion. All this falls under the heading of *differential equations*.

1.5 Chapter Exercises

Use the methods of this Chapter to evaluate the following functions at the indicated point(s) or symbol.

1. $f(x) = 3x^2 - 2x + 1$. What is $f(-1)$? $f(0)$? $f(+1)$? $f(-1/2)$?
2. $g(t) = t^3 \cos t$. What is the value of $g(x^2 + 1)$?
3. $h(z) = z + 2 \sin z - \cos(z + 2)$. Evaluate $h(z + 3)$.
4. $f(x) = \cos x$. Find the value of

$$\frac{f(x+h) - f(x)}{h}$$

whenever $h \neq 0$. Simplify this expression as much as you can!

- Use a trigonometric identity for $\cos(A + B)$ with $A = x$, $B = h$.

Solve the following inequalities for the stated variable.

5. $\frac{3}{x} > 6$, x
6. $3x + 4 \geq 0$, x
7. $\frac{3}{2x - 1} \leq 0$, x
8. $x^2 > 5$, x
9. $t^2 < \sqrt{5}$, t
10. $\sin^2 x \leq 1$, x
11. $z^p \geq 2$, z , if $p > 0$
12. $x^2 - 9 \leq 0$, x

Remove the absolute value (see Section 1.3 and Equation 1.3).

13. $f(x) = |x + 3|$
14. $g(t) = |t - 0.5|$
15. $g(t) = |1 - t|$
16. $f(x) = |2x - 1|$
17. $f(x) = |1 - 6x|$
18. $f(x) = |x^2 - 4|$
19. $f(x) = |3 - x^3|$
20. $f(x) = |x^2 - 2x + 1|$
21. $f(x) = |2x - x^2|$
22. $f(x) = |x^2 + 2|$
23. If x is an angle expressed in radians and $1 \leq x \leq \pi/2$ show that

$$\frac{\cos x}{x^{p-1}} \geq \cos x, \quad p \leq 1$$

24. Use your calculator to tabulate the values of the quantity

$$\left(1 + \frac{1}{n}\right)^n$$

for $n = 1, 2, 3, \dots, 10$ (See Exercise 17 of Exercise Set 3). Do the numbers you get seem to be getting close to something?

25. Use the AG-inequality to show that if $0 \leq x \leq \pi/2$, then

$$\frac{\sin x + \cos x}{\sqrt{2}} \geq \sqrt{\sin 2x}.$$

Suggested Homework Set 4. Work out problems 2, 4, 12, 17, 19, 21, 24

1.6 Using Computer Algebra Systems (CAS),

Use your favorite Computer Algebra System (CAS), like Maple, MatLab, etc., or even a graphing calculator to answer the following questions:

Evaluate the functions at the following points:

- $f(x) = \sqrt{x}$, for $x = -2, -1, 0, 1.23, 1.414, 2.7$. What happens when $x < 0$? Conclude that the natural domain of f is $[0, +\infty)$.
- $g(x) = \sin(x\sqrt{2}) + \cos(x\sqrt{3})$, for $x = -4.37, -1.7, 0, 3.1415, 12.154, 16.2$. Are there any values of x for which $g(x)$ is not defined as a real number? Explain.
- $f(t) = \sqrt[3]{t}$, for $t = -2.1, 0, 1.2, -4.1, 9$. Most CAS define power functions only when the base is positive, which is not the case if $t < 0$. In this case the natural domain of f is $(-\infty, +\infty)$ even though the CAS wants us to believe that it is $[0, \infty)$. So, be careful when reading off results using a CAS.
- $g(x) = \frac{x+1}{x-1}$. Evaluate $g(-1), g(0), g(0.125), g(1), g(1.001), g(20), g(1000)$. Determine the behavior of g near $x = 1$. To do this use values of x just less than 1 and then values of x just larger than 1.
- Define a function f by

$$f(t) = \begin{cases} \frac{t + \sqrt{t}}{\sqrt{t}}, & \text{if } t > 0, \\ 1, & \text{if } t = 0. \end{cases}$$

Evaluate $f(1), f(0), f(2.3), f(100.21)$. Show that $f(t) = \sqrt{t} + 1$ for every value of $t \geq 0$.

- Let $f(x) = 1 + 2 \cos^2(\sqrt{x+2}) + 2 \sin^2(\sqrt{x+2})$.
 - Evaluate $f(-2), f(0.12), f(-1.6), f(3.2), f(7)$.
 - Explain your results.
 - What is the natural domain of f ?
 - Can you conclude something simple about this function? Is it a constant function? Why?
- To solve the inequality $|2x - 1| < 3$ use your CAS to
 - Plot the graphs of $y = |2x - 1|$ and $y = 3$ and superimpose them on one another
 - Find their points of intersection, and
 - Solve the inequality (see the figure below)

The answer is: $-1 < x < 2$.

Evaluate the following inequalities graphically using a CAS:

- $|3x - 2| < 5$
- $|2x - 2| < 4.2$
- $|(1.2)x - 3| > 2.61$
- $|1.3 - (2.5)x| = 0.5$
- $|1.5 - (5.14)x| > 2.1$

8. Find an interval of x 's such that

- $|\cos x| < \frac{1}{2}$
- $\sin x + 2 \cos x < 1$
- $\sin(x\sqrt{2}) - \cos x > -\frac{1}{2}$.

Hint: Plot the functions on each side of the inequality separately, superimpose their graphs, estimate their points of intersection visually, and solve the inequality.

9. Plot the values of

$$f(x) = x \sin\left(\frac{1}{x}\right)$$

for small x 's such as $x = 0.1, 0.001, -0.00001, 0.000001, -0.00000001$ etc. Guess what happens to the values of $f(x)$ as x gets closer and closer to zero (regardless of the direction, *i.e.*, regardless of whether $x > 0$ or $x < 0$.)

10. Let $f(x) = 4x - 4x^2$, for $0 \leq x \leq 1$. Use the Box method to evaluate the following terms, called the *iterates* of f for $x = x_0 = 0.5$:

$$f(0.5), f(f(0.5)), f(f(f(0.5))), f(f(f(f(0.5)))) \dots$$

where each term is the image of the preceding term under f . Are these values approaching any specific value? Can you find values of $x = x_0$ (*e.g.*, $x_0 = 0.231, 0.64, \dots$) for which these iterations actually seem to be approaching some specific number? This is an example of a *chaotic sequence* and is part of an exciting area of mathematics called "Chaos".

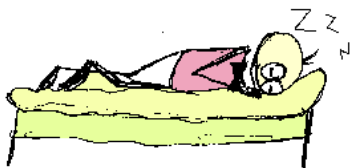
11. Plot the graphs of $y = x^2, (1.2)x^2, 4x^2, (10.6)x^2$ and compare these graphs with those of

$$y = x^{\frac{1}{2}}, (1.2)x^{\frac{1}{2}}, 4x^{\frac{1}{2}}, (10.6)x^{\frac{1}{2}}.$$

Use this graphical information to guess the general shape of graphs of the form $y = x^p$ for $p > 1$ and for $0 < p < 1$. Guess what happens if $p < 0$?

12. Plot the graphs of the family of functions $f(x) = \sin(ax)$ for $a = 1, 10, 20, 40, 50$.

- Estimate the value of those points in the interval $0 \leq x \leq \pi$ where $f(x) = 0$ (these x 's are called "zeros" of f).
- How many are there in each case?
- Now find the position and the number of exact zeros of f inside this interval $0 \leq x \leq \pi$.



NOTES:

Chapter 2

Limits and Continuity

The Big Picture

The notion of a ‘limit’ permeates the universe around us. In the simplest cases, the *speed of light*, denoted by ‘ c ’, in a vacuum is the upper *limit* for the velocities of any object. Photons always travel with speed c but electrons can never reach this speed exactly no matter how much energy they are given. That’s life! In another vein, let’s look at the speed barrier for the 100m dash in Track & Field. World records rebound and are broken in this, the most illustrious of all races. But there *must be a limit* to the time in which one can run the 100m dash, right? For example, it is clear that none will ever run this in a record time of, say, 3.00 seconds! On the other hand, it *has* been run in a record time of 9.79 seconds. So, there must be a *limiting time* that no one will ever be able to reach but the records will get closer and closer to! It is the author’s guess that this limiting time is around 9.70 seconds. In a sense, this time interval of 9.70 seconds between the start of the race and its end, may be considered a *limit* of human locomotion. We just can’t seem to run at a constant speed of $100/9.70 = 10.3$ meters per second. Of course, the actual ‘speed limit’ of any human may sometimes be slightly higher than 10.3 *m/sec*, but, not over the whole race. If you look at the Records Table below, you can see why we *could* consider this number, 9.70, a limit.



0.	Maurice Greene	USA	9.79	99/06/16	Athens, Invitational
1.	Donovan Bailey	CAN	9.84	96/07/26	Atlanta, Olympics
2.	Leroy Burrell	USA	9.85	94/07/06	Lausanne
3.	Carl Lewis	USA	9.86	91/08/25	Tokyo, Worlds
4.	Frank Fredericks	NAM	9.86	96/07/03	Lausanne
5.	Linford Christie	GBR	9.87	93/08/15	Stuttgart, Worlds
6.	Ato Boldon	TRI	9.89	97/05/10	Modesto
7.	Maurice Green	USA	9.90	97/06/13	Indianapolis
8.	Dennis Mitchell	USA	9.91	96/09/07	Milan
9.	Andre Cason	USA	9.92	93/08/15	Stuttgart, Worlds
10.	Tim Montgomery	USA	9.92	97/06/13	Indianapolis

On the other hand, in the world of temperature we have another ‘limit’, namely, something called *absolute zero* equal to $-273^{\circ}C$, or, by definition, $0K$ where K stands for Kelvin. This temperature is a lower *limit* for the temperature of any object under normal conditions. Normally, we may remove as much heat as we want from an object but we’ll never be able to remove all of it, so to speak, so the object will never attain this limiting temperature of $0 K$.

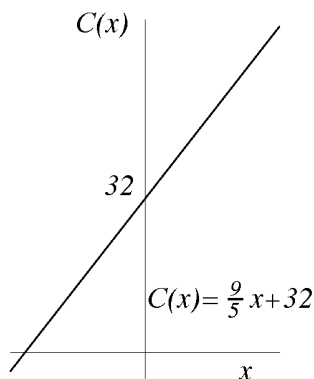


Figure 14.

These are physical examples of limits and long ago some guy called **Karl Weierstrass** (1815-1897), decided he would try to make sense out of all this limit stuff mathematically. So he worked really hard and created this method which we now call the *epsilon-delta* method which most mathematicians today use to prove that such and such a number is, in fact, the limit of some given function. We don't always have to prove it when we're dealing with applications, but if you want to know how to use this method you can look at the *Advanced Topics* later on. Basically, Karl's idea was that you could call some number L a 'limit' of a given function if the values of the function managed to get close, really close, always closer and closer to this number L but never really reach L . He just made this last statement meaningful using symbols.

In many practical situations functions may be given in different formats: that is, their graphs may be unbroken curves or even broken curves. For example, the function C which converts the temperature from degrees Centigrade, x , to degrees Fahrenheit, $C(x)$, is given by the straight line $C(x) = \frac{9}{5}x + 32$ depicted in Fig. 14.

This function's graph is an unbroken curve and we call such graphs the **graph of a continuous function** (as the name describes).

Example 42. If a taxi charges you \$3 as a flat fee for stepping in and 10 cents for every minute travelled, then the graph of the cost $c(t)$, as a function of time t (in minutes), is shown in Fig. 15.

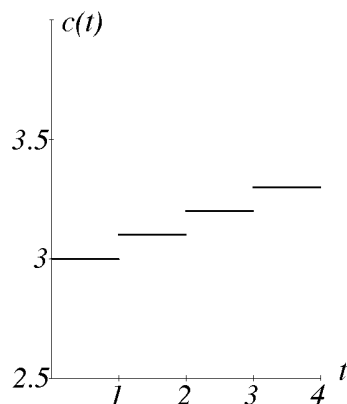


Figure 15.

When written out symbolically this function, c , in Fig. 15 is given by

$$c(t) = \begin{cases} 3, & 0 \leq t < 1 \\ 3.1, & 1 \leq t < 2 \\ 3.2, & 2 \leq t < 3 \\ 3.3, & 3 \leq t < 4 \\ \dots & \dots \end{cases}$$

or, more generally, as:

$$c(t) = 3 + \frac{n}{10}, \text{ if } n \leq t \leq n + 1$$

where $n = 0, 1, 2, \dots$

In this case, the graph of c is a broken curve and this is an example of a **discontinuous function** (because of the 'breaks' it cannot be continuous). It is also called a **step-function** for obvious reasons.

These two examples serve to motivate the notion of continuity. Sometimes functions describing real phenomena are not continuous but we "turn" them into continuous functions as they are easier to visualize graphically.

Example 43. For instance, in Table 2.1 above we show the plot of the frequency of Hard X-rays versus time during a Solar Flare of 6th March, 1989:

The actual X-ray count per centimeter per second is an integer and so the plot should consist of points of the form $(t, c(t))$ where t is in seconds and $c(t)$ is the X-ray count, which is an *integer*. These points are grouped tightly together over small time intervals in the graph and "consecutive" points are joined by a line segment (which is quite short, though). The point is that even though these signals are *discrete* we tend to *interpolate* between these data points by using these small line segments. It's a fair thing to do but is it the *right* thing to do? Maybe nature doesn't like straight lines! The resulting graph of $c(t)$ is now the graph of a **continuous function** (there are no "breaks").

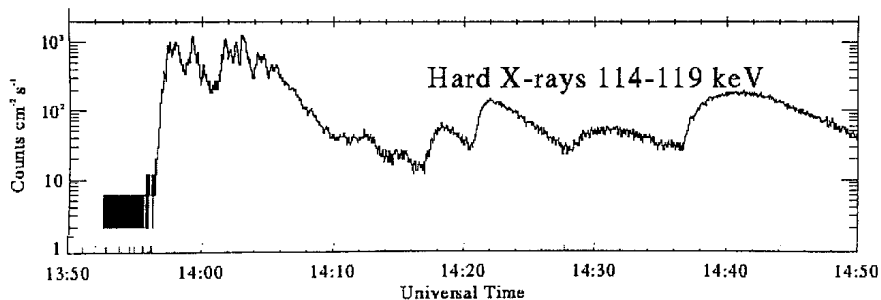


Table 2.1: The Mathematics of Solar Flares

As you can gather from Fig. 15, the size of the **break in the graph of $c(t)$** is given by **subtracting neighbouring values of the function c around $t = a$.**

To make this idea more precise we define the **limit from the left** and the **limit from the right** of function f at the point $x = a$, see Tables 2.2 & 2.3 (and an optional chapter for the rigorous definitions).

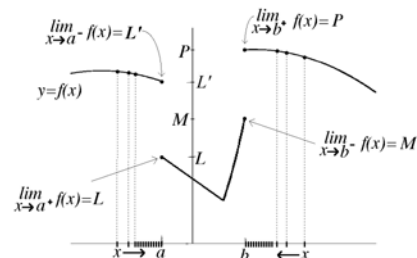


Figure 16.

2.1 One-Sided Limits of Functions

Limits from the Right

We say that the function f has a **limit from the right at $x = a$** (or the right-hand limit of f exists at $x = a$) whose value is L and denote this symbolically by

$$f(a + 0) = \lim_{x \rightarrow a^+} f(x) = L$$

if BOTH of the following statements are satisfied:

1. Let $x > a$ and x be very close to $x = a$.
2. As x approaches a (“from the right” because “ $x > a$ ”), the values of $f(x)$ approach the value L .

(For a more rigorous definition see the **Advanced Topics**, later on.)

Table 2.2: One-Sided Limits From the Right

For example, the function H defined by

$$H(x) = \begin{cases} 1, & \text{for } x \geq 0 \\ 0, & \text{for } x < 0 \end{cases}$$

called the **Heaviside Function** (named after *Oliver Heaviside*, (1850 - 1925) an electrical engineer) has the property that

$$\lim_{x \rightarrow 0^+} H(x) = 1$$

Why? This is because we can set $a = 0$ and $f(x) = H(x)$ in the definition (or in Table 2.2) and apply it as follows:



- a) Let $x > 0$ and x be very close to 0;
 b) As x approaches 0 we need to ask the question: “What are the values, $H(x)$, doing?”

Well, we know that $H(x) = 1$ for **any** $x > 0$, so, as long as $x \neq 0$, the values $H(x) = 1$, (see Fig. 17), so this will be true “in the limit” as x approaches 0.

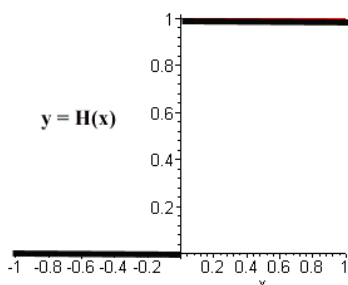
Limits from the Left

We say that the function f has a **limit from the left at $x = a$** (or the left-hand limit of f exists at $x = a$) and is equal to L and denote this symbolically by

$$f(a-0) = \lim_{x \rightarrow a^-} f(x) = L$$

if BOTH of the following statements are satisfied:

1. Let $x < a$ and x be very close to $x = a$.
2. As x approaches a (“from the left” because “ $x < a$ ”), the values of $f(x)$ approach the value L .



The graph of the Heaviside Function, $H(x)$.

Figure 17.

Table 2.3: One-Sided Limits From the Left

Returning to our Heaviside function, $H(x)$, (see Fig. 17), defined earlier we see that

$$\lim_{x \rightarrow 0^-} H(x) = 0$$

Why? In this case we set $a = 0$, $f(x) = H(x)$ in the definition (or Table 2.3), as before:

- a) Let $x < 0$ and x very close to 0;
 b) As x approaches 0 the values $H(x) = 0$, right? (This is because $x < 0$, and by definition, $H(x) = 0$ for such x). The same must be true of the “limit” and so we have

$$\lim_{x \rightarrow 0^-} H(x) = 0$$

OK, but how do you find these limits?

In practice, the idea is to choose some specific values of x near a (to the ‘right’ or to the ‘left’ of a) and, using your calculator, find the corresponding values of the function near a .

Example 44.

Returning to the Taxi problem, Example 42 above, find the values of $c(1+0)$, $c(2-0)$ and $c(4-0)$, (See Fig. 15).

Solution By definition, $c(1+0) = \lim_{t \rightarrow 1^+} c(t)$. But this means that we want the values of $c(t)$ as $t \rightarrow 1$ from the right, *i.e.*, the values of $c(t)$ for $t > 1$ (just slightly bigger than 1) and $t \rightarrow 1$. Referring to Fig. 15 and Example 42 we see that $c(t) = 3.1$ for such t 's and so $c(1+0) = 3.1$. In the same way we see that $c(2-0) = \lim_{t \rightarrow 2^-} c(t) = 3.1$ while $c(4-0) = \lim_{t \rightarrow 4^-} c(t) = 3.3$.

Example 45.

The function f is defined by:

$$f(x) = \begin{cases} x + 1, & x < -1 \\ -2x, & -1 \leq x \leq +1 \\ x^2, & x > +1 \end{cases}$$

Evaluate the following limits whenever they exist and justify your answers.

$$\text{i) } \lim_{x \rightarrow -1^-} f(x); \text{ ii) } \lim_{x \rightarrow 0^+} f(x); \text{ iii) } \lim_{x \rightarrow 1^+} f(x)$$

Solution i) We want a left-hand limit, right? This means that $x < -1$ and x should be very close to -1 (according to the definition in Table 2.3).

Now as x approaches -1 (from the ‘left’, i.e., with x always less than -1) we see that $x + 1$ approaches 0, that is, $f(x)$ approaches 0. Thus,

$$\lim_{x \rightarrow -1^-} f(x) = 0.$$

ii) We want a right-hand limit here. This means that $x > 0$ and x must be very close to 0 (according to the definition in Table 2.2). Now for values of $x > 0$ and close to 0 the value of $f(x)$ is $-2x$... OK, this means that as x approaches 0 then $-2x$ approaches 0, or, equivalently $f(x)$ approaches 0. So

$$\lim_{x \rightarrow 0^+} f(x) = 0$$

iii) In this case we need $x > 1$ and x very close to 1. But for such values, $f(x) = x^2$ and so if we let x approach 1 we see that $f(x)$ approaches $(1)^2 = 1$. So,

$$\lim_{x \rightarrow 1^+} f(x) = 1.$$

See Figure 18, in the margin, where you can ‘see’ the values of our function, f , in the second column of the table while the x ’s which are approaching one (from the right) are in the first column. Note how the numbers in the second column get closer and closer to 1. This table is **not a proof** but it does make the limit we found believable.

Example 46.

Evaluate the following limits and explain your answers.

$$f(x) = \begin{cases} x + 4 & x \leq 3 \\ 6 & x > 3 \end{cases}$$

$$\text{i) } \lim_{x \rightarrow 3^+} f(x) \text{ and ii) } \lim_{x \rightarrow 3^-} f(x)$$

Solution i) We set $x > 3$ and x very close to 3. Then the values are all $f(x) = 6$, by definition, and these don’t change with respect to x . So

$$\lim_{x \rightarrow 3^+} f(x) = 6$$

ii) We set $x < 3$ and x very close to 3. Then the values $f(x) = x + 4$, by definition, and as x approaches 3, we see that $x + 4$ approaches $3 + 4 = 7$. So

$$\lim_{x \rightarrow 3^-} f(x) = 7$$

Numerical evidence for Example 45, (iii). You can think of ‘ x ’ as being the name of an athlete and ‘ $f(x)$ ’ as being their record at running the 100 m. dash.

x	$f(x)$
1.5000	2.2500
1.2500	1.5625
1.1000	1.2100
1.0500	1.1025
1.0033	1.0067
1.0020	1.0040
1.0012	1.0025
1.0010	1.0020
1.0001	1.0002
...	...

Figure 18.

Example 47.

Evaluate the following limits, if they exist.

$$f(x) = \begin{cases} x^2 & x > 0 \\ -x^2 & x \leq 0 \end{cases}$$

$$\text{i) } \lim_{x \rightarrow 0^-} f(x); \text{ ii) } \lim_{x \rightarrow 0^+} f(x)$$

Solution i) Let $x < 0$ and x very close to 0. Since $x < 0$, $f(x) = -x^2$ and $f(x)$ is very close to $-0^2 = 0$ since x is. Thus

$$\lim_{x \rightarrow 0^-} f(x) = 0$$

ii) Let $x > 0$ and x very close to 0. Since $x > 0$, $f(x) = x^2$ and $f(x)$ is very close to 0 too! Thus

$$\lim_{x \rightarrow 0^+} f(x) = 0$$

NOTE: In this example the graph of f has no breaks whatsoever since $f(0) = 0$. In this case we say that the function f is **continuous** at $x = 0$. Had there been a 'break' or some points 'missing' from the graph we would describe f as **discontinuous** whenever those 'breaks' or 'missing points' occurred.

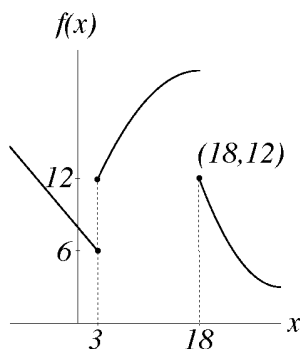


Figure 19.

Example 48.

Use the graph in Figure 19 to determine the value of the required limits.

$$\text{i) } \lim_{x \rightarrow 3^-} f(x); \text{ ii) } \lim_{x \rightarrow 3^+} f(x); \text{ iii) } \lim_{x \rightarrow 18^+} f(x)$$

Solution i) Let $x < 3$ and let x be very close to 3. The point $(x, f(x))$ which is **on** the curve $y = f(x)$ now approaches a definite point as x approaches 3. Which point? The graph indicates that this point is $(3, 6)$. Thus

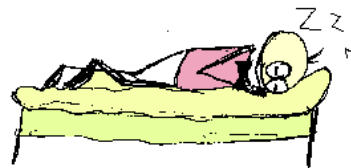
$$\lim_{x \rightarrow 3^-} f(x) = 6$$

ii) Let $x > 3$ and let x be very close to 3. In this case, as x approaches 3, the points $(x, f(x))$ travel down towards the point $(3, 12)$. Thus

$$\lim_{x \rightarrow 3^+} f(x) = 12$$

iii) Here we let $x > 18$ and let x be very close to 18. Now as x approaches 18 the points $(x, f(x))$ on the curve are approaching the point $(18, 12)$. Thus

$$\lim_{x \rightarrow 18^+} f(x) = 12$$



Exercise Set 4.

Evaluate the following limits and justify your conclusions.

1. $\lim_{x \rightarrow 2^+} (x + 2)$

7. $\lim_{x \rightarrow 0^+} x \sin x$

2. $\lim_{x \rightarrow 0^+} (x^2 + 1)$

8. $\lim_{x \rightarrow \pi^+} \left(\frac{\cos x}{x} \right)$

3. $\lim_{x \rightarrow 1^-} (1 - x^2)$

9. $\lim_{x \rightarrow 2^+} \left(\frac{x - 2}{x + 2} \right)$

4. $\lim_{t \rightarrow 2^+} \left(\frac{1}{t - 2} \right)$

10. $\lim_{x \rightarrow 1^-} \frac{x}{|x - 1|}$

5. $\lim_{x \rightarrow 0^+} (x|x|)$

11. $\lim_{x \rightarrow 1^-} \left(\frac{x - 1}{x + 2} \right)$

6. $\lim_{x \rightarrow 0^-} \frac{x}{|x|}$

12. $\lim_{x \rightarrow 3^+} \left(\frac{x - 3}{x^2 - 9} \right)$
(Hint: Factor the denominator)

13. Let the function f be defined as follows:

$$f(x) = \begin{cases} 1 - |x| & x < 1 \\ x & x \geq 1 \end{cases}$$

Evaluate i) $\lim_{x \rightarrow 1^-} f(x)$; ii) $\lim_{x \rightarrow 1^+} f(x)$

Conclude that the graph of $f(x)$ must have a 'break' at $x = 1$.

14. Let g be defined by

$$g(x) = \begin{cases} x^2 + 1 & x < 0 \\ 1 - x^2 & 0 \leq x \leq 1 \\ x & x > 1 \end{cases}$$

Evaluate

i). $\lim_{x \rightarrow 0^-} g(x)$ ii). $\lim_{x \rightarrow 0^+} g(x)$

iii). $\lim_{x \rightarrow 1^-} g(x)$ iv). $\lim_{x \rightarrow 1^+} g(x)$

v) Conclude that the graph of g has no breaks at $x = 0$ but it does have a break at $x = 1$.

15. Use the graph in Figure 20 to determine the value of the required limits. (The function f is composed of parts of 2 functions).

Evaluate

i). $\lim_{x \rightarrow -1^+} f(x)$ ii). $\lim_{x \rightarrow -1^-} f(x)$

iii). $\lim_{x \rightarrow 1^-} f(x)$ iv). $\lim_{x \rightarrow 1^+} f(x)$

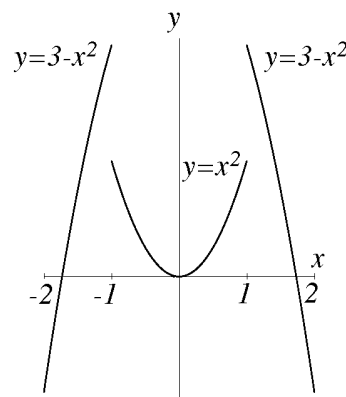


Figure 20.

2.2 Two-Sided Limits and Continuity

At this point we know how to determine the values of the limit from the left (or right) of a given function f at a point $x = a$. We have also seen that whenever

$$\lim_{x \rightarrow a^+} f(x) \neq \lim_{x \rightarrow a^-} f(x)$$

then there is a ‘break’ in the graph of f at $x = a$. The **absence of breaks or holes in the graph** of a function is what the notion of **continuity** is all about.

One of the key mathematical figures during the first millennium was a monk called **Alcuin of York** (735 - 804) who was Charlemagne’s scribe and general advisor. He wrote a very influential book in Latin which contained many mathematical problems passed on from antiquity. In this book of his you’ll find the following (paraphrased) problem:

A dog chases a rabbit who has a head-start of 150 feet. All you know is that every time the dog leaps 9 feet, the rabbit leaps 7 feet. How many leaps will it take for the dog to pass the rabbit?



Definition of the limit of a function at $x = a$.

We say that a function f has the (two-sided) limit L as x approaches a if

$$\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = L$$

When this happens, we write (for brevity)

$$\lim_{x \rightarrow a} f(x) = L$$

and read this as: the limit of $f(x)$ as x approaches a is L (L may be infinite here).

NOTE: So, in order for a limit to exist both the right- and left-hand limits must exist and be equal. Using this notion we can now define the ‘continuity of a function f at a point $x = a$.’

We say that f is **continuous** at $x = a$ if all the following conditions are satisfied:

1. f is defined at $x = a$ (i.e., $f(a)$ is finite)
2. $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) (= L, \text{ their common value})$ and
3. $L = f(a)$.

NOTE: These three conditions must be satisfied in order for a function f to be continuous at a given point $x = a$. If any one or more of these conditions is not satisfied we say that f is **discontinuous** at $x = a$. In other words, we see from the Definition above (or in Table 2.7) that the one-sided limits from the left and right must be equal in order for f to be continuous at $x = a$ but that this equality, in itself, is **not enough to guarantee continuity** as there are 2 other conditions that need to be satisfied as well.

Example 49.

Show that the given function is continuous at the given points, $x = 1$ and $x = 2$, where f is defined by

$$f(x) = \begin{cases} x + 1 & 0 \leq x \leq 1 \\ 2x & 1 < x \leq 2 \\ x^2 & x > 2 \end{cases}$$

Solution To show that f is continuous at $x = 1$ we need to verify 3 conditions (according to Table 2.7):

1. **Is f defined at $x = 1$?** Yes, its value is $f(1) = 1 + 1 = 2$ by definition.
2. **Are the one-sided limits equal?** Let's check this: (See Fig. 21)

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} \underbrace{(x + 1)}_{\text{because } f(x)=x+1 \text{ for } x \leq 1} = 1 + 1 = 2$$

Moreover,

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \underbrace{(2x)}_{\text{because } f(x)=2x \text{ for } x > 1, \text{ and close to } 1} = 2 \cdot 1 = 2$$

The one-sided limits are equal to each other and their common value is $L = 2$.

3. **Is $L = f(1)$?** By definition $f(1) = 1 + 1 = 2$, so OK, this is true, because $L = 2$.

Thus, by definition f is continuous at $x = 1$.

We proceed in the same fashion for $x = 2$. Remember, we still have to verify 3 conditions ...

1. **Is f defined at $x = 2$?** Yes, because its value is $f(2) = 2 \cdot 2 = 4$.
2. **Are the one-sided limits equal?** Let's see:

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} \underbrace{x^2}_{\text{because } f(x)=x^2 \text{ for } x > 2} = 2^2 = 4$$

and

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} \underbrace{(2x)}_{\text{because } f(x)=2x \text{ for } x \leq 2 \text{ and close to } 2} = 2 \cdot 2 = 4$$

So they are both equal and their common value, $L = 4$.

3. **Is $L = f(2)$?** We know that $L = 4$ and $f(2) = 2 \cdot 2 = 4$ by definition so, OK.

Thus, by definition (Table 2.7), f is continuous at $x = 2$.

Remarks:

1. The **existence of the limit of a function f at $x = a$ is equivalent to requiring that both one-sided limits be equal (to each other).**
2. The existence of the limit of a function f at $x = a$ doesn't imply that f is continuous at $x = a$.
Why? Because the value of this limit may be different from $f(a)$, or, worse still, $f(a)$ may be infinite.
3. It follows from (1) that

$$\boxed{\text{If } \lim_{x \rightarrow a^+} f(x) \neq \lim_{x \rightarrow a^-} f(x), \text{ then } \lim_{x \rightarrow a} f(x) \text{ does not exist,}}$$

EXAMPLES

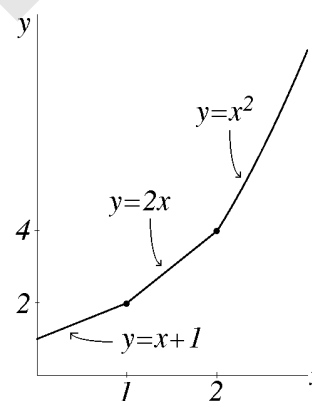
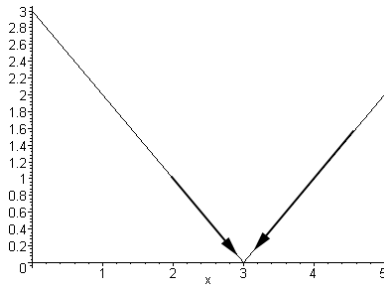


Figure 21.



The function $f(x) = |x - 3|$

Figure 22.

so, in particular, f cannot be continuous at $x = a$.

Example 50. Show that the function f defined by $f(x) = |x - 3|$ is continuous at $x = 3$ (see Figure 22).

Solution By definition of the absolute value we know that

$$f(x) = |x - 3| = \begin{cases} x - 3 & x \geq 3 \\ 3 - x & x < 3 \end{cases}$$

(Remember: $|symbol| = symbol$ if $symbol \geq 0$ and $|symbol| = -symbol$ if $symbol < 0$ where ‘symbol’ is any expression involving some variable. . .) OK, now

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (x - 3) = 3 - 3 = 0$$

and

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (3 - x) = 3 - 3 = 0$$

so $\lim_{x \rightarrow 3} f(x)$ exists and is equal to 0 (by definition).

Is $0 = f(3)$? Yes (because $f(3) = |3 - 3| = |0| = 0$). Of course $f(3)$ is defined. We conclude that f is continuous at $x = 3$.

Remark: In practice it is easier to remember the statement:

$$f \text{ is continuous at } x = a \text{ if } \lim_{x \rightarrow a} f(x) = f(a)$$

whenever all the ‘symbols’ here have meaning (*i.e.* the limit exists, $f(a)$ exists etc.).

Example 51. Determine whether or not the following functions have a limit at the indicated point.

The **Arabic numerals** or those 10 basic symbols we use today in the world of mathematics seem to have been accepted in Europe and subsequently in the West, sometime during the period 1482-1494, as can be evidenced from old merchant records from the era (also known as the *High Renaissance* in art). Prior to this, merchants and others used Roman numerals (X=10, IX=9, III = 3, etc.) in their dealings.

- $f(x) = x^2 + 1$ at $x = 0$
- $f(x) = 1 + |x - 1|$ at $x = 1$
- $f(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$ at $t = 0$
- $f(x) = \frac{1}{x}$ at $x = 0$
- $g(t) = \frac{t}{t + 1}$ at $t = 0$

Solution a)

$$\begin{aligned} \lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} (x^2 + 1) = 0^2 + 1 = 1 \\ \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^-} (x^2 + 1) = 0^2 + 1 = 1 \end{aligned}$$

Thus $\lim_{x \rightarrow 0} f(x)$ exists and is equal to 1. *i.e.* $\lim_{x \rightarrow 0} f(x) = 1$.

$$\begin{aligned}
 \text{b) } \quad \lim_{x \rightarrow 1^+} f(x) &= \lim_{x \rightarrow 1^+} (1 + |x - 1|) \\
 &= \lim_{x \rightarrow 1^+} (1 + (x - 1)) \quad (\text{because } |x - 1| = x - 1 \text{ as } x > 1) \\
 &= \lim_{x \rightarrow 1^+} x \\
 &= 1 \\
 \lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^-} (1 + |x - 1|) \\
 &= \lim_{x \rightarrow 1^-} (1 + (1 - x)) \quad (\text{because } |x - 1| = 1 - x \text{ as } x < 1) \\
 &= \lim_{x \rightarrow 1^-} (2 - x) \\
 &= 2 - 1 \\
 &= 1
 \end{aligned}$$

Thus $\lim_{x \rightarrow 1} f(x)$ exists and is equal to 1, *i.e.* $\lim_{x \rightarrow 1} f(x) = 1$.

$$\begin{aligned}
 \text{c) } \quad \lim_{t \rightarrow 0^+} f(t) &= \lim_{t \rightarrow 0^+} (1) \quad (\text{as } f(t) = 1 \text{ for } t > 0) \\
 &= 1 \\
 \lim_{t \rightarrow 0^-} f(t) &= \lim_{t \rightarrow 0^-} (0) \quad (\text{as } f(t) = 0 \text{ for } t < 0) \\
 &= 0
 \end{aligned}$$

Since $\lim_{t \rightarrow 0^+} f(t) \neq \lim_{t \rightarrow 0^-} f(t)$, it follows that $\lim_{t \rightarrow 0} f(t)$ does not exist. (In particular, f cannot be continuous at $t = 0$.)

$$\text{d) } \quad \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty$$

because “division by zero” does not produce a real number, in general. On the other hand

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty \quad (\text{since } x < 0)$$

Since $\lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x)$ the limit does not exist at $x = 0$.

$$\begin{aligned}
 \text{e) } \quad \lim_{t \rightarrow 0^+} g(t) &= \lim_{t \rightarrow 0^+} \frac{t}{t + 1} = \frac{0}{0 + 1} = 0 \\
 \lim_{t \rightarrow 0^-} g(t) &= \lim_{t \rightarrow 0^-} \frac{t}{t + 1} = \frac{0}{0 + 1} = 0
 \end{aligned}$$

and so $\lim_{t \rightarrow 0} g(t)$ exists and is equal to 0, *i.e.* $\lim_{t \rightarrow 0} g(t) = 0$.

The rigorous method of handling these examples is presented in an optional Chapter, **Advanced Topics**. Use of your calculator will be helpful in determining some limits but cannot substitute a theoretical proof. The reader is encouraged to consult the *Advanced Topics* for more details.

Remark:

It follows from Table 2.4 that continuous functions themselves have similar properties, being based upon the notion of limits. For example it is true that:

1. The sum or difference of two continuous functions (at $x = a$) is again continuous (at $x = a$).



The **Sandwich Theorem** states that, if

$$g(x) \leq f(x) \leq h(x)$$

for all (sufficiently) large x and for some (extended) real number A , and

$$\lim_{x \rightarrow a} g(x) = A, \quad \lim_{x \rightarrow a} h(x) = A$$

then f also has a limit at $x = a$ and

$$\lim_{x \rightarrow a} f(x) = A$$

In other words, f is “sandwiched” between two values that are ultimately the same and so f must also have the same limit.

2. The product or quotient of two continuous functions (at $x = a$) is also continuous (provided the quotient has a non-zero denominator at $x = a$).
3. A multiple of two continuous functions (at $x = a$) is again a continuous function (at $x = a$).

Properties of Limits of Functions

Let f, g be two given functions, $x = a$ be some (finite) point. The following statements hold (but will not be proved here):

Assume $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ both exist and are **finite**.

Then

- a) **The limit of a sum is the sum of the limits.**

$$\lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

- b) **The limit of a difference is the difference of the limits.**

$$\lim_{x \rightarrow a} (f(x) - g(x)) = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$$

- c) **The limit of a multiple is the multiple of the limit.**

$$\text{If } c \text{ is any number, then } \lim_{x \rightarrow a} cf(x) = c \lim_{x \rightarrow a} f(x)$$

- d) **The limit of a quotient is the quotient of the limits.**

$$\text{If } \lim_{x \rightarrow a} g(x) \neq 0 \text{ then } \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$$

- e) **The limit of a product is the product of the limits.**

$$\lim_{x \rightarrow a} f(x)g(x) = \left(\lim_{x \rightarrow a} f(x) \right) \left(\lim_{x \rightarrow a} g(x) \right)$$

- f) **If $f(x) \leq g(x)$ then $\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x)$**



Table 2.4: Properties of Limits of Functions

Now, the Properties in Table 2.4 and the following Remark allow us to make some very important observations about some classes of functions, such as polynomials.

How? Well, let's take the simplest polynomial $f(x) = x$. It is easy to see that for some given number $x = a$ and setting $g(x) = x$, Table 2.4 Property (e), implies that the function $h(x) = f(x)g(x) = x \cdot x = x^2$ is also continuous at $x = a$. In the same way we can show that $k(x) = f(x)h(x) = x \cdot x^2 = x^3$ is also continuous at $x = a$, and so on.

In this way we can prove (using, in addition, Property (a)), that **any polynomial whatsoever is continuous at $x = a$, where a is any real number**. We summarize this and other such consequences in Table 2.8.

SUMMARY: One-Sided Limits from the Right

We say that the function f has a **limit from the right at $x = a$** (or the right-hand limit of f exists at $x = a$) whose value is L and denote this symbolically by

$$\lim_{x \rightarrow a^+} f(x) = L$$

if BOTH the following statements are satisfied:

1. Let $x > a$ and x be very close to $x = a$.
2. As x approaches a (“from the right” because “ $x > a$ ”), the values of $f(x)$ approach the value L .

(For a more rigorous definition see the **Advanced Topics**)

Table 2.5: SUMMARY: One-Sided Limits From the Right

SUMMARY: One-Sided Limits from the Left

We say that the function f has a **limit from the left at $x = a$** (or the left-hand limit of f exists at $x = a$) and is equal to L and denote this symbolically by

$$\lim_{x \rightarrow a^-} f(x) = L$$

if BOTH the following statements are satisfied:

1. Let $x < a$ and x be very close to $x = a$.
2. As x approaches a (“from the left” because “ $x < a$ ”), the values of $f(x)$ approach the value L .

Table 2.6: SUMMARY: One-Sided Limits From the Left

Exercise Set 5.

Determine whether the following limits exist. Give reasons.

1. $\lim_{x \rightarrow 0} f(x)$ where $f(x) = \begin{cases} x + 2 & x \leq 0 \\ x & x > 0 \end{cases}$
2. $\lim_{x \rightarrow 1} (x + 3)$
3. $\lim_{x \rightarrow -2} \left(\frac{x + 2}{x} \right)$
4. $\lim_{x \rightarrow 0} x \sin x$
5. $\lim_{x \rightarrow 1} f(x)$ where $f(x) = \begin{cases} \sin(x - 1) & 0 \leq x \leq 1 \\ |x - 1| & x > 1 \end{cases}$
6. $\lim_{x \rightarrow 0} \left(\frac{x + 1}{x} \right)$
7. $\lim_{x \rightarrow 0} \left(\frac{2}{x} \right)$

Nicola Oresme, (1323-1382), Bishop of Lisieux, in Normandy, wrote a tract in 1360 (this is before the printing press) where, for the first time, we find the introduction of perpendicular xy -axes drawn on a plane. His work is likely to have influenced **René Descartes** (1596-1650), the founder of modern *Analytic Geometry*.

SUMMARY: Continuity of f at $x = a$.

We say that f is **continuous** at $x = a$ if all the following conditions are satisfied:

1. f is defined at $x = a$ (i.e., $f(a)$ is finite)
2. $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) (= L, \text{ their common value})$ and
3. $L = f(a)$.

These three conditions must be satisfied in order for a function f to be continuous at a given point $x = a$. If any one or more of these conditions is not satisfied we say that f is **discontinuous** at $x = a$.

Table 2.7: SUMMARY: Continuity of a Function f at a Point $x = a$

8. $\lim_{x \rightarrow 1} \left(\frac{x}{x+1} \right)$

9. $\lim_{x \rightarrow 2} (2 + |x - 2|)$

10. $\lim_{x \rightarrow 0} f(x)$ where $f(x) = \begin{cases} 3 & x \leq 0 \\ 2 & x > 0 \end{cases}$

11. Are the following functions continuous at 0? Give reasons.

a) $f(x) = |x|$

b) $g(t) = t^2 + 3t + 2$

c) $h(x) = 3 + 2|x|$

d) $f(x) = \frac{2}{x+1}$

e) $f(x) = \frac{x^2 + 1}{x^2 - 2}$

12. **Hard** Let f be defined by

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Show that f is continuous at $x = 0$.*(Hint: Do this in the following steps:*

a) Show that for $x \neq 0$, $|x \sin\left(\frac{1}{x}\right)| \leq |x|$.

b) Use (a) and the Sandwich Theorem to show that

$$0 \leq \lim_{x \rightarrow 0} \left| x \sin\left(\frac{1}{x}\right) \right| \leq 0$$

and so

$$\lim_{x \rightarrow 0} \left| x \sin\left(\frac{1}{x}\right) \right| = 0$$

c) Conclude that $\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right) = 0$.

d) Verify the other conditions of continuity.)



Some Continuous Functions

Let $x = a$ be a given point.

- a) The polynomial p of degree n , with fixed coefficients, given by

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$$

is continuous at any real number $x = a$.

- b) The rational function, r , where $r(x) = \frac{p(x)}{q(x)}$ where p, q are both polynomials is continuous at $x = a$ provided $q(a) \neq 0$ or equivalently, provided $x = a$ is not a root of $q(x)$. Thus

$$r(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \dots + b_0}$$

is continuous at $x = a$ provided the denominator is not equal to zero at $x = a$.

- c) If f is a continuous function, so is its **absolute value** function, $|f|$, and if

$$\lim_{x \rightarrow a} |f(x)| = 0, \quad \text{then}$$

$$\lim_{x \rightarrow a} f(x) = 0$$

(The proof of (c) uses the ideas in the Advanced Topics chapter.)

Table 2.8: Some Continuous Functions

What about discontinuous functions?

In order to show that a function is discontinuous somewhere we need to show that at least one of the three conditions in the definition of continuity (Table 2.7) is not satisfied.

Remember, to show that f is continuous requires the verification of all three conditions in Table 2.7 whereas to show some function is discontinuous only requires that **one** of the three conditions for continuity is not satisfied.

**Example 52.**

Determine which of the following functions are discontinuous somewhere. Give reasons.

a) $f(x) = \begin{cases} x & x \leq 0 \\ 3x + 1 & x > 0 \end{cases}$

b) $f(x) = \frac{x}{|x|}$, $f(0) = 1$

c) $f(x) = \begin{cases} x^2 & x \neq 0 \\ 1 & x = 0 \end{cases}$

d) $f(x) = \frac{1}{|x|}$, $x \neq 0$

Solution a) Note that $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (x) = 0$
 while $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (3x + 1) = 1$

Thus the $\lim_{x \rightarrow 0} f(x)$ does not exist and so f cannot be continuous at $x = 0$, or, equivalently, f is discontinuous at $x = 0$.

How can a function f be discontinuous at $x = a$? If any one (or more) of the following occurs ...

1. $f(a)$ is not defined (e.g., it is infinite, or we are dividing by 0, or extracting the root of a negative number, ...) (See Example 52 (d))
2. If either one of the left- and right-limits of f at $x = a$ is infinite (See Example 52 (d))
3. $f(a)$ is defined but the left- and right-limits at $x = a$ are unequal (See Example 52 (a), (b))
4. $f(a)$ is defined, the left- and right-limits are equal to L but $L = \pm\infty$
5. $f(a)$ is defined, the left- and right-limits are equal to L but $L \neq f(a)$ (See Example 52 (c))

Then f is discontinuous at $x = a$.

What about the other points, $x \neq 0$?

Well, if $x \neq 0$, and $x < 0$, then $f(x) = x$ is a polynomial, right? Thus f is continuous at each point x where $x < 0$. On the other hand, if $x \neq 0$ and $x > 0$ then $f(x) = 3x + 1$ is also a polynomial. Once again f is continuous at each point x where $x > 0$.

Conclusion: f is continuous at every point x except at $x = 0$.

b) Since $f(0) = 1$ is defined, let's check for the existence of the limit at $x = 0$. (You've noticed, of course, that at $x = 0$ the function is of the form $\frac{0}{0}$ **which is not defined as a real number** and this is why an additional condition was added there to make the function defined for **all** x and not just those $x \neq 0$.)

Now,

$$\begin{aligned} \lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} \frac{x}{|x|} = \lim_{x \rightarrow 0^+} \frac{x}{x} = \lim_{x \rightarrow 0^+} (1) \\ &= 1 \end{aligned}$$

(because $x \neq 0$)

$$\begin{aligned} \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^-} \frac{x}{|x|} = \lim_{x \rightarrow 0^-} \frac{x}{(-x)} = \lim_{x \rightarrow 0^-} (-1) \\ &= -1 \end{aligned}$$

(since $|x| = -x$ if $x < 0$ by definition). Since the one-sided limits are different it follows that

$$\lim_{x \rightarrow 0} f(x) \text{ does not exist.}$$

Thus f is discontinuous at $x = 0$.

What about the other points? Well, for $x \neq 0$, f is nice enough. For instance, if $x > 0$ then

$$f(x) = \frac{x}{|x|} = \frac{x}{x} = +1$$

for each such $x > 0$. Since f is a constant it follows that f is continuous for $x > 0$. On the other hand, if $x < 0$, then $|x| = -x$ so that

$$f(x) = \frac{x}{|x|} = \frac{x}{(-x)} = -1$$

and once again f is continuous for such $x < 0$.

Conclusion: f is continuous for each $x \neq 0$ and at $x = 0$, f is discontinuous.

The graph of this function is shown in Figure 23.

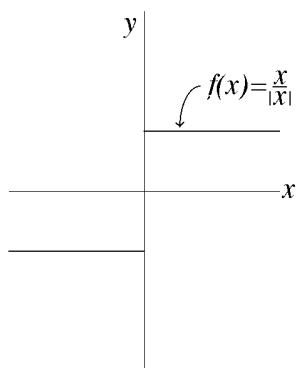


Figure 23.

c) Let's look at f for $x \neq 0$ first, (it doesn't really matter how we start). For $x \neq 0$, $f(x) = x^2$ is a polynomial and so f is continuous for each such $x \neq 0$, (Table 2.8).

What about $x = 0$? We are given that $f(0) = 1$ so f is defined there. What about the limit of f as x approaches $x = 0$. Does this limit exist?

Let's see

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x^2 = 0^2 = 0$$

and

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} x^2 = 0^2 = 0$$

OK, so $\lim_{x \rightarrow 0} f(x)$ exists and is equal to 0. But note that

$$\lim_{x \rightarrow 0} f(x) = 0 \neq f(0) = 1$$

So, in this case, f is discontinuous at $x = 0$, (because even though conditions (1) and (2) of Table 2.7 are satisfied the final condition (3) is not!)

d) In this case we see that

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = +\infty$$

and $f(0) = +\infty$ as well! Ah, but now $f(0)$ is not defined as a real number (thus violating condition (1)). Thus f is discontinuous at $x = 0 \dots$ and the other points? Well, for $x < 0$, $f(x) = -\frac{1}{x}$ is a quotient of two polynomials and any x (since $x \neq 0$) is not a root of the denominator. Thus f is a continuous function for such $x < 0$. A similar argument applies if $x > 0$.

Conclusion: f is continuous everywhere except at $x = 0$ where it is discontinuous.

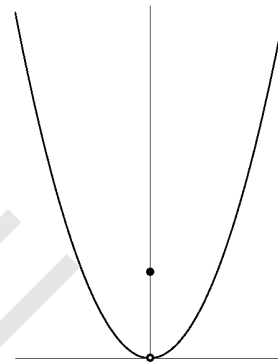
We show the graph of the functions defined in (c), (d) in Figure 24.

Exercise Set 6.

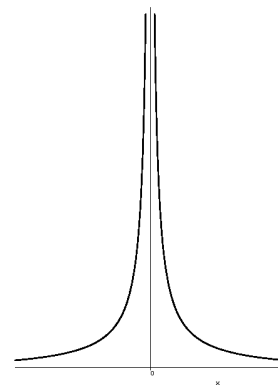
Determine the points of discontinuity of each of the following functions.

1. $f(x) = \frac{|x|}{x} + 1$ for $x \neq 0$ and $f(0) = 2$
2. $g(x) = \begin{cases} x & x < 0 \\ 1 + x^2 & x \geq 0 \end{cases}$
3. $f(x) = \frac{x^2 + 3x + 3}{x^2 - 1}$
(Hint: Find the zeros of the denominator.)
4. $f(x) = \begin{cases} x^3 + 1 & x \neq 0 \\ 2 & x = 0 \end{cases}$
5. $f(x) = \frac{1}{x} + \frac{1}{x^2}$ for $x \neq 0$, $f(0) = +1$
6. $f(x) = \begin{cases} 1.62 & x < 0 \\ 2x & x \geq 0 \end{cases}$

Before proceeding with a study of some trigonometric limits let's recall some fundamental notions about trigonometry. Recall that the measure of angle called the **radian** is equal to $\frac{360^\circ}{2\pi} \approx 57^\circ$. It is also that angle whose arc is numerically equal to the radius of the given circle. (So 2π radians correspond to 360° , π radians correspond to 180° , 1 radian corresponds to $\approx 57^\circ$, etc.) Now, to find the area of a



The graph of Example 52(c)



The graph of Example 52(d)

Figure 24.

Continuity of various trigonometric functions

(**Recall:** Angles x are in radians)

1. The functions f, g defined by $f(x) = \sin x, g(x) = \cos x$ are continuous everywhere (i.e., for each real number x).
2. The functions $h(x) = \tan x$ and $k(x) = \sec x$ are continuous at every point which is not an odd multiple of $\frac{\pi}{2}$. At such points h, k are discontinuous. (i.e. at $-\frac{\pi}{2}, \frac{3\pi}{2}, -\frac{3\pi}{2}, \frac{5\pi}{2}, \dots$)
3. The functions ‘csc’ and ‘cot’ are continuous whenever x is not a multiple of π , and discontinuous whenever x is a multiple of π . (i.e. at $x = \pi, -\pi, 2\pi, -5\pi$, etc.)

Table 2.9: Continuity of Various Trigonometric Functions

sector of a circle of radius r subtending an angle θ at the center we note that the area is proportional to this central angle so that

$$\begin{aligned} \frac{2\pi}{\text{Area of circle}} &= \frac{\theta}{\text{Area of sector}} \\ \text{Area of sector} &= \frac{\theta}{2\pi}(\pi r^2) \\ &= \frac{r^2\theta}{2} \end{aligned}$$

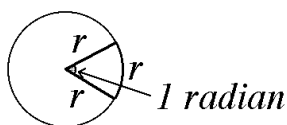


Figure 25.

We conclude that the area of a sector subtending an angle θ at the center is given by $\frac{r^2\theta}{2}$ where θ is in radians and summarize this in Table 2.10.

The area of a sector subtending an angle θ (in radians) at the center of a circle of radius r is given by

$$\text{Area of a sector} = \frac{r^2\theta}{2}$$

Table 2.10: Area of a Sector of a Circle

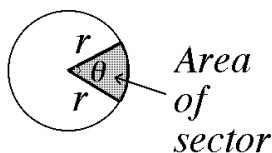


Figure 26.

Next we find some relationships between triangles in order to deduce a very important limit in the study of calculus.

We begin with a circle C of radius 1 and a sector subtending an angle, $x < \frac{\pi}{2}$ in radians at its center, labelled O . Label the extremities of the sector along the arc by A and B as in the adjoining figure, Fig. 27.

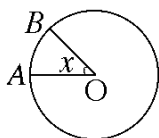


Figure 27.

At A produce an altitude which meets OB extended to C . Join AB by a line segment. The figure now looks like Figure 28.

We call the ‘triangle’ whose side is the arc AB and having sides AO, OB a “**curvilinear triangle**” for brevity. (It is also a “sector”!)

Let's compare areas. Note that

$$\begin{aligned} \text{Area of } \triangle ACO &> \text{Area of curvilinear triangle} \\ &> \text{Area of } \triangle ABO \\ \text{Now the area of } \triangle ACO &= \frac{1}{2}(1)|AC| = \frac{1}{2} \tan x \\ \text{Area of curvilinear triangle} &= \text{Area of the sector with central angle } x \\ &= \frac{1}{2}(1^2) \cdot x \text{ (because of Table 2.10 above)} \\ &= \frac{x}{2} \end{aligned}$$

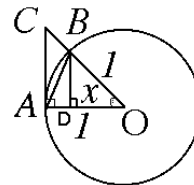


Figure 28.

Finally, from Figure 28,

$$\begin{aligned} \text{Area of triangle } ABO &= \frac{1}{2}(\text{altitude from base } AO)(\text{base length}) \\ &= \frac{1}{2}(\text{length of } BD) \cdot (1) = \frac{1}{2}(\sin x) \cdot (1) \\ &= \frac{\sin x}{2} \end{aligned}$$

(by definition of the sine of the angle x .) Combining these inequalities we get

$$\frac{1}{2} \tan x > \frac{x}{2} > \frac{\sin x}{2} \text{ (for } 0 < x < \frac{\pi}{2}, \text{ remember?)}$$

or

$$\sin x < x < \tan x$$

from which we can derive

$$\cos x < \frac{\sin x}{x} < 1 \text{ for } 0 < x \leq \frac{\pi}{2}$$

since all quantities are positive. This is a fundamental inequality in trigonometry.

We now apply Table 2.4(f) to this inequality to show that

$$\underbrace{\lim_{x \rightarrow 0^+} \cos x}_1 \leq \lim_{x \rightarrow 0^+} \frac{\sin x}{x} \leq \underbrace{\lim_{x \rightarrow 0^+} 1}_1$$

and we conclude that

$$\lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1$$

If, on the other hand, $-\frac{\pi}{2} < x < 0$ (or x is a negative angle) then, writing $x = -x_0$, we have $\frac{\pi}{2} > x_0 > 0$. Next

$$\frac{\sin x}{x} = \frac{\sin(-x_0)}{-x_0} = \frac{-\sin(x_0)}{-x_0} = \frac{\sin x_0}{x_0}$$

EXAMPLES



where we have used the relation $\sin(-x_0) = -\sin x_0$ valid for any angle x_0 (in radians, as usual). Hence

$$\begin{aligned}\lim_{x \rightarrow 0^-} \left(\frac{\sin x}{x} \right) &= \lim_{x \rightarrow 0^-} \frac{\sin x_0}{x_0} \\ &= \lim_{-x_0 \rightarrow 0^-} \frac{\sin x_0}{x_0} \\ &= \lim_{x_0 \rightarrow 0^+} \frac{\sin x_0}{x_0} \quad (\text{because if } -x_0 < 0 \text{ then } x_0 > 0 \\ &\quad \text{and } x_0 \text{ approaches } 0^+) \\ &= 1.\end{aligned}$$

Since both one-sided limits are equal it follows that

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

Another important limit like the one in Table 2.11 is obtained by using the basic

If the symbol \square represents any continuous function then, so long as we can let $\square \rightarrow 0$, we have

$$\lim_{\square \rightarrow 0} \frac{\sin \square}{\square} = 1$$

Table 2.11: Limit of $(\sin \square)/\square$ as $\square \rightarrow 0$

identity

$$1 - \cos \theta = \frac{1 - \cos^2 \theta}{1 + \cos \theta} = \frac{\sin^2 \theta}{1 + \cos \theta}$$

Dividing both sides by θ and rearranging terms we find

$$\frac{1 - \cos \theta}{\theta} = \frac{\sin \theta}{\theta} \cdot \frac{\sin \theta}{1 + \cos \theta}$$

Now, we know that

$$\lim_{\square \rightarrow 0} \frac{1 - \cos \square}{\square} = 0.$$

Table 2.12: Limit of $(1 - \cos \square)/\square$ as $\square \rightarrow 0$

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

and

$$\lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{1 + \cos \theta} \right)$$

exists (because it is the limit of the quotient of 2 continuous functions, the denominator not being 0 as $\theta \rightarrow 0$). Furthermore it is easy to see that

$$\lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{1 + \cos \theta} \right) = \frac{\sin 0}{1 + \cos 0} = \frac{0}{1 + 1} = 0$$



It now follows from Table 2.4(e) that

$$\begin{aligned}\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta} &= \left(\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \right) \left(\lim_{\theta \rightarrow 0} \frac{\sin \theta}{1 + \cos \theta} \right) = 1 \cdot 0 \\ &= 0\end{aligned}$$

and we conclude that

$$\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta} = 0$$

If you want, you can replace ' θ ' by ' x ' in the above formula or any other '*symbol*' as in Table 2.11. Hence, we obtain Table 2.12,

NOTES:

SAMPLE

Example 53.

Evaluate the following limits.

- a) $\lim_{x \rightarrow 0} \frac{\sin(3x)}{x}$
 b) $\lim_{x \rightarrow 0} \frac{1 - \cos(2x)}{x}$
 c) $\lim_{x \rightarrow 0} \frac{\sin(2x)}{\sin(3x)}$
 d) $\lim_{x \rightarrow 0^+} \frac{\sin(\sqrt{x})}{\sqrt{x}}$

One of the first complete introductions to *Trigonometry* was written by one **Johannes Müller of Königsberg**, (1436 - 1476), also known as *Regiomontanus*. The work, written in Latin, is entitled *De Triangulis omnimodis* first appeared in 1464.

Solution a) We use Table 2.11. If we let $\square = 3x$, we also need the symbol \square in the denominator, right? In other words, $x = \frac{\square}{3}$ and so

$$\frac{\sin 3x}{x} = \frac{\sin(\square)}{\left(\frac{\square}{3}\right)} = 3 \cdot \frac{\sin(\square)}{\square}$$

Now, as $x \rightarrow 0$ it is clear that, since $\square = 3x$, $\square \rightarrow 0$ as well. Thus

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin(3x)}{x} &= \lim_{\square \rightarrow 0} 3 \cdot \frac{\sin(\square)}{\square} \\ &= 3 \lim_{\square \rightarrow 0} \frac{\sin(\square)}{\square} \quad (\text{by Table 2.4(c)}) \\ &= 3 \cdot 1 \quad (\text{by Table 2.11}) \\ &= 3 \end{aligned}$$

b) We use Table 2.12 because of the form of the problem for $\square = 2x$ then $x = \frac{\square}{2}$. So

$$\frac{1 - \cos(2x)}{x} = \frac{1 - \cos \square}{\frac{\square}{2}} = 2 \cdot \frac{1 - \cos \square}{\square}$$

As $x \rightarrow 0$, we see that $\square \rightarrow 0$ too! So

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1 - \cos(2x)}{x} &= \lim_{\square \rightarrow 0} \left(2 \cdot \frac{1 - \cos(\square)}{\square} \right) \\ &= 2 \lim_{\square \rightarrow 0} \left(\frac{1 - \cos(\square)}{\square} \right) = 2 \cdot 0 = 0 \end{aligned}$$

c) This type of problem is not familiar at this point and all we have is Table 2.11 as reference . . . The idea is to rewrite the quotient as something that is *more familiar*. For instance, using plain algebra, we see that

$$\frac{\sin 2x}{\sin 3x} = \left(\frac{\sin 2x}{2x} \right) \left(\frac{2x}{3x} \right) \left(\frac{3x}{\sin 3x} \right),$$

so that some of the $2x$'s and $3x$ -cross-terms **cancel out** leaving us with the original expression.

OK, now as $x \rightarrow 0$ it is clear that $2x \rightarrow 0$ and $3x \rightarrow 0$ too! So,

$$\lim_{x \rightarrow 0} \frac{\sin 2x}{\sin 3x} = \left(\lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \right) \left(\lim_{x \rightarrow 0} \frac{2x}{3x} \right) \left(\lim_{3x \rightarrow 0} \frac{3x}{\sin 3x} \right)$$

(because “the limit of a product is the product of the limits” cf., Table 2.4.) Therefore

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin 2x}{\sin 3x} &= \left(\lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \right) \lim_{x \rightarrow 0} \frac{2x}{3x} \left(\lim_{3x \rightarrow 0} \frac{3x}{\sin 3x} \right) \\ &= 1 \cdot \frac{2}{3} \cdot 1 = \frac{2}{3} \end{aligned}$$

(by Tables 2.11& 2.4). Note that the middle-term, $\frac{2x}{3x} = \frac{2}{3}$ since $x \neq 0$.

Using the ‘Box’ method we can rewrite this argument more briefly as follows: We have **two** symbols, namely ‘ $2x$ ’ and ‘ $3x$ ’, so if we are going to use Table 2.11 we need to introduce these symbols into the expression as follows: (**Remember**, \square and \triangle are just ‘symbols’...).

Let $\square = 2x$ and $\triangle = 3x$. Then

$$\frac{\sin 2x}{\sin 3x} = \frac{\sin \square}{\sin \triangle} = \left(\frac{\sin \square}{\square}\right)\left(\frac{\square}{\triangle}\right)\left(\frac{\triangle}{\sin \triangle}\right)$$

So that \square ’s and \triangle ’s **cancel out** leaving the original expression.

OK, now as $x \rightarrow 0$ it is clear that $\square \rightarrow 0$ and $\triangle \rightarrow 0$ too! So

$$\lim_{x \rightarrow 0} \frac{\sin 2x}{\sin 3x} = \left(\lim_{\square \rightarrow 0} \frac{\sin \square}{\square}\right)\left(\lim_{x \rightarrow 0} \frac{\square}{\triangle}\right)\left(\lim_{\triangle \rightarrow 0} \frac{\triangle}{\sin \triangle}\right)$$

(because “the limit of a product is the product of the limits” , cf., Table 2.4.)
Therefore

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin 2x}{\sin 3x} &= \left(\lim_{\square \rightarrow 0} \frac{\sin \square}{\square}\right) \lim_{x \rightarrow 0} \frac{2x}{3x} \left(\lim_{\triangle \rightarrow 0} \frac{\triangle}{\sin \triangle}\right) \\ &= 1 \cdot \frac{2}{3} \cdot 1 = \frac{2}{3} \end{aligned}$$

(by Tables 2.11& 2.4).

d) In this problem we let $\square = \sqrt{x}$. As $x \rightarrow 0^+$ we know that $\sqrt{x} \rightarrow 0^+$ as well. Thus

$$\lim_{x \rightarrow 0^+} \frac{\sin \sqrt{x}}{\sqrt{x}} = \lim_{\square \rightarrow 0^+} \frac{\sin \square}{\square} = 1$$

(by Table 2.11).

Philosophy:

Learning mathematics has a lot to do with learning the rules of the interaction between symbols, some recognizable (such as 1 , 2 , $\sin x$, ...) and others not (such as \square , \triangle , etc.) Ultimately these are all ‘symbols’ and we need to recall **how** they interact with one another.

Sometimes it is helpful to replace the commonly used symbols ‘ y ’, ‘ z ’, etc. for variables, by other, not so commonly used ones, like \square , \triangle or ‘*squiggle*’ etc. It doesn’t matter **how** we denote something, what’s important is **how it interacts** with other symbols.



NOTES:

Limit questions can be approached in the following way.

You want to find

$$\lim_{x \rightarrow a} f(x).$$

Option 1 Take the value to which x tends, *i.e.* $x = a$, and evaluate the expression (function) at that value, *i.e.* $f(a)$.

Three possibilities arise:

- You obtain a number like $\frac{B}{A}$, with $A \neq 0$ and the question is answered (if the function is continuous at $x = a$), the answer being $\frac{B}{A}$.
- You get $\frac{B}{0}$, with $B \neq 0$ which implies that the limit exists and is plus infinity ($+\infty$) if $B > 0$ and minus infinity ($-\infty$) if $B < 0$.
- You obtain something like $\frac{0}{0}$ which means that the limit being sought may be “in disguise” and we need to move onto Option 2 below.

Option 2 If the limit is of the form $\frac{0}{0}$ proceed as follows:

We need to **play around** with the expression, that is you may have to factor some terms, use trigonometric identities, substitutions, simplify, rationalize the denominator, multiply and divide by the same symbol, etc. until you can return to Option 1 and repeat the procedure there.

Option 3 If 1 and 2 fail, then check the left and right limits.

- If they are equal, the limit exists and go to Option 1.
- If they are unequal, the limit does not exist. Stop here, that's your answer.

EXAMPLES



Table 2.13: Three Options to Solving Limit Questions

Exercise Set 7.

Determine the following limits if they exist. Explain.

- $\lim_{x \rightarrow \pi} \frac{\sin x}{x - \pi}$
(Hint: Write $\square = x - \pi$. Note that $x = \square + \pi$ and as $x \rightarrow \pi$, $\square \rightarrow 0$.)
- $\lim_{x \rightarrow \frac{\pi}{2}} (x - \frac{\pi}{2}) \tan x$
(Hint: $(x - \frac{\pi}{2}) \tan x = \frac{(x - \frac{\pi}{2})}{\cos x} \sin x$. Now set $\square = x - \frac{\pi}{2}$, so that $x = \square + \frac{\pi}{2}$ and note that, as $x \rightarrow \frac{\pi}{2}$, $\square \rightarrow 0$.)
- $\lim_{x \rightarrow 0} \frac{\sin(4x)}{2x}$

4. $\lim_{x \rightarrow 0} \frac{1 - \cos(3x)}{4x}$
5. $\lim_{x \rightarrow 0} \frac{\sin(4x)}{\sin(2x)}$
6. $\lim_{x \rightarrow 1^+} \frac{\sin \sqrt{x-1}}{\sqrt{x-1}}$

Web Links

For a very basic introduction to Limits see:
en.wikibooks.org/wiki/Calculus/Limits

Section 2.1: For one-sided limits and quizzes see:

www.math.montana.edu/frankw/ccp/calculus/estlimit/onesided/learn.htm

More about limits can be found at:

www.plu.edu/~heathdj/java/calcl/Epsilon.html (a neat applet)
www.ping.be/~ping1339/limth.htm

The proofs of the results in Table 2.4 can be found at:

www.math.montana.edu/frankw/ccp/calculus/estlimit/addition/learn.htm
www.math.montana.edu/frankw/ccp/calculus/estlimit/conmult/learn.htm
www.math.montana.edu/frankw/ccp/calculus/estlimit/divide/learn.htm
www.math.montana.edu/frankw/ccp/calculus/estlimit/multiply/learn.htm

Exercise Set 8.

Find the following limits whenever they exist. Explain.

1. $\lim_{x \rightarrow 2} \frac{x-2}{x}$
2. $\lim_{x \rightarrow 0^+} \sqrt{x} \cos x$
3. $\lim_{x \rightarrow 3} \left(\frac{x-3}{x^2-9} \right)$
4. $\lim_{x \rightarrow \frac{\pi}{2}} \left(x - \frac{\pi}{2} \right) \sec x$
5. $\lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{2x-\pi}{\cos x} \right)$
6. $\lim_{x \rightarrow 2^-} \sin(\pi x)$
7. $\lim_{x \rightarrow 0} \frac{\sin(3x)}{x}$
8. $\lim_{x \rightarrow \pi^+} \left(\frac{\cos x}{x-\pi} \right)$
9. $\lim_{x \rightarrow \pi^-} \left(\frac{\cos x}{x-\pi} \right)$
10. $\lim_{x \rightarrow 0^+} x |x|$

Hints:

- 3) Factor the denominator (Table 2.13, Option 2).
- 4) Write $\square = x - \frac{\pi}{2}$, $x = \square + \frac{\pi}{2}$ and simplify (Table 2.13, Option 2).

- 5) Let $\alpha = x - \frac{\pi}{2}$, $x = \alpha + \frac{\pi}{2}$ and use a formula for the cosine of the sum of two angles.
 9) See Table 2.13, Option 1(b).

Find the points of discontinuity, if any, of the following functions f :

11. $f(x) = \frac{\cos x}{x - \pi}$
 12. $f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ -1 & x = 0 \end{cases}$
 13. $f(x) = x^3 + x^2 - 1$
 14. $f(x) = \frac{x^2 + 1}{x^2 - 1}$
 15. $f(x) = \frac{x - 2}{|x^2 - 4|}$

Evaluate the following limits, whenever they exist. Explain.

16. $\lim_{x \rightarrow 0} \frac{\cos x - \cos 2x}{x^2}$

(Hint: Use the trigonometric identity

$$\cos A - \cos B = -2 \sin \left(\frac{A+B}{2} \right) \sin \left(\frac{A-B}{2} \right)$$

along with Table 2.11.)

17. $\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^2}$

(Hint: Factor the term 'tan x ' out of the numerator and use Tables 2.11 & 2.12.)

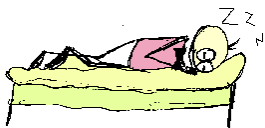
18. $\lim_{x \rightarrow 1} \frac{x^2 + 1}{(x - 1)^2}$

19. Find values of a and b such that

$$\lim_{x \rightarrow \pi} \frac{ax + b}{2 \sin x} = \frac{\pi}{4}$$

(Hint: It is necessary that $a\pi + b = 0$, why? Next, use the idea of Exercise Set 7 #1.)

20. $\lim_{x \rightarrow 0^+} \frac{1}{\sqrt{x+1} - \sqrt{x}}$



2.3 Important Theorems About Continuous Functions

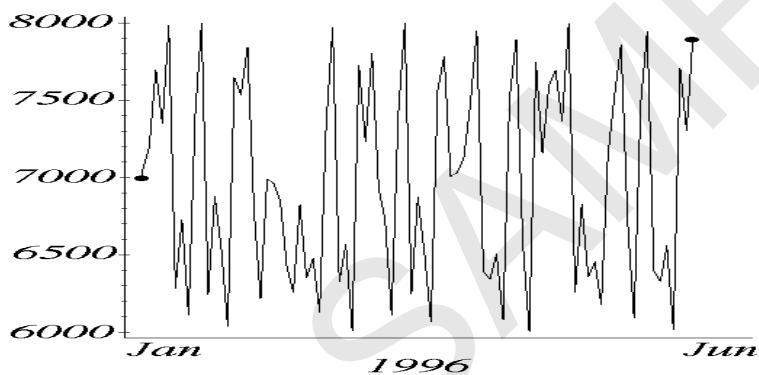
There are two main results (one being a consequence of the other) in the basic study of continuous functions. These are based on the property that the graph of a continuous function on a given interval has no ‘breaks’ in it.

Basically one can think of such a graph as a string which joins 2 points, say $(a, f(a))$ to $(b, f(b))$ (see Figure 29).

In Figure 29(a), the graph may have “sharp peaks” and may also look “smooth” and still be the graph of a continuous function (as is Figure 29(b)).

The **Intermediate Value Theorem** basically says that if you are climbing a mountain and you stop at 1000 meters and you want to reach 5000 meters, then at some future time you will pass, say the 3751 meter mark! This is obvious, isn’t it? But this basic observation allows you to understand this deep result about continuous functions.

For instance, the following graph may represent the fluctuations of your local Stock Exchange over a period of 1 year.



Assume that the index was 7000 points on Jan. 1, 1996 and that on June 30 it was 7900 points. Then sometime during the year the index passed the 7513 point mark at least once . . .

OK, so what does this theorem say mathematically?

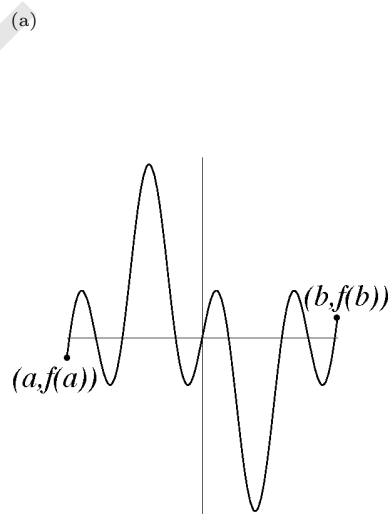
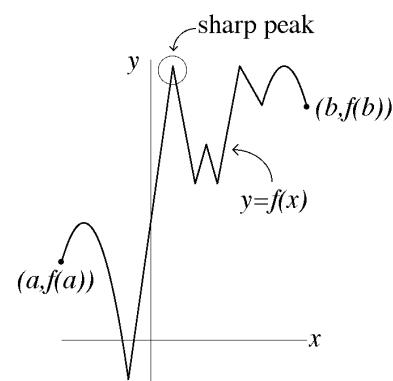
Intermediate Value Theorem (IVT)

Let f be continuous at each point of a closed interval $[a, b]$. Assume

1. $f(a) \neq f(b)$;
2. Let z be a point between $f(a)$ and $f(b)$.

Then there is at least one value of c between a and b such that

$$f(c) = z$$



(b)

Figure 29.

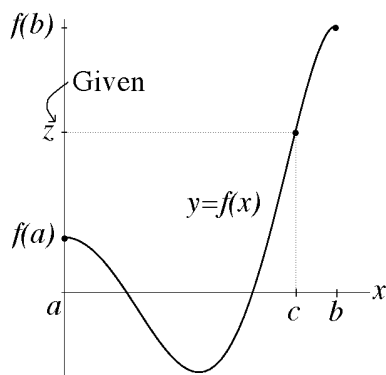


Figure 30.

The idea behind this Theorem is that **any** horizontal line that intersects the graph of a continuous function must intersect it at a point of its domain! This sounds and looks obvious (see Figure 30), but it's NOT true if the graph is NOT that of a continuous function (see Example 31). One of the most important **consequences** of this Intermediate Value Theorem (IVT) is sometimes called **Bolzano's Theorem** (after *Bernhard Bolzano (1781-1848)* mathematician, priest and philosopher).

Theorem 2.3.1. Let f be continuous on a closed interval $[a, b]$ (i.e., at each point x in $[a, b]$). If $f(a)f(b) < 0$, then there is at least one point c between a and b such that $f(c) = 0$.

Bolzano's Theorem is especially useful in determining the **location of roots** of polynomials or general (continuous) functions. Better still, it is also helpful in determining **where the graphs of functions intersect** each other.

For example, at which point(s) do the graphs of the functions given by $y = \sin x$ and $y = x^2$ intersect? In order to find this out you need to equate their values, so that $\sin x = x^2$ which then means that $x^2 - \sin x = 0$ so the points of intersection are roots of the function whose values are given by $y = x^2 - \sin x$.

Example 54. Show that there is one root of the polynomial $p(x) = x^3 + 2$ in the interval $-2 \leq x \leq -1$.

Solution We note that $p(-2) = -6$ and $p(-1) = 1$. So let $a = -2$, $b = -1$ in Bolzano's Theorem. Since $p(-2) < 0$ and $p(-1) > 0$ it follows that $p(x_0) = 0$ for some x_0 in $[-2, -1]$ which is what we needed to show.

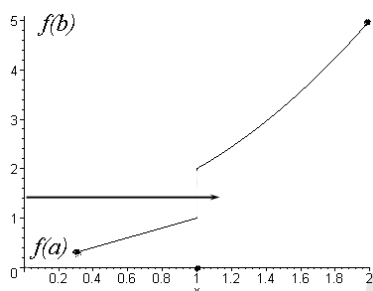


Figure 31.

Remark:

If you're not given the interval where the root of the function may be you need to find it! Basically **you look for points a and b where $f(a) < 0$ and $f(b) > 0$** and then you can refine your estimate of the root by "narrowing down" your interval.

Example 55. The distance between 2 cities A and B is 270 km. You're driving along the superhighway between A and B with speed limit 100 km/h hoping to get to your destination as soon as possible. You quickly realize that after one and one-half hours of driving you've travelled 200 km so you decide to stop at a rest area to relax. All of a sudden a police car pulls up to yours and the officer hands you a speeding ticket! Why?

Solution Well, the officer didn't actually **see** you speeding but saw you leaving A . Had you been travelling at the speed limit of 100 km/h it should have taken you 2 hours to get to the rest area. The officer quickly realized that somewhere along the highway you must have travelled at speeds of around 133 km/hr $\left(= \left(\frac{200 \text{ km}}{90 \text{ min}} \right) 60 \text{ min} \right)$. As a check notice that if you were travelling at a constant speed of say 130 km/h then you would have travelled a distance of only $130 \times 1.5 = 195$ km short of your mark.

A typical graph of your journey appears in Figure 32. Note that your "speed" must be related to the amount of "steepness" of the graph. The faster you go, the

“steeper” the graph. This motivates the notion of a **derivative** which you’ll see in the next chapter.

Philosophy

Actually one uses a form of the Intermediate Value Theorem almost daily. For instance, do you find yourself asking: “Well, based on this and that, such and such must happen somewhere between ‘this’ and ‘that’?”

When you’re driving along in your car you make decisions based on your speed, right? Will you get to school or work on time? Will you get to the store on time? You’re always assuming (correctly) that your **speed is a continuous function of time** (of course you’re not really **thinking** about this) and you make these quick mental calculations which will verify whether or not you’ll get “there” on time. Basically you know what time you started your trip and you have an idea about when it should end and then figure out where you have to be in between. . .

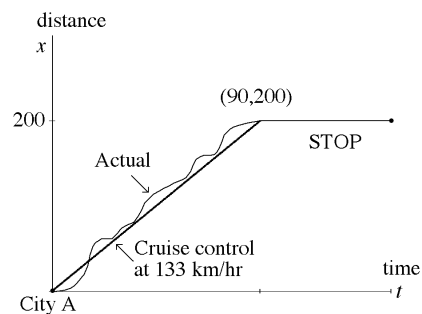
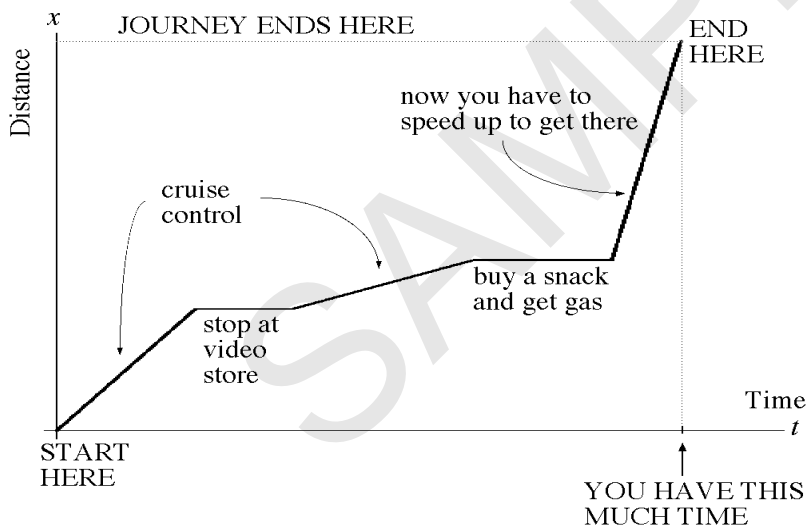


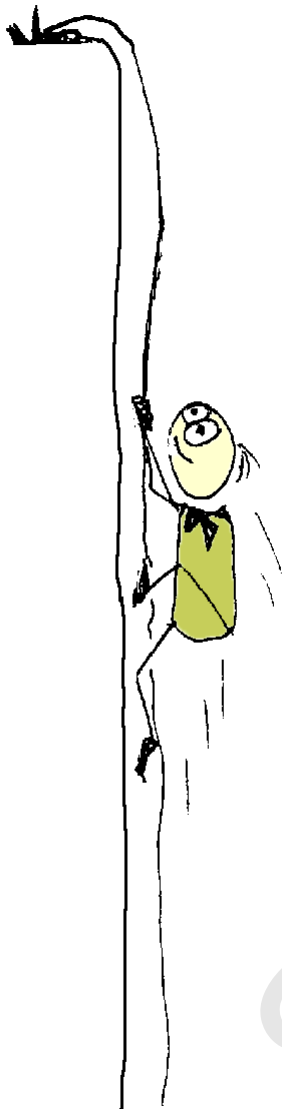
Figure 32.

Another result which you know about continuous functions is this: **If f is continuous on a closed interval $[a, b]$, then it has a maximum value and a minimum value, and these values are attained by some points in $[a, b]$.**



Since total distance travelled is a continuous function of time it follows that there exists at least one time t at which you were at the video store (this is true) and some other time t at which you were “speeding” on your way to your destination (also true!)... all applications of the IVT.

Finally, we should mention that since the definition of a continuous function depends on the notion of a limit it is immediate that many of the properties of limits should reflect themselves in similar properties of continuous functions. For example, from Table 2.4 we see that sums, differences, and products of continuous functions are continuous functions. The same is true of quotients of continuous functions provided the denominator is not zero at the point in question!

**Web Links**

For an application of the IVT to Economics see :

<http://hadm.sph.sc.edu/Courses/Econ/irr/irr.html>

For proofs of the main theorems here see:

www.cut-the-knot.com/Generalization/ivt.html

www.cut-the-knot.com/fta/brodie.html

NOTES:

SAMPLE

2.4 Evaluating Limits at Infinity

In this section we introduce some basic ideas as to when the variable tends to **plus infinity** ($+\infty$) or **minus infinity** ($-\infty$). Note that limits at infinity are always one-sided limits, (why?). This section is intended to be a prelude to a later section on **L'Hospital's Rule** which will allow you to evaluate many of these limits by a neat trick involving the function's **derivatives**.

For the purposes of evaluating limits at infinity, the symbol ' ∞ ' has the following properties:

PROPERTIES

1. It is an 'extended' real number (same for ' $-\infty$ ').
2. For any real number c (including 0), and $r > 0$,

$$\lim_{x \rightarrow \infty} \frac{c}{x^r} = 0$$

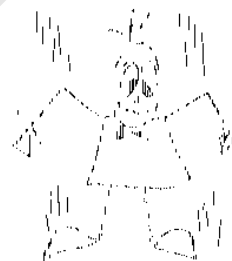
(Think of this as saying that $\frac{c}{\infty^r} = 0$ and $\infty^r = \infty$ for $r > 0$.)

3. The symbol $\frac{\infty}{\infty}$ is undefined and **can only be defined in the limiting sense** using the procedure in Table 2.13, Stage 2, some insight and maybe a little help from your calculator. We'll be using this procedure a little later when we attempt to evaluate limits at $\pm\infty$ using extended real numbers.

Table 2.14: Properties of $\pm\infty$

Basically, the limit symbol " $x \rightarrow \infty$ " means that the real variable x can be made "larger" than any real number!

A similar definition applies to the symbol " $x \rightarrow -\infty$ " except that now the real variable x may be made "smaller" than any real number. The next result is very useful in evaluating limits involving oscillating functions where it may not be easy to find the limit.



The **Sandwich Theorem** (mentioned earlier) is also valid for **limits at infinity**, that is, if

$$g(x) \leq f(x) \leq h(x)$$

for all (sufficiently) large x and for some (extended) real number A ,

$$\lim_{x \rightarrow \infty} g(x) = A, \quad \lim_{x \rightarrow \infty} h(x) = A$$

then f has a limit at infinity and

$$\lim_{x \rightarrow \infty} f(x) = A$$

Table 2.15: The Sandwich Theorem

Example 56. Evaluate the following limits at infinity.

a) $\lim_{x \rightarrow \infty} \frac{\sin(2x)}{x}$

- b) $\lim_{x \rightarrow \infty} \frac{3x^2 - 2x + 1}{x^2 + 2}$
 c) $\lim_{x \rightarrow -\infty} \frac{1}{x^3 + 1}$
 d) $\lim_{x \rightarrow \infty} (\sqrt{x^2 + x + 1} - x)$

EXAMPLES



Solution a) Let $f(x) = \frac{\sin(2x)}{x}$. Then $|f(x)| = \frac{|\sin(2x)|}{x}$ and $|f(x)| \leq \frac{1}{x}$ since $|\sin(2x)| \leq 1$ for every real number x . Thus

$$0 \leq \lim_{x \rightarrow \infty} |f(x)| \leq \lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

(where we have set $g(x) = 0$ and $h(x) = \frac{1}{x}$ in the statement of the Sandwich Theorem.) Thus,

$$\lim_{x \rightarrow \infty} |f(x)| = 0$$

which means that

$$\lim_{x \rightarrow \infty} f(x) = 0$$

(See Table 2.8 (c)).

b) Factor the term ' x^2 ' out of both numerator and denominator. Thus

$$\begin{aligned} \frac{3x^2 - 2x + 1}{x^2 + 2} &= \frac{x^2(3 - \frac{2}{x} + \frac{1}{x^2})}{x^2(1 + \frac{2}{x^2})} \\ &= \frac{(3 - \frac{2}{x} + \frac{1}{x^2})}{(1 + \frac{2}{x^2})} \end{aligned}$$

Now

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{3x^2 - 2x + 1}{x^2 + 2} &= \frac{\lim_{x \rightarrow \infty} (3 - \frac{2}{x} + \frac{1}{x^2})}{\lim_{x \rightarrow \infty} (1 + \frac{2}{x^2})} \\ &\text{(because the limit of a quotient is the quotient of the limits)} \\ &= \frac{3 - 0 + 0}{1 + 0} \\ &= 3 \end{aligned}$$

where we have used the Property 2 of limits at infinity, Table 2.14.

c) Let $f(x) = \frac{1}{x^3 + 1}$. We claim $\lim_{x \rightarrow -\infty} f(x) = 0 \dots$ Why?

Well, as $x \rightarrow -\infty$, $x^3 \rightarrow -\infty$ too, right? Adding 1 won't make any difference, so $x^3 + 1 \rightarrow -\infty$ too (remember, this is true because $x \rightarrow -\infty$). OK, now $x^3 + 1 \rightarrow -\infty$ which means $(x^3 + 1)^{-1} \rightarrow 0$ as $x \rightarrow -\infty$.

d) As it stands, letting $x \rightarrow \infty$ in the expression $x^2 + x + 1$ also gives ∞ . So $\sqrt{x^2 + x + 1} \rightarrow \infty$ as $x \rightarrow \infty$. So we have to calculate a "difference of two infinities" *i.e.*,

$$\begin{array}{c} f(x) = \sqrt{x^2 + x + 1} - x \\ \swarrow \quad \searrow \\ \infty \text{ as } x \rightarrow \infty \quad \infty \text{ as } x \rightarrow \infty \end{array}$$

There is no way of doing this so we have to simplify the expression (see Table 2.13, Stage 2) by rationalizing the expression ... So,

$$\begin{aligned}\sqrt{x^2 + x + 1} - x &= \frac{(\sqrt{x^2 + x + 1} - x)(\sqrt{x^2 + x + 1} + x)}{\sqrt{x^2 + x + 1} + x} \\ &= \frac{(x^2 + x + 1) - x^2}{\sqrt{x^2 + x + 1} + x} \\ &= \frac{x + 1}{\sqrt{x^2 + x + 1} + x}\end{aligned}$$

The form still isn't good enough to evaluate the limit directly. (We would be getting a form similar to $\frac{\infty}{\infty}$ if we took limits in the numerator and denominator separately.)

OK, so we keep simplifying by factoring out 'x's from both numerator and denominator ... Now,

$$\begin{aligned}\sqrt{x^2 + x + 1} - x &= \frac{x + 1}{\sqrt{x^2 + x + 1} + x} \\ &= \frac{x(1 + \frac{1}{x})}{x(\sqrt{1 + \frac{1}{x} + \frac{1}{x^2}} + 1)} \\ &= \frac{1 + \frac{1}{x}}{\sqrt{1 + \frac{1}{x} + \frac{1}{x^2}} + 1}\end{aligned}$$

OK, now we can let $x \rightarrow \infty$ and we see that

$$\begin{aligned}\lim_{x \rightarrow \infty} (\sqrt{x^2 + x + 1} - x) &= \lim_{x \rightarrow \infty} \frac{1 + \frac{1}{x}}{\sqrt{1 + \frac{1}{x} + \frac{1}{x^2}} + 1} \\ &= \frac{1 + 0}{\sqrt{1 + 0 + 0} + 1} \\ &= \frac{1}{2}\end{aligned}$$

As a **quick check** let's use a calculator and some large values of x : e.g. $x = 10$, 100, 1000, 10000, ... This gives the values: $f(10) = 0.53565$, $f(100) = 0.50373$, $f(1000) = 0.50037$, $f(10000) = 0.500037$, ... which gives a sequence whose limit appears to be $0.500\dots = \frac{1}{2}$, which is our theoretical result.



Exercise Set 9.

Evaluate the following limits (a) numerically and (b) theoretically.

- $\lim_{x \rightarrow \infty} \frac{\sin(3x)}{2x}$ (Remember: x is in **radians** here.)
- $\lim_{x \rightarrow -\infty} \frac{x}{x^3 + 2}$
- $\lim_{x \rightarrow \infty} \frac{x^3 + 3x - 1}{x^3 + 1}$
- $\lim_{x \rightarrow \infty} \sqrt{x}(\sqrt{x+1} - \sqrt{x})$
- $\lim_{x \rightarrow -\infty} \frac{\cos x}{x^2}$

6. **Hard** Show that

$$\lim_{x \rightarrow \infty} \sin x$$



does **not** exist by giving a **graphical** argument.

(*Hint:* Use the ideas developed in the Advanced Topics to **prove this theoretically.**)

2.5 How to Guess a Limit

We waited for this part until you learned about limits in general. Here we'll show you a quick and quite reliable way of *guessing* or *calculating* some limits at infinity (or “minus” infinity). Strictly speaking, you still need to ‘prove’ that your guess *is* right, even though it *looks* right. See Table 2.16. Later on, in Section 3.12, we will see a method called *L'Hospital's Rule* that can be used effectively, under some mild conditions, to evaluate limits involving *indeterminate forms*.

OK, now just a few words of caution before you start manipulating infinities. If an operation between infinities and reals (or another infinity) is not among those listed in Table 2.16, it is called an **indeterminate form**.

The most common indeterminate forms are:

$$0 \cdot (\pm\infty), \quad \pm\frac{\infty}{\infty}, \quad \infty - \infty, \quad (\pm\infty)^0, \quad 1^{\pm\infty}, \quad \frac{0}{0}, \quad 0^0$$

When you meet these forms in a limit you can't do much except simplify, rationalize, factor, etc. and then see if the form becomes “determinate”.

FAQ about Indeterminate Forms

Let's have a closer look at these *indeterminate* forms: They are called indeterminate because we cannot assign a single real number (once and for all) to any one of those expressions. For example,

Question 1: Why can't we define $\frac{\infty}{\infty} = 1$? After all, this looks okay ...

Answer 1: If that were true then,

$$\lim_{x \rightarrow \infty} \frac{2x}{x} = 1,$$

but this is impossible because, for any real number x no matter how large,

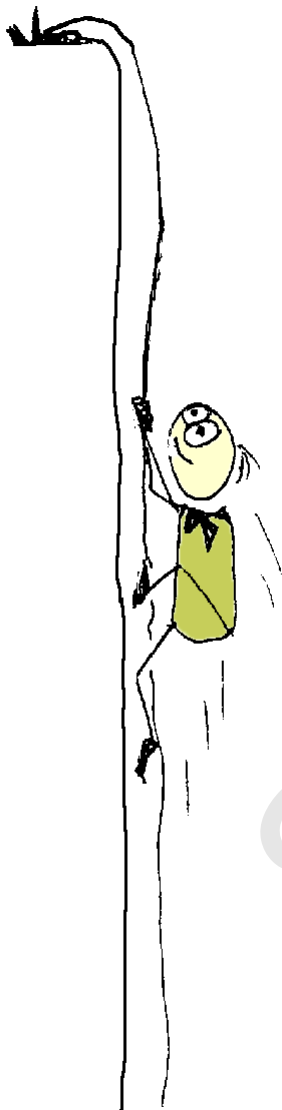
$$\frac{2x}{x} = 2,$$

and so, in fact,

$$\lim_{x \rightarrow \infty} \frac{2x}{x} = 2,$$

and so we can't define $\frac{\infty}{\infty} = 1$. Of course, we can easily modify this example to show that if r is any real number, then

$$\lim_{x \rightarrow \infty} \frac{rx}{x} = r,$$



which seems to imply that $\frac{\infty}{\infty} = r$. But r is also arbitrary, and so these numbers r can't all be equal because we can choose the r 's to be different! This shows that we cannot define the quotient $\frac{\infty}{\infty}$. Similar reasoning shows that we cannot define the quotient $-\frac{\infty}{\infty}$.

Question 2: All right, but surely $1^\infty = 1$, since $1 \times 1 \times 1 \times \dots = 1$?

Answer 2: No. The reason for this is that there is an infinite number of 1's here and this statement about multiplying 1's together is only true if there is a *finite* number of 1's. Here, we'll give some **numerical evidence** indicating that $1^\infty \neq 1$, necessarily.

Let $n \geq 1$ be a positive integer and look at some of the values of the expression

$$\left(1 + \frac{1}{n}\right)^n, \quad n = 1, 2, 3, \dots, 10,000$$

These values below:

n	$\left(1 + \frac{1}{n}\right)^n$	value
$n = 1$	$\left(1 + \frac{1}{1}\right)^1$	2
$n = 2$	$\left(1 + \frac{1}{2}\right)^2$	2.25
$n = 3$	$\left(1 + \frac{1}{3}\right)^3$	2.37
$n = 4$	$\left(1 + \frac{1}{4}\right)^4$	2.44
$n = 5$	$\left(1 + \frac{1}{5}\right)^5$	2.48
$n = 10$	$\left(1 + \frac{1}{10}\right)^{10}$	2.59
$n = 50$	$\left(1 + \frac{1}{50}\right)^{50}$	2.69
$n = 100$	$\left(1 + \frac{1}{100}\right)^{100}$	2.7048
$n = 1,000$	$\left(1 + \frac{1}{1000}\right)^{1000}$	2.7169
$n = 10,000$	$\left(1 + \frac{1}{10000}\right)^{10000}$	2.71814
...



Well, you can see that the values do not appear to be approaching 1! In fact, they seem to be getting closer to some number whose value is around 2.718. More on this special number later, in Chapter 4. Furthermore, we saw in Exercise 17, of Exercise Set 3, that these values must always lie between 2 and 3 and so, once again cannot converge to 1. This shows that, generally speaking, $1^\infty \neq 1$. In this case one can show that, in fact,

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = 2.7182818284590\dots$$

is a special number called *Euler's Number*, (see Chapter 4).

Question 3: What about $\infty - \infty = 0$?

Answer 3: No. This isn't true either since, to be precise, ∞ is NOT a real number, and so we cannot apply real number properties to it. The simplest example that shows that this difference between two infinities is not zero is the following. Let n be an integer (not infinity), for simplicity. Then

$$\begin{aligned}
 \infty - \infty &= \lim_{n \rightarrow \infty} [n - (n - 1)] \\
 &= \lim_{n \rightarrow \infty} [n - n + 1] \\
 &= \lim_{n \rightarrow \infty} 1 \\
 &= 1.
 \end{aligned}$$

The same argument can be used to find examples where $\infty - \infty = r$, where r is any given real number. It follows that we cannot assign a real number to the expression $\infty - \infty$ and so this is an indeterminate form.

Question 4: Isn't it true that $\frac{0}{0} = 0$?

Answer 4: No, this isn't true either. See the example in Table 2.11 and the discussion preceding it. The results there show that

$$\begin{aligned}
 \frac{0}{0} &= \frac{\sin 0}{0} \\
 &= \lim_{x \rightarrow 0} \frac{\sin x}{x} \\
 &= 1
 \end{aligned}$$

in this case. So we cannot assign a real number to the quotient "zero over zero".

Question 5: Okay, but it must be true that $\infty^0 = 1$!?

Answer 5: Not generally. An example here is harder to construct but it can be done using the methods in Chapter 4.

The Numerical Estimation of a Limit

At this point we'll be guessing limits of indeterminate forms by performing numerical calculations. See Example 59, below for their *theoretical*, rather than *numerical* calculation.

Example 57.

Guess the value of each of the following limits at infinity:

- a) $\lim_{x \rightarrow \infty} \frac{\sin(2x)}{x}$
 b) $\lim_{x \rightarrow -\infty} \frac{x^2}{x^2 + 1}$
 c) $\lim_{x \rightarrow \infty} (\sqrt{x+1} - \sqrt{x})$

Solution a) Since $x \rightarrow \infty$, we only need to try out **really large values** of x . So, just set up a table such as the one below and look for a pattern ...

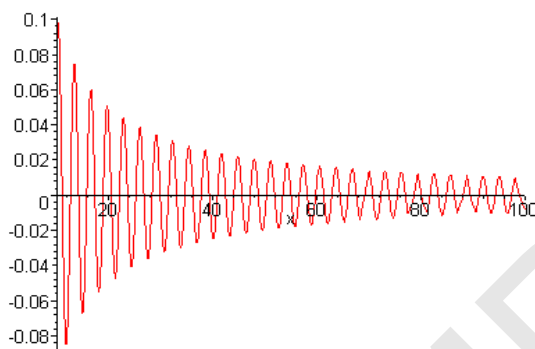
Some values of x	The values of $f(x) = \frac{\sin 2x}{x}$
10	.0913
100	-.00873
1,000	0.000930
10,000	0.0000582
100,000	-0.000000715
1,000,000	-0.000000655
...	...

We note that even though the values of $f(x)$ here alternate in sign, they are always getting smaller. In fact, they seem to be approaching $f(x) = 0$, as $x \rightarrow \infty$. This is our guess and, on this basis, we can claim that

$$\lim_{x \rightarrow \infty} \frac{\sin 2x}{x} = 0.$$

See Example 59 a), for another way of seeing this.

Below you'll see a graphical depiction (made by using your favorite software package or the Plotter included with this book), of the function $f(x)$ over the interval $[10, 100]$.



Note that the *oscillations* appear to be dying out, that is, they are getting smaller and smaller, just like the oscillations of your car as you pass over a bump! We guess that the value of this limit is 0.

b) Now, since $x \rightarrow -\infty$, we only need to try out really *small* (and negative) values of x . So, we set up a table like the one above and look for a pattern in the values.

Some values of x	The values of $f(x) = \frac{x^2}{x^2 + 1}$
-10	0.9900990099
-100	0.9999000100
-1,000	0.9999990000
-10,000	0.9999999900
-100,000	0.9999999999
-1,000,000	1.0000000000
...	...

In this case the values of $f(x)$ all have the same sign, they are always *positive*. Furthermore, they seem to be approaching $f(x) = 1$, as $x \rightarrow \infty$. This is our guess and, on this numerical basis, we can claim that

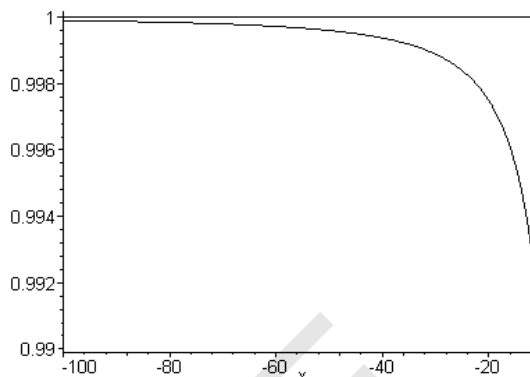
$$\lim_{x \rightarrow -\infty} \frac{x^2}{x^2 + 1} = 1.$$

EXAMPLES



See Example 59 b), for another way of seeing this.

A graphical depiction of this function $f(x)$ over the interval $[-100, -10]$ appears below.



In this example, the values of the function appear to increase steadily towards the line whose equation is $y = 1$. So, we guess that the value of this limit is 1.

c) Once again $x \rightarrow +\infty$, we only need to try out really *large* (and positive) values of x . Our table looks like:

Some values of x	The values of $f(x) = \sqrt{x+1} - \sqrt{x}$
10	0.15434713
100	0.04987562
1,000	0.01580744
10,000	0.00499999
100,000	0.00158120
1,000,000	0.00050000
...	...

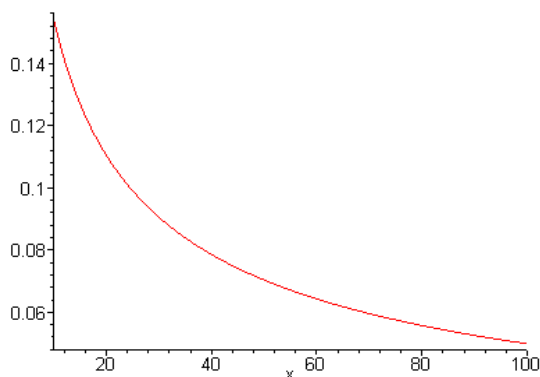
In this case the values of $f(x)$ all have the same sign, they are always *positive*. Furthermore, they seem to be approaching $f(x) = 0$, as $x \rightarrow \infty$. We can claim that

$$\lim_{x \rightarrow \infty} (\sqrt{x+1} - \sqrt{x}) = 0.$$

See Example 59 d), for another way of seeing this. A graphical depiction of this function $f(x)$ over the interval $[10, 100]$ appears below.

Note that larger values of x are not necessary since we have a *feeling* that they'll just be closer to our limit. We can believe that the values of $f(x)$ are always getting closer to 0 as x gets larger. So 0 should be the value of this limit. In the graph below we see that the function is getting smaller and smaller as x increases but it

always stays positive. Nevertheless, its values never reach the number 0 exactly, but only in the *limiting sense* we described in this section.



Watch out!



This numerical way of “guessing” limits **doesn’t always work!** It works well when the function *has* a limit, but it doesn’t work if the limit doesn’t exist (see the previous sections).

For example, the function $f(x) = \sin x$ has NO limit as $x \rightarrow \infty$. But how do you know this? The table could give us a hint;

Some values of x	The values of $f(x) = \sin x$
10	-0.5440211109
100	-0.5063656411
1,000	+0.8268795405
10,000	-0.3056143889
100,000	+0.0357487980
1,000,000	-0.3499935022
...	...

As you can see, these values do not seem to have a pattern to them. They don’t seem to “converge” to any particular value. We should be suspicious at this point and claim that the limit doesn’t exist. But remember: **Nothing can replace a rigorous (theoretical) argument for the existence or non-existence of a limit!** Our guess may not coincide with the reality of the situation as the next example will show!

Now, we’ll manufacture a function **with the property that, based on our numerical calculations, it seems to have a limit (actually = 0) as $x \rightarrow \infty$, but, in reality, its limit is SOME OTHER NUMBER!**

Example 58. Evaluate the following limit using your calculator,

$$\lim_{x \rightarrow \infty} \left(\frac{1}{x} + 10^{-12} \right).$$

Solution Setting up the table gives us:

Some values of x	The values of $f(x) = \left(\frac{1}{x} + 10^{-12}\right)$
10	0.100000000
100	0.010000000
1,000	0.001000000
10,000	0.000100000
100,000	0.000010000
999,999,999	0.000000001
...	...

Well, if we didn't know any better we would think that this limit should be 0. But this is **only because we are limited by the number of digits displayed upon our calculator!** The answer, based upon our knowledge of limits, should be the number 10^{-12} (but this number would display as 0 on most hand-held calculators). That's the real problem with using calculators for finding limits. You must be careful!!

Web Links

More (solved) examples on limits at infinity at:

<http://tutorial.math.lamar.edu/Classes/CalcI/LimitsAtInfinityI.aspx>

<http://www.sosmath.com/calculus/limcon/limcon04/limcon04.html>



Finding Limits using Extended Real Numbers (Optional)

At this point we'll be guessing limits of indeterminate forms by performing a new arithmetic among *infinite quantities*! In other words, we'll define addition and multiplication of infinities and then use these ideas to actually *find* limits at (plus or minus) infinity. This material is not standard in Calculus Texts and so can be omitted if so desired. However, it does offer an alternate method for actually guessing limits correctly every time!

Operations on the Extended Real Number Line

The **extended real number line** is the collection of all (usual) real numbers plus **two new symbols**, namely, $\pm\infty$ (**called extended real numbers**) which have the following properties:

Let x be any real number. Then

- a) $x + (+\infty) = (+\infty) + x = +\infty$
- b) $x + (-\infty) = (-\infty) + x = -\infty$
- c) $x \cdot (+\infty) = (+\infty) \cdot x = +\infty$ if $x > 0$
- d) $x \cdot (-\infty) = (-\infty) \cdot x = -\infty$ if $x > 0$
- e) $x \cdot (+\infty) = (+\infty) \cdot x = -\infty$ if $x < 0$
- f) $x \cdot (-\infty) = (-\infty) \cdot x = +\infty$ if $x < 0$

The operation $0 \cdot (\pm\infty)$ is undefined and requires further investigation.

Operations between $+\infty$ and $-\infty$

- g) $(+\infty) + (+\infty) = +\infty$
- h) $(-\infty) + (-\infty) = -\infty$
- i) $(+\infty) \cdot (+\infty) = +\infty = (-\infty) \cdot (-\infty)$
- j) $(+\infty) \cdot (-\infty) = -\infty = (-\infty) \cdot (+\infty)$

Quotients and powers involving $\pm\infty$

- k) $\frac{x}{\pm\infty} = 0$ for **any** real x
- l) $\infty^r = \begin{cases} \infty & r > 0 \\ 0 & r < 0 \end{cases}$
- m) $a^\infty = \begin{cases} \infty & a > 1 \\ 0 & 0 \leq a < 1 \end{cases}$

Table 2.16: Properties of Extended Real Numbers

The **extended real number line** is, by definition, the ordinary (positive and negative) real numbers with the addition of two idealized points denoted by $\pm\infty$ (and called the points at infinity). The way in which infinite quantities interact with each other and with real numbers is summarized briefly in Table 2.16 above.

It is important to note that any basic operation that is not explicitly mentioned in Table 2.16 is to be considered an *indeterminate form*, unless it can be derived from one or more of the basic axioms mentioned there.

If you think that adding and multiplying 'infinity' is nuts, you should look at the work of **Georg Cantor** (1845-1918), who actually developed an arithmetic of *transfinite cardinal numbers*, (or numbers that are infinite). He showed that 'different infinities' exist and actually set up rules of arithmetic for them. His work appeared in 1833.

As an example, the totality of all the integers (one type of infinite number), is different from the totality of all the numbers in the interval $[0, 1]$ (another 'larger' infinity). In a very specific sense, there are more "real numbers" than "integers".



Evaluating Limits of Indeterminate Forms

OK, now what? Well, you want

$$\lim_{x \rightarrow \pm\infty} f(x)$$

Basically, you look at $f(\pm\infty)$ respectively.

This is an expression involving “infinities” which you simplify (if you can) using the rules of arithmetic of the extended real number system listed in Table 2.16. If you get an indeterminate form you need to factor, rationalize, simplify, separate terms etc. until you get something more manageable.

Example 59. Evaluate the following limits involving indeterminate forms:

- $\lim_{x \rightarrow \infty} \frac{\sin 2x}{x}$
- $\lim_{x \rightarrow \infty} \frac{x^2}{x^2 + 1}$
- $\lim_{x \rightarrow -\infty} \frac{x^3 + 1}{x^3 - 1}$
- $\lim_{x \rightarrow \infty} \sqrt{x+1} - \sqrt{x}$
- $\lim_{x \rightarrow 0^+} \frac{x}{\sin x}$

Solution a) Let $f(x) = \frac{\sin(2x)}{x}$, then $f(\infty) = \frac{\sin(2\infty)}{\infty}$. Now use Table 2.16 on the previous page. Even though $\sin(2\infty)$ doesn’t really have a meaning, we can safely take it that the “ $\sin(2\infty)$ is something less than or equal to 1”, because the sine of any *finite* angle has this property. So $f(\infty) = \frac{\text{something}}{\infty} = 0$, by property (k), in Table 2.16.

We conclude our guess which is:

$$\lim_{x \rightarrow \infty} \frac{\sin 2x}{x} = 0$$

Remember: This is just an educated guess; you really have to *prove* this to be sure. This method of guessing is far better than the numerical approach of the previous subsection since it gives the right answer in case of Example 58, where the numerical approach failed!

b) Let $f(x) = \frac{x^2}{x^2+1}$. Then $f(\infty) = \frac{\infty}{\infty}$ by properties (l) and (a) in Table 2.16. So we have to simplify, etc. There is no other recourse ... Note that

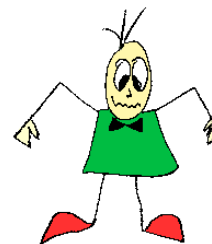
$$\begin{aligned} f(x) &= \frac{x^2}{x^2+1} = 1 - \frac{1}{x^2+1} \\ \text{So } \lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} 1 - \frac{1}{x^2+1} \\ &= 1 - \frac{1}{\infty} \quad (\text{by property (l) and (a)}) \\ &= 1 - 0 \quad (\text{by property (k)}) \\ &= 1 \end{aligned}$$

c) Let $f(x) = \frac{x^3 + 1}{x^3 - 1}$. Then

$$f(-\infty) = \frac{(-\infty)^3 + 1}{(-\infty)^3 - 1} = \frac{-\infty^3 + 1}{-\infty^3 - 1} = \frac{-\infty + 1}{-\infty - 1} = \frac{-\infty}{-\infty} = \frac{\infty}{\infty}$$

is an indeterminate form! So we have to simplify ... Dividing the numerator by the denominator using long division, we get

$$\begin{aligned} \frac{x^3 + 1}{x^3 - 1} &= 1 + \frac{2}{x^3 - 1} \\ \text{Hence } \lim_{x \rightarrow -\infty} \frac{x^3 + 1}{x^3 - 1} &= \lim_{x \rightarrow \infty} \left(1 + \frac{2}{x^3 - 1}\right) \\ &= 1 + \frac{2}{\infty} \quad (\text{by property (e) and (a)}) \\ &= 1 + 0 \quad (\text{by property (k), in Table 2.16}) \\ &= 1 \end{aligned}$$



d) Let $f(x) = \sqrt{x+1} - \sqrt{x}$. Then

$$\begin{aligned} f(\infty) &= \sqrt{\infty+1} - \sqrt{\infty} \\ &= \sqrt{\infty} - \sqrt{\infty} \quad (\text{by property (a)}) \\ &= \infty - \infty \quad (\text{by property (k), in Table 2.16}) \end{aligned}$$

It follows that $f(\infty)$ is an indeterminate form. Let's simplify ... By rationalizing the numerator we know that

$$\begin{aligned} \sqrt{x+1} - \sqrt{x} &= \frac{(x+1) - x}{\sqrt{x+1} + \sqrt{x}} \\ &= \frac{1}{\sqrt{x+1} + \sqrt{x}} \end{aligned}$$

So,

$$\begin{aligned} \lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x+1} + \sqrt{x}} \\ &= \frac{1}{\sqrt{\infty+1} + \sqrt{\infty}} \\ &= \frac{1}{\sqrt{\infty} + \sqrt{\infty}} \quad (\text{by property (a)}) \\ &= \frac{1}{\infty + \infty} \quad (\text{by property (h)}) \\ &= \frac{1}{\infty} \quad (\text{by property (g)}) \\ &= 0 \quad (\text{by property (k)}) \end{aligned}$$

d) In this case the function can be seen to be of *more than one indeterminate form*: For example,

$$0 \cdot \frac{1}{\sin 0} = 0 \cdot \frac{1}{0} = 0 \cdot \infty,$$

which is indeterminate (by definition), or

$$\frac{0}{\sin 0} = \frac{0}{0},$$

which is also indeterminate. But we have already seen in Table 2.11 that when this indeterminate form is interpreted as a limit, it is equal to 1.

2.6 Chapter Exercises

Use the methods of this Chapter to decide the continuity of the following functions at the indicated point(s).

1. $f(x) = 3x^2 - 2x + 1$, at $x = 1$
2. $g(t) = t^3 \cos(t)$, at $t = 0$
3. $h(z) = z + 2 \sin(z) - \cos(z + 2)$ at $z = 0$
4. $f(x) = 2 \cos(x)$ at $x = \pi$
5. $f(x) = |x + 1|$ at $x = -1$

Evaluate the limits of the functions from Exercises 1-5 above and justify your conclusions.

6. $\lim_{x \rightarrow 1} (3x^2 - 2x + 1)$
7. $\lim_{t \rightarrow 0} t^3 \cos(t)$
8. $\lim_{z \rightarrow 0} (z + 2 \sin(z) - \cos(z + 2))$
9. $\lim_{x \rightarrow \pi} 2 \cos(x)$
10. $\lim_{x \rightarrow -1} |x + 1|$

Evaluate the following limits

11. $\lim_{t \rightarrow 2^+} \left(\frac{t - 2}{t + 2} \right)$
12. $\lim_{x \rightarrow 4^+} \left(\frac{x - 4}{x^2 - 16} \right)$
13. $\lim_{t \rightarrow 2^+} \left(\frac{1}{t - 2} \right)$
14. $\lim_{x \rightarrow 1^+} \left(\frac{x - 1}{|x - 1|} \right)$
15. $\lim_{x \rightarrow 0^+} \left(1 + \frac{1}{x^3} \right)$

16. Let g be defined as

$$g(x) = \begin{cases} x^2 + 1 & x < 0 \\ 1 - |x| & 0 \leq x \leq 1 \\ x & x > 1 \end{cases}$$

Evaluate

- i). $\lim_{x \rightarrow 0^-} g(x)$ ii). $\lim_{x \rightarrow 0^+} g(x)$
- iii). $\lim_{x \rightarrow 1^-} g(x)$ iv). $\lim_{x \rightarrow 1^+} g(x)$

v) Conclude that the graph of g has no breaks at $x = 0$ but it does have a break at $x = 1$.

Determine whether the following limits exist. Give reasons

17. $\lim_{x \rightarrow 0} f(x)$ where $f(x) = \begin{cases} 2-x & x \leq 0 \\ x+1 & x > 0 \end{cases}$
18. $\lim_{x \rightarrow 1} |x-3|$
19. $\lim_{x \rightarrow -2} \left(\frac{x+2}{x+1} \right)$
20. $\lim_{x \rightarrow 0} x^2 \sin x$
21. $\lim_{x \rightarrow 1} f(x)$ where $f(x) = \begin{cases} \frac{\sin(x-1)}{x-1}, & 0 \leq x < 1 \\ 1, & x = 1 \\ |x-1| & x > 1 \end{cases}$

Determine the points of discontinuity of each of the following functions.

22. $f(x) = \frac{|x|}{x} - 1$ for $x \neq 0$ and $f(0) = 1$
23. $g(x) = \begin{cases} \frac{x}{|x|} & x < 0 \\ 1+x^2 & x \geq 0 \end{cases}$
24. $f(x) = \frac{x^2 - 3x + 2}{x^3 - 1}$, for $x \neq 1$; $f(1) = -1/3$.
25. $f(x) = \begin{cases} x^4 - 1 & x \neq 0 \\ -0.99 & x = 0 \end{cases}$
26. $f(x) = 1.65 + \frac{1}{x^2}$ for $x \neq 0$, $f(0) = +1$

Determine whether the following limits exist. If the limits exist, find their values in the extended real numbers.

27. $\lim_{x \rightarrow 0} \frac{\sin(ax)}{bx}$, where $a \neq 0, b \neq 0$
28. $\lim_{x \rightarrow 0} \frac{\cos(2x)}{|x|}$
29. $\lim_{x \rightarrow 0} \frac{x \sin(x)}{\sin(2x)}$
30. $\lim_{x \rightarrow 3^-} \frac{\sin \sqrt{3-x}}{\sqrt{3-x}}$
31. $\lim_{x \rightarrow 0} \frac{bx}{\sin(ax)}$, where $a \neq 0, b \neq 0$
32. $\lim_{x \rightarrow \infty} \frac{\cos 3x}{4x}$
33. $\lim_{x \rightarrow -\infty} x \sin x$
34. $\lim_{x \rightarrow \infty} \sqrt{x^2 + 1} - x$

35. Use Bolzano's Theorem and your pocket calculator to prove that the function f defined by $f(x) = x \sin x + \cos x$ has a root in the interval $[-5, 1]$.

36. Use Bolzano's Theorem and your pocket calculator to prove that the function f defined by $f(x) = x^3 - 3x + 2$ has a root in the interval $[-3, 0]$. Can you find it?

Are there any others? (Idea: Find smaller and smaller intervals and keep applying Bolzano's Theorem)

37. Find an interval of x 's containing the x -coordinates of the point of intersection of the curves $y = x^2$ and $y = \sin x$. Later on, when we study **Newton's Method** you'll see how to calculate these intersection points very accurately.

Hint: Use Bolzano's Theorem on the function $y = x^2 - \sin x$ over an appropriate interval (you need to find it).

Suggested Homework Set 5. Problems 1, 9, 12, 17, 22, 27, 34, 36



Chapter 3

The Derivative of a Function

The Big Picture

This chapter contains material which is fundamental to the further study of Calculus. Its basis dates back to the great Greek scientist Archimedes (287-212 B.C.) who first considered the problem of the *tangent line*. Much later, attempts by the key historical figures Kepler (1571-1630), Galileo (1564-1642), and Newton (1642-1727) among others, to understand the motion of the planets in the solar system and thus the speed of a moving body, led them to the problem of *instantaneous velocity* which translated into the mathematical idea of a *derivative*. Through the geometric notion of a tangent line we will introduce the concept of the **ordinary derivative of a function**, itself another function with certain properties. Its interpretations in the physical world are so many that this book would not be sufficient to contain them all. Once we know what a derivative is and how it is used we can formulate many problems in terms of these, and the natural concept of an **ordinary differential equation** arises, a concept which is central to most applications of Calculus to the sciences and engineering. For example, the motion of every asteroid, planet, star, comet, or other celestial object is governed by a differential equation. Once we can solve these equations we can describe the motion. Of course, this is hard in general, and if we can't solve them exactly we can always approximate the solutions which give the orbits by means of some, so-called, **numerical approximations**. This is the way it's done these days ... We can send probes to Mars because we have a very good idea of where they should be going in the first place, because we know the mass of Mars (itself an amazing fact) with a high degree of accuracy.

Most of the time we realize that things are in motion and this means that certain physical quantities are changing. These changes are best understood through the derivative of some underlying function. For example, when a car is moving its distance from a given point is changing, right? The "rate at which the distance changes" is the derivative of the distance function. This brings us to the notion of "instantaneous velocity". Furthermore, when a balloon is inflated, its volume is changing and the "rate" at which this volume is changing is approximately given by the derivative of the original volume function (its units would be $meters^3/sec$). In a different vein, the stock markets of the world are full of investors who delve into **stock options** as a means of furthering their investments. Central to all this business is the **Black-Sholes equation**, a complicated differential equation, which won their discoverer(s) a Nobel Prize in Economics a few years ago.



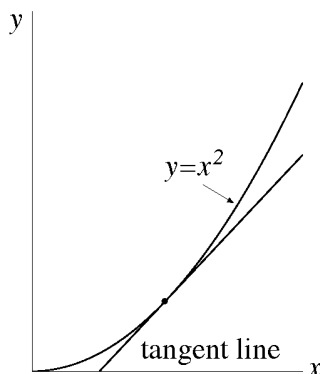


Figure 33.

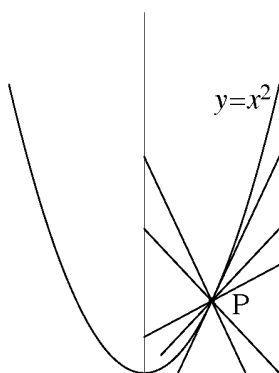


Figure 34.

Review

Look over your notes on functions in Chapter 1, especially the whole thing dealing with functions f being evaluated at abstract symbols other than x , like $f(x + 2)$. To really understand derivatives you should review Chapter 2, in particular the part on the definition of continuity and right/left-handed limits.

3.1 Motivation

We begin this chapter by motivating the notation of the **derivative of a function**, itself another function with certain properties.

First, we'll define the notion of a **tangent to a curve**. In the phrase that describes it, a tangent at a given point P on the graph of the curve $y = f(x)$ is a straight line segment which intersects the curve $y = f(x)$ at P and is 'tangent' to it (think of the ordinary tangents to a circle, see Figure 33).

Example 60. Find the equation of the line tangent to the curve $y = x^2$ at the point $(1, 1)$.

Solution Because of the shape of this curve we can see from its graph that every straight line crossing this curve will do so in at most two points, and we'll actually show this below. Let's choose a point P , say, $(1, 1)$ on this curve for ease of exposition. We'll **find the equation of the tangent line to P** and we'll do this in the following steps:

1. Find the equation of all the straight lines through P .
2. Show that there exists, among this set of lines, a **unique** line which is tangent to P .

OK, the equation of every line through $P(1, 1)$ has the form

$$y = m(x - 1) + 1$$

where m is its slope, right? (Figure 34).

Since we want the straight line to intersect the curve $y = x^2$, we must set $y = x^2$ in the preceding equation to find

$$x^2 = m(x - 1) + 1$$

or the quadratic

$$x^2 - mx + (m - 1) = 0$$

Finding its roots gives 2 solutions (the two x -coordinates of the point of intersection we spoke of earlier), namely,

$$x = m - 1 \text{ and } x = 1$$

The second root $x = 1$ is clear to see as all these straight lines go through $P(1, 1)$. The first root $x = m - 1$ gives a new root which is related to the **slope** of the straight line through $P(1, 1)$.

OK, we want only **one point of intersection**, right? (Remember, we're looking for a **tangent**). This means that the two roots must coincide! So we set $m - 1 = 1$ (as the two roots are equal) and this gives $m = 2$.

Thus the line whose slope is 2 and whose equation is

$$y = 2(x - 1) + 1 = 2x - 1$$

is the equation of the line tangent to $P(1, 1)$ for the curve $y = x^2$. Remember that at the point $(1, 1)$ this line has slope $m = 2$. This will be useful later.

OK, but this is only an example of a tangent line to a curve . . . How do you define this in general?

Well, let's take a function f , look at its graph and choose some point $P(x_0, y_0)$ on its graph where $y_0 = f(x_0)$. Look at a nearby point $Q(x_0 + h, f(x_0 + h))$. **What is the equation of the line joining P to Q?** Its form is

$$y - y_0 = m(x - x_0)$$

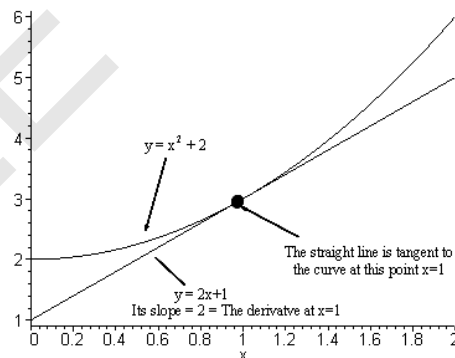
But $y_0 = f(x_0)$ and m , the slope, is equal to the quotient of the difference between the y-coordinates and the x-coordinates (of Q and P), that is,

$$m = \frac{f(x_0 + h) - f(x_0)}{(x_0 + h) - x_0} = \frac{f(x_0 + h) - f(x_0)}{h}.$$

OK, so the equation of this line is

$$y = \left\{ \frac{f(x_0 + h) - f(x_0)}{h} \right\} (x - x_0) + f(x_0)$$

From this equation you can see that the slope of this line must change with “ h ”. So, if we let h approach 0 as a limit, this line may approach a “limiting line” and it is this limiting line that we call the **tangent line to the curve $y = f(x)$ at $P(x_0, y_0)$** (see the figure in the margin on the right). The **slope** of this “tangent line” to the curve $y = f(x)$ at (x_0, y_0) defined by



$$m = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

(whenever this limit exists and is finite) is called the **derivative of f at x_0** . It is a number!!

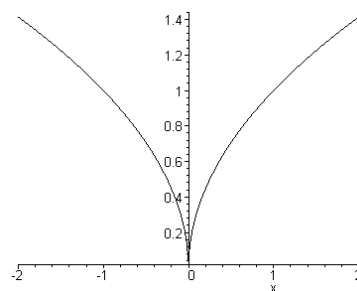
Notation for Derivatives The following notations are all adopted universally for the derivatives of f at x_0 :

$$f'(x_0), \frac{df}{dx}(x_0), D_x f(x_0), Df(x_0)$$

All of these have the same meaning.

Consequences!

1. If the limit as $h \rightarrow 0$ does not exist as a two-sided limit or it is infinite we say that the **derivative does not exist**. This is equivalent to saying that there is no uniquely defined tangent line at $(x_0, f(x_0))$, (Example 63).
2. The derivative, $f'(x_0)$ when it exists, is **the slope of the tangent line** at $(x_0, f(x_0))$ on the graph of f .
3. There's nothing special about these tangent lines to a curve in the sense that **the same line can be tangent to other points on the same curve**. (The simplest example occurs when $f(x) = ax + b$ is a straight line. Why?)



This graph has a vertical tangent line, namely $x = 0$, at the origin.

In this book we will use the symbols ' $f'(x_0)$, $Df(x_0)$ ' to mean the derivative of f at x_0 where

$$\begin{aligned} f'(x_0) &= \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} \\ &= \text{The slope of the tangent line at } x = x_0 \\ &= \text{The instantaneous rate of change of } f \text{ at } x = x_0. \end{aligned}$$

whenever this (two-sided) limit exists and is finite.

Table 3.1: Definition of the Derivative as a Limit

4. If either one or both one-sided limits defined by $f'_\pm(x_0)$ is infinite, the tangent line at that point $P(x_0, f(x_0))$ is **vertical** and given by the equation $x = x_0$, (See the margin).

The key idea in finding the derivative using Table 3.1, here, is always to

SIMPLIFY first, THEN pass to the LIMIT

The concept of a left and right-derivative of f at $x = x_0$ is defined by the left and right limits of the expression on the right in Table 3.1. So, for example,

$$\begin{aligned} f'_-(x_0) &= \lim_{h \rightarrow 0^-} \frac{f(x_0 + h) - f(x_0)}{h}, \\ f'_+(x_0) &= \lim_{h \rightarrow 0^+} \frac{f(x_0 + h) - f(x_0)}{h}, \end{aligned}$$

define the left and right-derivative of f at $x = x_0$ respectively, whenever these limits exist and are finite.

Example 61.

In Example 60 we showed that the slope of the tangent line to the curve $y = x^2$ at $(1, 1)$ is equal to 2. Show that the derivative of f where $f(x) = x^2$ at $x = 1$ is also equal to 2 (using the limit definition of the derivative, Table 3.1).

Solution By definition, the derivative of f at $x = 1$ is given by

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$$

provided this limit exists and is finite. OK, then calculate

$$\begin{aligned} \frac{f(1+h) - f(1)}{h} &= \frac{(1+h)^2 - 1^2}{h} \\ &= \frac{1 + 2h + h^2 - 1}{h} \\ &= 2 + h \end{aligned}$$

Since this is true for each value of $h \neq 0$ we can let $h \rightarrow 0$ and find

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} &= \lim_{h \rightarrow 0} (2 + h) \\ &= 2 \end{aligned}$$

and so $f'(1) = 2$, as well. Remember, this also means that the slope of the tangent line at $x = 1$ is equal to 2, which is what we found earlier.

Example 62.

Find the slope of the tangent line at $x = 2$ for the curve whose equation is $y = 1/x$.

Solution OK, we set $f(x) = 1/x$. As we have seen above, the slope's value, m_{tan} , is given by

$$m_{tan} = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h}.$$

Remember to simplify this ratio as much as possible (without the “lim” symbol). For $h \neq 0$ we have,

$$\begin{aligned} \frac{f(2+h) - f(2)}{h} &= \frac{\frac{1}{(2+h)} - \frac{1}{2}}{h} \\ &= \frac{\frac{2}{2(2+h)} - \frac{(2+h)}{2(2+h)}}{h} \\ &= \frac{2 - (2+h)}{2h(2+h)} \\ &= \frac{-h}{2h(2+h)} \\ &= \frac{-1}{2(2+h)}, \quad \text{since } h \neq 0. \end{aligned}$$

Since this is true for each $h \neq 0$, we can pass to the limit to find,

$$\begin{aligned} m_{tan} &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{-1}{2(2+h)} \\ &= -\frac{1}{4}. \end{aligned}$$

Example 63. We give examples of the following:

- A function f whose derivative does not exist (as a two-sided limit).
- A function f with a vertical tangent line to its graph $y = f(x)$ at $x = 0$, (‘infinite’ derivative at $x = 0$, i.e., both one-sided limits of the derivative exist but are infinite).
- A function f with a horizontal tangent line to its graph $y = f(x)$ at $x = 0$, (the derivative is equal to zero in this case).

Solution a) Let

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

This function is the same as $f(x) = |x|$, the absolute value of x , right? The idea is that in order for the two-sided (or ordinary) limit of the “derivative” to exist at some point, it is necessary that both one-sided limits (from the right and the left) each exist and both be equal, remember? The point is that **this function’s derivative has both one-sided limits existing at $x = 0$ but unequal**. Why? Let’s use Table 3.1 and try to find its “limit from the right” at $x = 0$.

For this we suspect that we need $h > 0$, as we want the limit from the **right**, and we’re using the same notions of right and left limits drawn from the theory of continuous functions.

$$\begin{aligned} \frac{f(0+h) - f(0)}{h} &= \frac{f(h) - f(0)}{h} \\ &= \frac{h - 0}{h} \quad (\text{because } f(h) = h \text{ if } h > 0) \\ &= 1, \quad (\text{since } h \neq 0). \end{aligned}$$

Leonardo da Vinci, 1452-1519, who has appeared in a recent film on Cinderella, is the ideal of the Italian *Risorgimento*, the Renaissance: Painter, inventor, scientist, engineer, mathematician, pathologist etc., he is widely accepted as a universal genius, perhaps the greatest ever. What impresses me the most about this extremely versatile man is his ability to assimilate nature into a quantifiable whole, his towering mind, and his insatiable appetite for knowledge. He drew the regular polytopes (three-dimensional equivalents of the regular polygons) for his friend **Fra Luca Pacioli**, priest and mathematician, who included the hand-drawn sketches at the end of the original manuscript of his book on the golden number entitled *De divina proportione*, published in 1509, and now in Torino, Italy.



This is true for each possible value of $h > 0$. So,

$$\lim_{h \rightarrow 0^+} \frac{f(h) - f(0)}{h} = 1,$$

and so this limit from the right, also called the **right derivative of f at $x = 0$** , exists and is equal to 1. This also means that as $h \rightarrow 0$, the slope of the tangent line to the graph of $y = |x|$ approaches the value 1.

OK, now let's find its limit from the **left** at $x = 0$. For this we want $h < 0$, right? Now

$$\begin{aligned} \frac{f(0+h) - f(0)}{h} &= \frac{f(h) - f(0)}{h} \\ &= \frac{-h - 0}{h} \quad (\text{because } f(h) = -h \text{ if } h < 0) \\ &= -1 \quad (\text{since } h \neq 0) \end{aligned}$$

This is true for each possible value of $h < 0$. So,

$$\lim_{h \rightarrow 0^-} \frac{f(h) - f(0)}{h} = -1.$$

This, so-called, **left-derivative of f at $x = 0$** exists and its value is -1 , a different value than 1 (which is the value of the right derivative of our f at $x = 0$). Thus

$$\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$$

does not exist as a two-sided limit. The graph of this function is shown in Figure 35. Note the cusp/ sharp point/v-shape at the origin of this graph. This graphical phenomenon guarantees that the derivative does not exist there.

Note that there is **no uniquely defined tangent line** at $x = 0$ (as both $y = x$ and $y = -x$ should qualify, so there is no actual "tangent line").

Solution b) We give an example of a function whose derivative is infinite at $x = 0$, say, so that its tangent line is $x = 0$ (if its derivative is infinite at $x = x_0$, then its tangent line is the vertical line $x = x_0$).

**SIMPLIFY first, then
GO to the LIMIT**

Define f by

$$f(x) = \begin{cases} \sqrt{x}, & x \geq 0, \\ -\sqrt{-x}, & x < 0. \end{cases}$$

The graph of f is shown in Figure 36.

Let's calculate its left- and right-derivative at $x = 0$. For $h < 0$, at $x_0 = 0$,

$$\begin{aligned} \frac{f(0+h) - f(0)}{h} &= \frac{f(h) - f(0)}{h} \\ &= \frac{-\sqrt{-h} - 0}{h} \quad (\text{because } f(h) = -\sqrt{-h} \text{ if } h < 0) \\ &= \frac{-\sqrt{-h}}{-(-h)} \\ &= \frac{1}{\sqrt{-h}}. \end{aligned}$$

So we obtain,

$$\begin{aligned} \lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} &= \lim_{h \rightarrow 0^-} \frac{1}{\sqrt{-h}} \\ &= +\infty, \end{aligned}$$

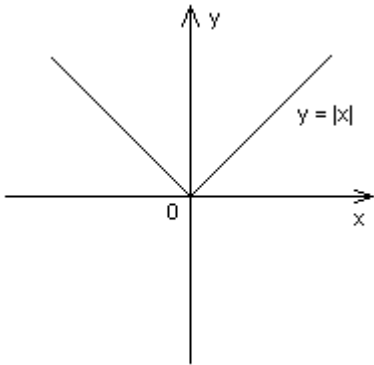


Figure 35.

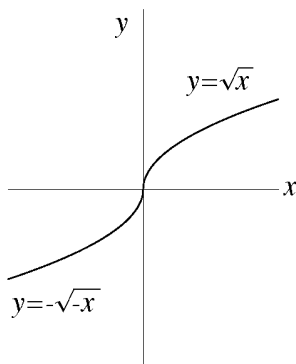


Figure 36.

$f'(x_0)$	Tangent Line Direction	Remarks
+	$\swarrow \nearrow$	“rises”, bigger means steeper up
-	$\nwarrow \searrow$	“falls”, smaller means steeper down
0	\longleftrightarrow	horizontal tangent line
$\pm\infty$	\updownarrow	vertical tangent line

Table 3.2: Geometrical Properties of the Derivative

and, similarly, for $h > 0$,

$$\begin{aligned}\lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} &= \lim_{h \rightarrow 0^+} \frac{1}{\sqrt{h}} \\ &= +\infty.\end{aligned}$$

Finally, we see that

$$\lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h}$$

both exist and are equal to $+\infty$.

Note: The line $x = 0$ acts as the ‘tangent line’ to the graph of f at $x = 0$.

Solution c) For an example of a function with a horizontal tangent line at some point (*i.e.* $f'(x) = 0$ at, say, $x = 0$) consider f defined by $f(x) = x^2$ at $x = 0$, see Figure 37. Its derivative $f'(0)$ is given by

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = 0$$

and since the derivative of f at $x = 0$ is equal to the slope of the tangent line there, it follows that the tangent line is horizontal, and given by $y = 0$.

Example 64. On the surface of our moon, an object P falling from rest will fall a distance of approximately $5.3t^2$ feet in t seconds. Find its instantaneous velocity at $t = a$ sec, $t = 1$ sec, and at $t = 2.6$ seconds.

Solution We’ll need to calculate its instantaneous velocity, let’s call it, “ v ”, at $t = a$ seconds. Since, in this case, $f(t) = 5.3t^2$, we have, by definition,

$$v = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

Now, for $h \neq 0$,

$$\begin{aligned}\frac{f(a+h) - f(a)}{h} &= \frac{5.3(a+h)^2 - 5.3a^2}{h} \\ &= \frac{5.3a^2 + 10.6ah + 5.3h^2 - 5.3a^2}{h} \\ &= 10.6a + 5.3h\end{aligned}$$

So,

$$v = \lim_{h \rightarrow 0} (10.6a + 5.3h) = 10.6a$$

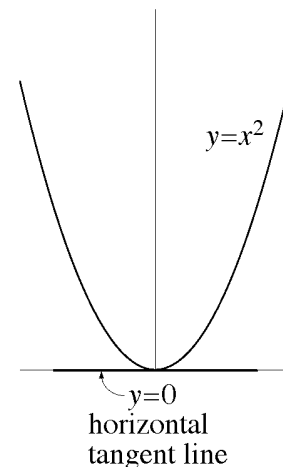


Figure 37.

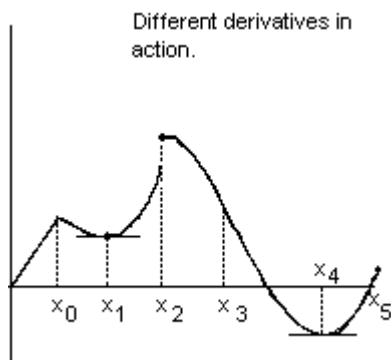


Figure 38.

$f'(x_0)$ does not exist (right and left derivatives not equal)

$f'(x_1) = 0$, (horizontal tangent line).

$f'(x_2)$ does not exist (left derivative at $x = x_2$ is infinite).

$f'(x_3) < 0$, (tangent line “falls”)

$f'(x_4) = 0$, (horizontal tangent line)

$f'(x_5) > 0$, (tangent line “rises”).

Table 3.3: Different Derivatives in Action: See Figure 38

feet per second. It follows that its instantaneous velocity at $t = 1$ second is given by $(10.6) \cdot (1) = 10.6$ feet per second, obtained by setting $a = 1$ in the formula for v . Similarly, $v = (10.6) \cdot (2.6) = 27.56$ feet per second. From this and the preceding discussion, you can conclude that an object falling from rest on the surface of the moon will fall at approximately *one-third* the rate it does on earth (neglecting air resistance, here).

Example 65. How long will it take the falling object of Example 64 to reach an instantaneous velocity of 50 feet per second?

Solution We know from Example 64 that $v = 10.6a$, at $t = a$ seconds. Since, we want $10.6a = 50$ we get $a = \frac{50}{10.6} = 4.72$ seconds.

Example 66. Different derivatives in action, see Figure 38, and Table 3.3.

Example 67. Evaluate the derivative of the function f defined by $f(x) = \sqrt{5x + 1}$ at $x = 3$.

Solution By definition,

$$f'(3) = \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h}.$$

Now, we try to simplify as much as possible before passing to the limit. For $h \neq 0$,

$$\begin{aligned} \frac{f(3+h) - f(3)}{h} &= \frac{\sqrt{5(3+h)+1} - \sqrt{5(3)+1}}{h} \\ &= \frac{\sqrt{16+5h} - 4}{h}. \end{aligned}$$

Now, to simplify this last expression, we *rationalize* the numerator (by multiplying both the numerator *and* denominator by $\sqrt{16+5h} + 4$). Then we'll find,

$$\begin{aligned} \frac{\sqrt{16+5h} - 4}{h} &= \left\{ \frac{\sqrt{16+5h} - 4}{h} \right\} \left\{ \frac{\sqrt{16+5h} + 4}{\sqrt{16+5h} + 4} \right\} \\ &= \frac{16 + 5h - 16}{h(\sqrt{16+5h} + 4)} \\ &= \frac{5}{\sqrt{16+5h} + 4}, \quad \text{since } h \neq 0. \end{aligned}$$

Think BIG here: Remember that rationalization gives

$$\sqrt{\square} - \sqrt{\triangle} = \frac{\square - \triangle}{\sqrt{\square} + \sqrt{\triangle}}$$

for any two positive symbols, \square, \triangle .

We can't simplify this any more, and now the expression "looks good" if we set $h = 0$ in it, so we can pass to the limit as $h \rightarrow 0$, to find,

$$\begin{aligned} f'(3) &= \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} \\ &= \lim_{h \rightarrow 0} \frac{5}{\sqrt{16+5h} + 4} \\ &= \frac{5}{\sqrt{16} + 4} \\ &= \frac{5}{8}. \end{aligned}$$

Summary

The derivative of a function f at a point $x = a$, (or $x = x_0$), denoted by $f'(a)$, or $\frac{df}{dx}(a)$, or $Df(a)$, is defined by the two equivalent definitions

$$\begin{aligned} f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}. \end{aligned}$$

whenever either limit exists (in which case so does the other). You get the second definition from the first by setting $h = x - a$, so that the statement " $h \rightarrow 0$ " is the same as " $x \rightarrow a$ ".

The **right-derivative** (resp. **left-derivative**) is defined by the right- (resp. left-) limits

$$\begin{aligned} f'_+(a) &= \lim_{h \rightarrow 0^+} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{x - a}, \end{aligned}$$

and

$$\begin{aligned} f'_-(a) &= \lim_{h \rightarrow 0^-} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{x \rightarrow a^-} \frac{f(x) - f(a)}{x - a}. \end{aligned}$$

NOTES

**SIMPLIFY first, then
GO to the LIMIT**

Exercise Set 10.

Evaluate the following limits

1. $\lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h}$ where $f(x) = x^2$
2. $\lim_{h \rightarrow 0} \frac{f(-1+h) - f(-1)}{h}$ where $f(x) = |x|$
3. $\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$ where $f(x) = \begin{cases} 0 & x = 0 \\ \frac{1}{x} & x \neq 0 \end{cases}$
- 4.a) $\lim_{h \rightarrow 0^-} \frac{f(1+h) - f(1)}{h}$
- b) $\lim_{h \rightarrow 0^+} \frac{f(1+h) - f(1)}{h}$ where $f(x) = \begin{cases} x+1 & x \geq 1 \\ x & 0 \leq x < 1 \end{cases}$
5. $\lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h}$ where $f(x) = \sqrt{x}$
HINT: Rationalize the numerator and simplify.
6. $\lim_{h \rightarrow 0} \frac{f(-2+h) - f(-2)}{h}$ where $f(x) = -x^2$

Find the slope of the tangent line to the graph of f at the given point.

7. $f(x) = 3x + 2$ at $x = 1$
8. $f(x) = 3 - 4x$ at $x = -2$
9. $f(x) = x^2$ at $x = 3$
10. $f(x) = |x|$ at $x = 1$
11. $f(x) = x|x|$ at $x = 0$
HINT: Consider the left and right derivatives separately.
12. $f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$ (at $x = 0$. Remember Heaviside's function?)

Determine whether or not the following functions have a derivative at the indicated point. Explain.

13. $f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$ at $x = 0$
14. $f(x) = \sqrt{x+1}$ at $x = -1$
HINT: Graphing this function may help.
15. $f(x) = |x^2|$ at $x = 0$
16. $f(x) = \sqrt{6-2x}$ at $x = 1$
17. $f(x) = \frac{1}{x^2}$ at $x = 1$
18. A function f is defined by

$$f(x) = \begin{cases} x & 0 \leq x < 1 \\ x+2 & 1 \leq x < 2 \\ 8-x^2 & 2 \leq x < 3. \end{cases}$$

- a) What is $f'(1)$? Explain.
- b) Does $f'(2)$ exist? Explain.
- c) Evaluate $f'(\frac{5}{2})$.

NOTES:

3.2 Working with Derivatives

By now you know how to find the derivative of a given function (and you can actually check to see whether or not it **has** a derivative at a given point). You also understand the relationship between the derivative and the slope of a tangent line to a given curve (otherwise go to Section 3.1).

Sometimes it is useful to define the derivative $f'(a)$ of a given function at $x = a$ as

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \quad (3.1)$$

provided the (two-sided) limit exists and is finite. Do you see why this definition is equivalent to

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}?$$

Simply replace the symbol “ h ” by “ $x - a$ ” and simplify. As $h \rightarrow 0$ it is necessary that $x - a \rightarrow 0$ or $x \rightarrow a$.

Notation

When a given function f has a derivative at $x = a$ we say that “ f is differentiable at $x = a$ ” or briefly “ f is differentiable at a .”

If f is differentiable at every point x of a given interval, \mathbf{I} , we say that “ f is differentiable on \mathbf{I} .”

Example 68. The function f defined by $f(x) = x^2$ is differentiable everywhere on the real line (i.e., at each real number) and its derivative at x is given by $f'(x) = 2x$.

Example 69. The Power Rule. The function g defined by $g(x) = x^n$ where $n \geq 0$ is any given integer is differentiable at every point x . If $n < 0$ then it is differentiable everywhere except at $x = 0$. Show that its derivative is given by

$$\frac{d}{dx} x^n = nx^{n-1}.$$



Solution We need to recall the *Binomial Theorem*: This says that

$$(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n$$

for some integer n whenever $n \geq 1$, (there are $(n+1)$ terms in total). From this we get the well-known formulae

$$\begin{aligned} (x+h)^2 &= x^2 + 2xh + h^2, \\ (x+h)^3 &= x^3 + 3x^2h + 3xh^2 + h^3, \\ (x+h)^4 &= x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4. \end{aligned}$$

OK, by definition (and the Binomial Theorem), for $h \neq 0$,

$$\begin{aligned} \frac{g(x+h) - g(x)}{h} &= \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h} \\ &= nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}. \end{aligned}$$

Since $n \geq 1$ it follows that (because the limit of a sum is the sum of the limits),

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} &= \lim_{h \rightarrow 0} \left(nx^{n-1} + \frac{n(n-1)}{2} x^{n-2} h + \dots + h^{n-1} \right) \\ &= nx^{n-1} + \lim_{h \rightarrow 0} \left(\frac{n(n-1)}{2} x^{n-2} h + \dots + h^{n-1} \right) \\ &= nx^{n-1} + 0 \\ &= nx^{n-1}. \end{aligned}$$



Thus $g'(x)$ exists and $g'(x) = nx^{n-1}$.

Remark! Actually, more is true here. It is the case that **for every number** 'a' (integer or not), but a is NOT a variable like 'x, sin x, ...',

$$\frac{d}{dx} x^a = ax^{a-1} \text{ if } x > 0.$$

This formula is useful as it gives a simple expression for the derivative of any power of the independent variable, in this case, 'x'.

QUICKIES

- a) $f(x) = x^3$; $f'(x) = 3x^{3-1} = 3x^2$
- b) $f(t) = \frac{1}{t} = t^{-1}$, so $f'(t) = (-1)t^{-2} = -\frac{1}{t^2}$
- c) $g(z) = \frac{1}{z^2} = z^{-2}$, so $g'(z) = (-2)z^{-3} = -\frac{2}{z^3}$
- d) $f(x) = \sqrt{x} = x^{\frac{1}{2}}$, so $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$
- e) $f(x) = x^{-\frac{2}{3}}$; $f'(x) = -\frac{2}{3}x^{-\frac{5}{3}}$
- f) $f(x) = \text{constant}$, $f'(x) = 0$

Quick summary

Notation:

- By cf we mean the function whose values are given by

$$(cf)(x) = cf(x)$$

where c is a constant.

- By the symbols $f + g$ we mean the function whose values are given by

$$(f + g)(x) = f(x) + g(x)$$

A function f is said to be **differentiable at the point a** if its derivative $f'(a)$ exists there. This is equivalent to saying that both the left- and right-hand derivatives exist at a and are equal. A function f is said to be **differentiable everywhere** if it is differentiable at every point a of the real line.

For example, the function f defined by the absolute value of x , namely, $f(x) = |x|$, is differentiable at every point except at $x = 0$ where $f'_-(0) = -1$ and $f'_+(0) = 1$. On the other hand, the function g defined by $g(x) = x|x|$ is differentiable everywhere. Can you show this?

Properties of the Derivative

Let f, g be two differentiable functions at x and let c be a constant. Then $cf, f \pm g, fg$ are all differentiable at x and

a)
$$\frac{d}{dx}(cf) = c \frac{df}{dx} = cf'(x), \quad c \text{ is a constant.}$$

b)
$$\begin{aligned} \frac{d}{dx}(f \pm g) &= \frac{df}{dx} \pm \frac{dg}{dx}, & \text{Sum/Difference Rule} \\ &= f'(x) \pm g'(x) \end{aligned}$$

c)
$$\frac{d}{dx}(fg) = f'(x)g(x) + f(x)g'(x), \quad \text{Product Rule}$$

d) If for some x , the value $g(x) \neq 0$ then $\frac{f}{g}$ is differentiable at x and

$$\frac{d}{dx} \left(\frac{f}{g} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}, \quad \text{Quotient Rule}$$

where all the derivatives are evaluated at the point ' x '. Hints to the *proofs* or *verification* of these basic Rules may be found at the end of this section. They are left to the reader as a Group Project.

Example 70. Find the derivative, $f'(x)$ of the function f defined by $f(x) = 2x^3 - 5x + 1$. What is its value at $x = 1$?

Solution We use Example 69 and Properties (a) and (b) to see that

$$\begin{aligned} f'(x) &= \frac{d}{dx}(2x^3) + \frac{d}{dx}(-5x) + \frac{d}{dx}(1) = 2 \frac{d}{dx}(x^3) + (-5) \cdot \frac{d}{dx}(x) + 0 \\ &= 2 \cdot 3x^2 + (-5) \cdot 1 \\ &= 6x^2 - 5. \end{aligned}$$

So, $f'(x) = 6x^2 - 5$ and thus the derivative evaluated at $x = 1$ is given by $f'(1) = 6 \cdot (1)^2 - 5 = 6 - 5 = 1$.

Example 71. Given that $f(x) = \sqrt[3]{x} + \sqrt[3]{2} - 1$ find $f'(x)$ at $x = -1$.

Solution We rewrite all "roots" as powers and then use the Power Rule. So,

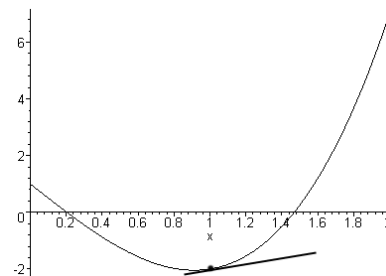
$$\begin{aligned} f'(x) &= \frac{d}{dx} \left(\sqrt[3]{x} + \sqrt[3]{2} - 1 \right) \\ &= \frac{d}{dx} \left(x^{1/3} + 2^{1/3} - 1 \right) = \frac{d}{dx} \left(x^{1/3} \right) + \frac{d}{dx} \left(2^{1/3} - 1 \right) \\ &= \frac{1}{3} x^{\frac{1}{3}-1} + 0 - 0 \\ &= \frac{1}{3} x^{-\frac{2}{3}}. \end{aligned}$$



Note that the formula

$$\frac{d}{dx}(fg) = \frac{df}{dx} \frac{dg}{dx}$$

is **NOT TRUE** in general. For example, if $f(x) = x, g(x) = 1$, then $f(x)g(x) = x$ and so $(fg)'(x) = 1$. On the other hand, $f'(x)g'(x) = 0$, and so this formula cannot be true.



The tangent line at $x = 1$ to the curve $f(x)$ defined in Example 70.

Figure 39.

Finally, $f'(-1) = \frac{1}{3}(-1)^{-\frac{2}{3}} = \frac{1}{3}(\sqrt[3]{-1})^{-2} = \frac{1}{3}(-1)^{-2} = -\frac{1}{3}$.

Example 72. Find the slope of the tangent line to the curve defined by the function $h(x) = (x^2 + 1)(x - 1)$ at the point $(1, 0)$ on its graph.

Solution Using the geometrical interpretation of the derivative (cf., Table 3.1), we know that this slope is equal to $h'(1)$. So, we need to calculate the derivative of h and then evaluate it at $x = 1$. Since h is made up of two functions we can use the Product Rule (Property (c), above). To this end we write $f(x) = (x^2 + 1)$ and $g(x) = x - 1$. Then $h(x) = f(x)g(x)$ and we want $h'(x)$. So, using the Product Rule we see that

$$\begin{aligned} h'(x) &= \frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x) \\ &= \frac{d}{dx}(x^2 + 1) \cdot (x - 1) + (x^2 + 1) \cdot \frac{d}{dx}(x - 1) \\ &= (2x + 0) \cdot (x - 1) + (x^2 + 1) \cdot (1 - 0) = 2x(x - 1) + x^2 + 1 \\ &= 3x^2 - 2x + 1. \end{aligned}$$

The required slope is now given by $h'(1) = 3 - 2 + 1 = 2$. See Figure 40.

Example 73. Find the equation of the tangent line to the curve defined by

$$h(t) = \frac{t}{t^2 + 1}$$

at the point $(0, 0)$ on its graph.

Solution Since the function h is a quotient of two functions we may use the Quotient Rule, (d). To this end, let $f(t) = t$ and $g(t) = t^2 + 1$. The idea is that we have to find $h'(t)$ at $t = 0$ since this will give the slope of the tangent line at $t = 0$, and then use the general equation of a line in the form $y = mx + b$ in order to get the actual equation of our tangent line passing through $(0, 0)$. OK, now

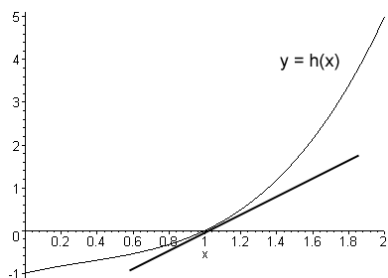
$$\begin{aligned} h'(t) &= \frac{f'(t)g(t) - f(t)g'(t)}{g^2(t)} \\ &= \frac{(1) \cdot (t^2 + 1) - (t) \cdot (2t)}{(t^2 + 1)^2} \\ &= \frac{1 - t^2}{(t^2 + 1)^2}. \end{aligned}$$

Next, it is clear that $h'(0) = 1$ and so the tangent line must have the equation $y = x + b$ for an appropriate point (x, y) on it. But $(x, y) = (0, 0)$ is on it, by hypothesis. So, we set $x = 0, y = 0$ in the general form, solve for b , and conclude that $b = 0$. Thus, the required equation is $y = x + 0 = x$, i.e., $y = x$, see Figure 41.

Example 74. At which points on the graph of $y = x^3 + 3x$ does the tangent line have slope equal to 9?

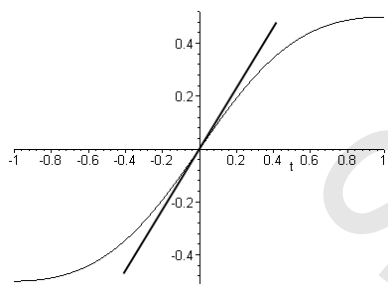
Solution This question is not as direct as the others, above. The idea here is to find the expression for the derivative of y and then set this expression equal to 9 and then solve for x . Now, $y'(x) = 3x^2 + 3$ and so $9 = y'(x) = 3x^2 + 3$ implies that $3x^2 = 6$ or $x = \pm\sqrt{2}$. Note the *two* roots here. So there are two points on the required graph where the slope is equal to 9. The y -coordinates are then given by setting $x = \pm\sqrt{2}$ into the expression for y . We find the points $(\sqrt{2}, 5\sqrt{2})$ and $(-\sqrt{2}, -5\sqrt{2})$, since $(\sqrt{2})^3 = 2\sqrt{2}$.

Example 75. If $f(x) = (x^2 - x + 1)(x^2 + x + 1)$ find $f'(0)$ and $f'(1)$.



The graph of the function h and its tangent line at $x = 1$. The slope of this straight line is equal to 2

Figure 40.



The tangent line $y = x$ through $(0, 0)$ for the function h in Example 73.

Figure 41.

Solution Instead of using the Product Rule we can simply expand the product noting that $f(x) = (x^2 - x + 1)(x^2 + x + 1) = x^4 + x^2 + 1$. So, $f'(x) = 4x^3 + 2x$ by the Power Rule, and thus, $f'(0) = 0$, $f'(1) = 6$.

Exercise Set 11.

Find the derivative of each of the following functions using any one of the Rules above: Show specifically which Rules you are using at each step. There is no need to simplify your final answer.

Example: If $f(x) = \frac{x^{0.3}}{x+1}$, then

$$\begin{aligned} f'(x) &= \frac{D(x^{0.3})(x+1) - x^{0.3}D(x+1)}{(x+1)^2}, \quad \text{by the Quotient Rule,} \\ &= \frac{(0.3)x^{-0.7}(x+1) - x^{0.3}(1)}{(x+1)^2}, \quad \text{by the Power Rule with } a = 2/3 \\ &= \frac{(0.3)x^{-0.7}(x+1) - x^{0.3}}{(x+1)^2}. \end{aligned}$$

1. $f(x) = x^{1.5}$
2. $f(t) = t^{-2}$
3. $g(x) = 6$
4. $h(x) = x^{\frac{2}{3}}$
5. $k(t) = t^{\frac{1}{5}}$
6. $f(x) = 4.52$
7. $f(t) = t^4$
8. $g(x) = x^{-3}$
9. $f(x) = x^{-1}$
10. $f(x) = x^\pi$
11. $f(t) = t^2 - 6$
12. $f(x) = 3x^2 + 2x - 1$
13. $f(t) = (t-1)(t^2 + 4)$
14. $f(x) = \sqrt{x}(3x^2 + 1)$
15. $f(x) = \frac{x^{0.5}}{2x+1}$
16. $f(x) = \frac{x-1}{x+1}$
17. $f(x) = \frac{x^3 - 1}{x^2 + x - 1}$
18. $f(x) = \frac{x^{\frac{2}{3}}}{\sqrt{x} + 3x^{\frac{3}{4}}}$

Group Project on Differentiation

Prove the Differentiation Rules in Section 3.2 using the definition of the derivative as a limit, the limit properties in Table 2.4, and some basic algebra. Assume throughout that f and g are differentiable at x and $g(x) \neq 0$. In order to prove the Properties proceed as follows using the hints given:

1. **Property a)**

Show that for any real number c , and $h \neq 0$, we have

$$(cf)'(x) = c \times \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h},$$

and complete the argument.

2. **Property b) The Sum/Difference Rule:** Show that for a given x and $h \neq 0$,

$$\frac{(f+g)(x+h) - (f+g)(x)}{h} = \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}.$$

Then use Table 2.4, a) and the definition of the derivatives.

3. **Property c) The Product Rule:** Show that for a given x and $h \neq 0$,

$$\frac{(fg)(x+h) - (fg)(x)}{h} = g(x) \frac{f(x+h) - f(x)}{h} + f(x+h) \frac{g(x+h) - g(x)}{h}.$$

Then use Table 2.4 e), the definition of the derivatives, and the continuity of f at x .

4. **Property d) The Quotient Rule:** First, show that for a given x and any h ,

$$\left(\frac{f}{g}\right)(x+h) - \left(\frac{f}{g}\right)(x) = \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}.$$

Next, rewrite the previous expression as

$$\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)} = \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)},$$

and then rewrite it as,

$$\begin{aligned} \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)} &= \\ \frac{(f(x+h) - f(x))g(x) - f(x)(g(x+h) - g(x))}{g(x+h)g(x)}. \end{aligned}$$

Now, let $h \rightarrow 0$ and use Table 2.4, d) and e), the continuity of g at x , and the definition of the derivatives.

Suggested Homework Set 6. Problems 1, 3, 6, 8, 18

3.3 The Chain Rule

This section is about a method that will enable you to find the derivative of complicated looking expressions, with some speed and simplicity. After a few examples you'll be using it without much thought ... It will become very natural. Many examples in nature involve variables which depend upon other variables. For example, the speed of a car depends on the amount of gas being injected into the carburetor, and this, in turn depends on the diameter of the injectors, etc. In this case we could ask the question: "How does the speed change if we vary the size of the injectors only?" and leave all the other variables the same. We are then led naturally to a study of the *composition* (not the same as the product), of various functions and their derivatives.

We recall the **composition of two functions**, (see Chapter 1), and the **limit-definition of the derivative** of a given function from Section 3.2. First, let's see if we can discover the form of the Rule that finds the derivative of the composition of two functions in terms of the individual derivatives. That is, we want an explicit Rule for finding

$$\frac{d}{dx}(f \circ g)(x) = \frac{d}{dx}f(g(x)),$$

in terms of f and $g(x)$.

We assume that f and g are both differentiable at some point that we call x_0 (and so g is also *continuous* there). Furthermore, we must assume that the range of g is contained in the domain of f (so that the composition *makes sense*). Now look at the quantity

$$k(x) = f(g(x)),$$

which is just shorthand for this composition. We want to calculate $k'(x_0)$. So, we need to examine the expression

$$\frac{k(x_0 + h) - k(x_0)}{h} = \frac{f(g(x_0 + h)) - f(g(x_0))}{h},$$

and see what happens when we let $h \rightarrow 0$. Okay, now let's **assume that g is not identically a constant function near $x = x_0$** . This means that $g(x) \neq g(x_0)$ for any x in a small interval around x_0 . Now,

$$\begin{aligned} \frac{k(x_0 + h) - k(x_0)}{h} &= \frac{f(g(x_0 + h)) - f(g(x_0))}{h} \\ &= \frac{f(g(x_0 + h)) - f(g(x_0))}{g(x_0 + h) - g(x_0)} \cdot \frac{g(x_0 + h) - g(x_0)}{h}. \end{aligned}$$

As $h \rightarrow 0$, $g(x_0 + h) \rightarrow g(x_0)$ because g is continuous at $x = x_0$. Furthermore,

$$\frac{g(x_0 + h) - g(x_0)}{h} \rightarrow g'(x_0),$$

since g is differentiable at the point $x = x_0$. Lastly,

$$\frac{f(g(x_0 + h)) - f(g(x_0))}{g(x_0 + h) - g(x_0)} \rightarrow f'(g(x_0))$$

since f is differentiable at $x = g(x_0)$ (use definition 3.1 with $x = g(x_0 + h)$ and $a = g(x_0)$ to see this). It now follows by the theory of limits that



$$\begin{aligned}
\lim_{h \rightarrow 0} \frac{k(x_0 + h) - k(x_0)}{h} &= \lim_{h \rightarrow 0} \frac{f(g(x_0 + h)) - f(g(x_0))}{g(x_0 + h) - g(x_0)} \cdot \frac{g(x_0 + h) - g(x_0)}{h}, \\
&= \lim_{h \rightarrow 0} \left(\frac{f(g(x_0 + h)) - f(g(x_0))}{g(x_0 + h) - g(x_0)} \right) \times \\
&\quad \times \lim_{h \rightarrow 0} \left(\frac{g(x_0 + h) - g(x_0)}{h} \right), \\
&= f'(g(x_0)) \cdot g'(x_0), \\
&= k'(x_0).
\end{aligned}$$

In other words we can believe that

$$k'(x_0) = f'(g(x_0)) \cdot g'(x_0),$$

and this is the formula we wanted. It's called the **Chain Rule**.

The Chain Rule: Summary

Let f, g be two differentiable functions with g differentiable at x and $g(x)$ in the domain of f' . Then $y = f \circ g$ is differentiable at x and

$$\frac{d}{dx}(f \circ g)(x) = \frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$$

The Chain Rule also says

$$Df(\square) = f'(\square) D\square,$$

where “Df = df/dx = f'(x)”. You can read this as: “Dee of f of box is f prime box dee box”. We call this the **Box formulation of the Chain Rule**.

Let's see what this means. When the composition $(f \circ g)$ is defined (and the range of g is contained in the domain of f') then $(f \circ g)'$ exists and

$$\begin{aligned}
\frac{d}{dx}(f \circ g)(x) &= \frac{df}{dx}(g(x)) \cdot g'(x) \\
\underbrace{\frac{d}{dx}f(g(x))}_{\text{derivative of composition}} &= \underbrace{f'(g(x)) \cdot g'(x)}_{\text{derivative of } f \text{ at } g(x) \cdot \text{derivative of } g \text{ at } x}
\end{aligned}$$

In other words, the derivative of a composition is found by differentiating the *outside* function first, (here, f), evaluating its derivative, (here f'), at the *inside* function, (here, $g(x)$), and finally multiplying this number, $f'(g(x))$, by the derivative of g at x .

The **Chain Rule** is one of the most useful and important rules in the theory of differentiation of functions as it will allow us to find the derivative of very complicated-looking expressions with ease. For example, using the Chain Rule we'll be able to show that

$$\frac{d}{dx}(x+1)^3 = 3(x+1)^2.$$

Without using the Chain Rule, the alternative is that we have to expand

$$(x+1)^3 = x^3 + 3x^2 + 3x + 1$$

using the Binomial Theorem and then use the Sum Rule along with the Power Rule to get the result which, incidentally, is identical to the stated one since

$$\frac{d}{dx}(x^3 + 3x^2 + 3x + 1) = 3x^2 + 6x + 3 = 3(x+1)^2.$$

An easy way to remember the Chain Rule is as follows:

Replace the symbol “ $g(x)$ ” by our box symbol, \square . Then the Chain Rule says that

$$\underbrace{\frac{d}{dx}f(\square)}_{\text{derivative of a composition}} = \underbrace{f'(\square)}_{\text{derivative of } f \text{ at } \square} \cdot \underbrace{\square'}_{\text{derivative of } \square \text{ at } x}$$

Symbolically, it can be shortened by writing that

$$Df(\square) = f'(\square) D\square, \quad \text{The Chain Rule}$$

where the \square may represent (or even *contain*) any other function(s) you wish. In words, it can be remembered by saying that the

$$\text{Derivative of } f \text{ of Box is } f \text{ prime Box dee-Box}$$

like a famous brand name for “sneakers”, (*i.e.* ‘dee-Box’).

Consequences of the Chain Rule!

It's **NOT TRUE** that

Let g be a differentiable function with $g(x) \neq 0$. Then $\frac{1}{g}$ is differentiable and by the Quotient Rule,

$$Df(g(x)) = f'(x)g'(x),$$

1. $\frac{d}{dx}\left(\frac{1}{g(x)}\right) = \frac{-1}{(g(x))^2} \cdot g'(x)$, or,
2. $\frac{d}{dx}(g(x))^a = a(g(x))^{a-1} \cdot g'(x)$ The Generalized Power Rule

whenever a is a real number and $g(x) > 0$. This Generalized Power Rule follows easily from the Chain Rule, above, since we can let $f(x) = x^a$, $g(x) = \square$. Then the composition $(f \circ g)(x) = g(x)^a = \square^a$. According to the Chain Rule,

$$\frac{d}{dx}f(\square) = f'(\square) \cdot \square'.$$

But, by the ordinary Power Rule, Example 69, we know that $f'(x) = ax^{a-1}$. Okay, now since $f'(\square) = a\square^{a-1}$, and $\square' = g'(x)$, the Chain Rule gives us the result.

An easy way to remember these formulae, once and for all, is by writing

$$\begin{aligned} D\square^{\text{power}} &= \text{power} \cdot \square^{(\text{power})-1} \cdot D\square && \text{Generalized Power Rule} \\ D\left(\frac{1}{\square}\right) &= \frac{-1}{(\square)^2} \cdot D\square, && \text{Reciprocal Rule} \end{aligned}$$



where \square may be some differentiable function of x , and we have used the modern notation “D” for the derivative with respect to x . Recall that the **reciprocal** of *something* is, by definition, “1 divided by that something”.

The Chain Rule can take on different forms. For example, let $y = f(u)$ and assume that the variable u is itself a function of another variable, say x , and we write this as $u = g(x)$. So $y = f(u)$ and $u = g(x)$. So y must be a function of x and it is reasonable to expect that y is a *differentiable* function of x if certain additional conditions on f and g are imposed. Indeed, let y be a differentiable function of u

and let u be a differentiable function of x . Then y is, in fact, a differentiable function of x . Now the question is:

“How does y vary with x ?” The result looks like this ...

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ \text{or} \\ y'(x) &= f'(u) \cdot g'(x)\end{aligned}$$

where we must replace all occurrences of the symbol ‘ u ’ in the above by the symbol ‘ $g(x)$ ’ after the differentiations are made.

Sophie Germain, 1776-1831, was the second of three children of a middle-class Parisian family. Somewhat withdrawn, she never married, and by all accounts lived at home where she worked on mathematical problems with a passion. Of the many stories which surround this gifted mathematician, there is this one ... Upon the establishment of the *École Polytechnique* in 1795, women were not allowed to attend the lectures so Sophie managed to get the lecture notes in mathematics by befriending students. She then had some great ideas and wrote this big essay called a *memoire* and then submitted it (under a male name) to one of the great French mathematicians of the time, **Joseph Lagrange**, 1736-1813, for his advice and opinion. Lagrange found much merit in the work and wished to meet its creator. When he did finally meet her he was delighted that the work had been written by a woman, and went on to introduce her to the great mathematicians of the time. She won a prize in 1816 dealing with the solution of a problem in two-dimensional harmonic motion, yet remained a lone genius all of her life.

Example 76. Let f be defined by $f(x) = 6x^{\frac{1}{2}} + 3$. Find $f'(x)$.

Solution We know $f(x) = 6x^{\frac{1}{2}} + 3$. So, if we let $x = \square$ we get

$$\begin{aligned}f'(x) &= 6 \frac{d}{dx} \square^{1/2} + \frac{d}{dx} 3 \text{ (by Properties (a) and (b))}, \\ &= 6 \cdot \frac{1}{2} \square^{-1/2} + 0, \\ &= 3x^{-1/2} \\ &= \frac{3}{\sqrt{x}}.\end{aligned}$$

Example 77. Let g be defined by $g(t) = t^5 - 4t^3 - 2$. What is $g'(0)$, the derivative of g evaluated at $t = 0$?

Solution

$$\begin{aligned}g'(t) &= \frac{d}{dt}(t^5) - 4 \frac{d}{dt}(t^3) - \frac{d}{dt}2 \text{ (by Property (b))} \\ &= 5t^4 - 4(3)t^2 - 0 \text{ (Power Rule)} \\ &= 5t^4 - 12t^2.\end{aligned}$$

But $g'(0)$ is $g'(t)$ with $t = 0$, right? So, $g'(0) = 5(0)^4 - 12(0)^2 = 0$.

Example 78. Let y be defined by $y(x) = (x^2 - 3x + 1)(2x + 1)$. Evaluate $y'(1)$.

Solution Let $f(x) = x^2 - 3x + 1$, $g(x) = 2x + 1$. Then $y(x) = f(x)g(x)$ and we want $y'(x)$... So, we can use the **Product Rule** (or you can multiply the polynomials out, collect terms and then differentiate each term). Now,

$$\begin{aligned}y'(x) &= f'(x)g(x) + f(x)g'(x) \\ &= (2x - 3 + 0)(2x + 1) + (x^2 - 3x + 1)(2 + 0) \\ &= (2x - 3)(2x + 1) + 2(x^2 - 3x + 1), \text{ so,} \\ \\ y'(1) &= (2(1) - 3)(2(1) + 1) + 2((1)^2 - 3(1) + 1) \\ &= -5.\end{aligned}$$

Example 79. Let y be defined by $y(x) = \frac{x^2 + 4}{x^3 - 4}$. Find the slope of the tangent line to the curve $y = y(x)$ at $x = 2$.

Solution We write $y(x) = \frac{f(x)}{g(x)}$ where $f(x) = x^2 + 4$, $g(x) = x^3 - 4$. We also need $f'(x)$ and $g'(x)$, since the Quotient Rule will come in handy here.

The next Table may be useful as we always need these 4 quantities when using the Quotient Rule:

$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
$x^2 + 4$	$2x$	$x^3 - 4$	$3x^2$

Now, by the Quotient Rule,

$$\begin{aligned} y'(x) &= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \\ &= \frac{2x(x^3 - 4) - (x^2 + 4)(3x^2)}{(x^3 - 4)^2}. \end{aligned}$$

No need to simplify here. We're really asking for $y'(2)$, right? Why? Think "slope of tangent line \Rightarrow derivative". Thus,

$$\begin{aligned} y'(2) &= \frac{4(4) - 8(12)}{16} \\ &= -5 \end{aligned}$$

and the required slope has value -5 .

Example 80. Let $y(x) = x^2 - \frac{6}{x-4}$. Evaluate $y'(0)$.

Solution Now

$$\begin{aligned} y'(x) &= \frac{d}{dx}(x^2) - 6 \frac{d}{dx}\left(\frac{1}{x-4}\right) \\ &\quad \text{(where we used Property (a), the Power Rule, and Consequence 1.)} \\ &= 2x - 6 \left[\frac{-1}{(x-4)^2} \frac{d}{dx}(x-4) \right] \\ &\text{since } \frac{d}{dx}\left(\frac{1}{\square}\right) = \frac{d}{dx}\left(\frac{1}{\square}\right) = \left(\frac{-1}{\square^2}\right) \square' \\ &\quad \text{where } \square = (x-4). \end{aligned}$$

All right, now

$$y'(x) = 2x + \frac{6}{(x-4)^2},$$

and so

$$\begin{aligned} y'(0) &= 2(0) + \frac{6}{(-4)^2}, \\ &= \frac{3}{8}. \end{aligned}$$

Example 81. Let y be defined by $y(x) = \frac{4-x^2}{x^2-2x-3}$. Evaluate $y'(x)$ at $x = 1$.

Solution Write $y(x) = \frac{f(x)}{g(x)}$. We need $y'(1)$, right? OK, now we have the table ...

$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
$4 - x^2$	$-2x$	$x^2 - 2x - 3$	$2x - 2$



$$\begin{aligned}
 y'(1) &= \frac{f'(1)g(1) - f(1)g'(1)}{(g(1))^2} \\
 &= \frac{(-2)(1 - 2 - 3) - (3)(2 - 2)}{(1 - 2 - 3)^2} \\
 &= \frac{8 - 0}{16} \\
 &= \frac{1}{2}.
 \end{aligned}$$

Example 82. Let y be defined by $y(x) = \frac{1.6}{(x+1)^{100}}$. Evaluate $y'(0)$.

Solution Write $y(x) = (1.6)(x+1)^{-100} = 1.6f(x)^{-100}$ where $f(x) = x+1$ (or, replace $f(x)$ by \square). Now use Property (a) and Consequence (2) to find that

$$\begin{aligned}
 y'(x) &= (1.6)(-100)f(x)^{-101} \cdot f'(x) \\
 &= -160(x+1)^{-101} \cdot (1) \\
 &= \frac{-160}{(x+1)^{101}},
 \end{aligned}$$

so $y'(0) = -160$.

EXAMPLES



On the other hand, you could have used Consequence (1) to get the result... For example, write $y(x)$ as $y(x) = \frac{1.6}{f(x)}$ where $f(x) = (x+1)^{100}$. Then

$$\begin{aligned}
 y'(x) &= \frac{-1.6}{(f(x))^2} \cdot f'(x) \quad (\text{by Consequence (1)}) \\
 &= \frac{-1.6}{(x+1)^{200}} (100(x+1)^{99}(1)) \quad (\text{by Consequence (2)}) \\
 &= \frac{-160}{(x+1)^{200}} (x+1)^{99} \\
 &= \frac{-160}{(x+1)^{101}}
 \end{aligned}$$

and so $y'(0) = -160$, as before.

Example 83. Let $y = u^5$ and $u = x^2 - 4$. Find $y'(x)$ at $x = 1$.

Solution Here $f(u) = u^5$ and $g(x) = x^2 - 4$. Now $f'(u) = 5u^4$ by the Power Rule and $g'(x) = 2x$... So,

$$\begin{aligned}
 y'(x) &= f'(u) \cdot g'(x) \\
 &= 5u^4 \cdot 2x \\
 &= 10xu^4 \\
 &= 10x(x^2 - 4)^4, \quad \text{since } u = x^2 - 4.
 \end{aligned}$$

At $x = 1$ we get

$$\begin{aligned}
 y'(1) &= 10(1)(-3)^4 \\
 &= 810.
 \end{aligned}$$

Since this value is 'large' for a slope the actual tangent line is very 'steep', close to 'vertical' at $x = 1$.

SHORTCUT

Write $y = \square^5$, then $y' = 5 \square^4 \square'$, by the Generalized Power Rule. Replacing the \square by $x^2 - 4$ we find, $y' = 5 (x^2 - 4)^4 2x = 10x(x^2 - 4)^4$, as before.



The point is, you don't have to memorize another formula. The "Box" formula basically gives all the different variations of the Chain Rule.

Example 84. Let $y = u^3$ and $u = (x^2 + 3x + 2)$. Evaluate $y'(x)$ at $x = 0$ and interpret your result geometrically.

Solution The Rule of Thumb is:

Whenever you see a function raised to the power of some number (NOT a variable), then put everything "between the outermost parentheses", so to speak, in a box, \square . The whole thing then looks like just a box raised to some power, and you can use the box formulation of the Chain Rule on it.

Chain Rule approach: Write $y = u^3$ where $u = x^2 + 3x + 2$. OK, now $y = f(u)$ and $u = g(x)$ where $f(u) = u^3$ and $g(x) = x^2 + 3x + 2$. Then the Chain Rule gives

$$\begin{aligned} y'(x) &= f'(u)g'(x) \\ &= 3u^2 \cdot (2x + 3) \\ &= 3(x^2 + 3x + 2)^2 (2x + 3). \end{aligned}$$

Since $u = x^2 + 3x + 2$, we have to replace each u by the original $x^2 + 3x + 2$. Don't worry, you don't have to simplify this. Finally,

$$\begin{aligned} y'(0) &= 3(3)(2)^2 \\ &= 36 \end{aligned}$$

and this is the slope of the tangent line to the curve $y = y(x)$ at $x = 0$.

Power Rule/Box approach: Write $y = \square^3$ where $\square = x^2 + 3x + 2$ and $a = 3$. Then

$$\begin{aligned} y'(x) &= 3\square^2 \cdot \square' \\ &= 3(x^2 + 3x + 2)^2(2x + 3) \end{aligned}$$

and so $y'(0) = 36$, as before.

Example 85. Let y be defined by $y(x) = (x + 2)^2(2x - 1)^4$. Evaluate $y'(-2)$.

Solution We have a product and some powers here. So we expect to use a combination of the Product Rule and the Power Rule. OK, we let $f(x) = (x + 2)^2$ and $g(x) = (2x - 1)^4$, use the Power Rule on f, g , and make the table:

$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
$(x + 2)^2$	$2(x + 2)$	$(2x - 1)^4$	$4(2x - 1)^3(2)$.

Using the Product Rule,

$$\begin{aligned} y'(x) &= f'(x)g(x) + f(x)g'(x) \\ &= 2(x + 2)(2x - 1)^4 + 8(x + 2)^2(2x - 1)^3 \end{aligned}$$

Finally, it is easy to see that $y'(-2) = 0$.

Example 86. Find an expression for the derivative of $y = \sqrt{f(x)}$ where $f(x) > 0$ is differentiable.

Solution OK, this ‘square root’ is really a power so we think “Power Rule”. We can speed things up by using boxes, so write $\square = f(x)$. Then, the Generalized Power Rule gives us,

$$\begin{aligned} y'(x) &= \frac{d}{dx} \square^{\frac{1}{2}} \\ &= \frac{1}{2} \square^{\frac{1}{2}-1} \cdot \square' \quad (\text{PowerRule}) \\ &= \frac{1}{2} \square^{-\frac{1}{2}} \square' \\ &= \frac{1}{2 \square^{\frac{1}{2}}} \square' \\ &= \frac{\square'}{2\sqrt{\square}} \\ &= \frac{f'(x)}{2\sqrt{f(x)}} \end{aligned}$$

SNAPSHOTS

Example 87. $f(x) = (x^{2/3} + 1)^2$, $f'(x)$?

Solution Let $\square = (x^{2/3} + 1) = x^{2/3} + 1$. So, $f(x) = \square^2$ and

$$\begin{aligned} f'(x) &= \overbrace{(2) \cdot (x^{2/3} + 1)^1}^{D(\square^2)} \cdot \overbrace{(2/3) \cdot x^{-1/3}}^{D(\square)} \\ &= \frac{4}{3} \cdot (x^{2/3} + 1) \cdot x^{-1/3} \\ &= \frac{4}{3} \cdot (x^{1/3} + x^{-1/3}). \end{aligned}$$



Example 88. $f(x) = \sqrt{\sqrt{x} + 1}$. Evaluate $f'(x)$.

Solution Let $\square = (\sqrt{x} + 1) = x^{1/2} + 1$. Then $f(x) = \sqrt{\square}$ so

$$\begin{aligned} f'(x) &= \overbrace{(1/2) \cdot (x^{1/2} + 1)^{-1/2}}^{D(\sqrt{\square})} \cdot \overbrace{(1/2) \cdot x^{-1/2}}^{D(\square)} \\ &= \frac{1}{4} \cdot (x^{1/2} + 1)^{-1/2} \cdot x^{-1/2} \\ &= \frac{1}{4\sqrt{x} \cdot (1 + \sqrt{x})} \end{aligned}$$

Example 89. $f(x) = \frac{\sqrt{x}}{\sqrt{1+x^2}}$. Find $f'(1)$.

Solution Simplify this first. Note that

$$\frac{\sqrt{x}}{\sqrt{1+x^2}} = \sqrt{\frac{x}{1+x^2}} = \square^{1/2}$$

where $\square = \frac{x}{1+x^2}$. So $f(x) = \sqrt{\square}$, and

$$\begin{aligned} f'(x) &= \overbrace{(1/2) \cdot \left(\frac{x}{1+x^2}\right)^{-1/2}}^{D(\sqrt{\square})} \cdot \overbrace{\frac{(x^2+1) \cdot 1 - (x) \cdot (2x)}{(1+x^2)^2}}^{D(\square)} \\ &= (1/2) \cdot \left(\frac{x}{1+x^2}\right)^{-1/2} \cdot \frac{1-x^2}{(1+x^2)^2} \\ &= \frac{x^{-1/2} \cdot (1-x^2)}{2(1+x^2)^{3/2}}, \end{aligned}$$

where we used the Generalized Power Rule to get $D(\sqrt{\square})$ and the Quotient Rule to evaluate $D(\square)$. So, $f'(1) = 0$.

Example 90. $f(x) = \pi \cdot \left(\frac{1}{x}\right)^{-2.718}$, where $\pi = 3.14159\dots$. Find $f'(1)$.

Solution Simplify this first, in the sense that you can turn negative exponents into positive ones by taking the reciprocal of the expression, right? In this case, note that $(1/x)^{-2.718} = x^{2.718}$. So the question now asks us to find the derivative of $f(x) = \pi \cdot x^{2.718}$. The Power Rule gives us $f'(x) = (2.718) \cdot \pi \cdot x^{1.718}$. So, $f'(1) = (2.718) \cdot \pi = 8.53882$.

Example 91. Find an expression for the derivative of $y = f(x^3)$ where f is differentiable.

Solution This looks mysterious but it really isn't. If you don't see an 'x' for the variable, replace all the symbols between the outermost parentheses by ' \square '. Then $y = f(\square)$ and you realize quickly that you need to differentiate a composition of two functions. This is where the Chain Rule comes into play. So,

$$\begin{aligned} y'(x) &= Df(\square) \\ &= f'(\square) \cdot D\square \\ &= f'(x^3) \cdot \frac{d}{dx}(x^3) \text{ (because } \square = x^3\text{)} \\ &= f'(x^3) \cdot 3x^2 \end{aligned}$$

So, we have shown that any function f for which $y = f(x^3)$ has a derivative $y'(x) = 3x^2 f'(x^3)$ which is the desired expression. Remember that $f'(x^3)$ means that you find the derivative of f , and every time you see an x you replace it by x^3 .

OK, but what does this $f'(x^3)$ really mean?

Let's look at the function f , say, defined by $f(x) = (x^2 + 1)^{10}$. Since $f(\square) = (\square^2 + 1)^{10}$ it follows that $f(x^3) = ((x^3)^2 + 1)^{10} = (x^6 + 1)^{10}$, where we replaced \square by x^3 (or you put x^3 IN the box, remember the *Box method?*).

The point is that this new function $y = f(x^3)$ has a derivative given by

$$y'(x) = 3x^2 \cdot f'(x^3),$$

which means that we find $f'(x)$, replace each one of the x 's by x^3 , and simplify (as much as possible) to get $y'(x)$. Now, we write $f(\square) = \square^{10}$, where $\square = x^2 + 1$. The Generalized Power Rule gives us

$$\begin{aligned} f'(x) &= D(\square^{10}) = (10)\square^9 (d\square) = (10)(x^2 + 1)^9 (2x), \\ &= (20x)(x^2 + 1)^9. \text{ So,} \\ y'(x) &= 3x^2 \cdot f'(x^3) = (3x^2) (20 x^3) (x^6 + 1)^9 \\ &= 60 x^5 (x^6 + 1)^9. \end{aligned}$$

The Generalized Power Rule takes the form

$$\frac{d}{dx} \square^r = r \square^{r-1} \frac{d}{dx} \square,$$

where the box symbol, \square , is just another symbol for some differentiable function of x .

Example 91 represents a, so-called, **transformation of the independent variable** (since the original 'x' is replaced by 'x³') and such transformations appear within the context of **differential equations** where they can be used to simplify very difficult looking differential equations to simpler ones.

EXAMPLES



A Short Note on Differential Equations

More importantly though, examples like the last one appear in the study of **differential equations** which are equations which, in some cases called **linear**, look like polynomial equations

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0 = 0$$

and each x is replaced by a symbol ' $D = \frac{d}{dx}$ ' where the related symbol $D^n = \frac{d^n}{dx^n}$ means the operation of taking the n^{th} derivative. This symbol, $D = \frac{d}{dx}$, has a special name: it's called a **differential operator** and its domain is a collection of functions while its range is also a collection of functions. In this sense, the concept of an operator is more general than that of a function. Now, the symbol D^2 is the derivative of the derivative and it is called the **second derivative**; the derivative of the second derivative is called the **third derivative** and denoted by D^3 , and so on. The coefficients a_m above are usually given functions of the independent variable, x .

Symbolically, we write these **higher-order derivatives** using Leibniz's notation:

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \frac{dy}{dx} = y''(x)$$

for the **second derivative of y**,

$$\frac{d^3 y}{dx^3} = \frac{d}{dx} \frac{d^2 y}{dx^2} = y'''(x)$$

for the **third derivative of y**, and so on. These higher order derivatives are very useful in determining the graphs of functions and in studying a 'function's behavior'. We'll be seeing them soon when we deal with **curve sketching**.

Example 92.

Let f be a function with the property that $f'(x) + f(x) = 0$ for every x . We'll meet such functions later when we discuss Euler's constant, $e \approx 2.71828\dots$, and the corresponding exponential function.

Show that the new function y defined by $y(x) = f(x^3)$ satisfies the differential equation $y'(x) + 3x^2 y(x) = 0$.

Solution We use Example 91. We already know that, by the Chain Rule, $y(x) = f(x^3)$ has its derivative given by $y'(x) = 3x^2 \cdot f'(x^3)$. So,

$$\begin{aligned} y'(x) + 3x^2 y(x) &= 3x^2 \cdot f'(x^3) + 3x^2 f(x^3) \\ &= 3x^2 (f'(x^3) + f(x^3)) \end{aligned}$$

But $f'(x) + f(x) = 0$ means that $f'(\Delta) + f(\Delta) = 0$, right? (Since it is true for any 'x' and so for any symbol ' Δ '). Replacing Δ by x^3 gives $f'(x^3) + f(x^3) = 0$ as a consequence, and the conclusion now follows ...

The function y defined by $y(x) = f(x^3)$, where f is any function with $f'(x) + f(x) = 0$, satisfies the equation $y'(x) + 3x^2 y(x) = 0$.

Example 93.

Find the second derivative $f''(x)$ given that $f(x) = (2x + 1)^{101}$. Evaluate $f''(-1)$.

Solution We can just use the Generalized Power Rule here. Let $\square = 2x + 1$. Then $\square' = 2$ and so $f'(x) = 101 \cdot \square^{100} \cdot \square' = 101 \cdot \square^{100} \cdot 2 = 202 \cdot \square^{100}$. Doing this one more time, we find $f''(x) = (202) \cdot (100) \cdot 2 \cdot \square^{99} = 40,400 \cdot \square^{99} = 40,400 \cdot (2x+1)^{99}$.

Finally, since $(-1)^{\text{odd number}} = -1$, we see that $f''(-1) = 40,400 \cdot (-1)^{99} = -40,400$.

Example 94. Find the second derivative $f''(x)$ of the function defined by $f(x) = (1 + x^3)^{-1}$. Evaluate $f''(0)$.

Solution Use the Generalized Power Rule again. Let $\square = x^3 + 1$. Then $\square' = 3x^2$ and so $f'(x) = (-1) \cdot \square^{-2} \cdot \square' = (-1) \cdot \square^{-2} \cdot (3x^2) = -(3x^2) \cdot \square^{-2} = -(3x^2) \cdot (1 + x^3)^{-2}$. To find the derivative of THIS function we can use the Quotient Rule. So,

$$\begin{aligned} f''(x) &= -\left\{ \frac{(1+x^3)^2 \cdot (6x) - (3x^2) \cdot (2)(1+x^3)^1(3x^2)}{(1+x^3)^4} \right\} \\ &= \frac{-6x}{(1+x^3)^2} + \frac{18x^4}{(1+x^3)^3} \\ &= \frac{6x \cdot (2x^3 - 1)}{(1+x^3)^3} \end{aligned}$$

It follows that $f''(0) = 0$.



Exercise Set 12.

Find the indicated derivatives.

1. $f(x) = \pi$, $f'(x) = ?$
2. $f(t) = 3t - 2$, $f'(0) = ?$
3. $g(x) = x^{\frac{2}{3}}$, $g'(x) = ?$ at $x = 1$
4. $y(x) = \sqrt{(x-4)^3}$, $y'(x) = ?$
5. $f(x) = \frac{1}{\sqrt{x^5}}$, $f'(x) = ?$
6. $g(t) = \sqrt[3]{t^2 + t - 2}$, $g'(t) = ?$
7. $f(x) = 3x^2$, $\frac{d^2 f}{dx^2} = ?$
8. $f(x) = x(x+1)^4$, $f'(x) = ?$
9. $y(x) = \frac{x^2 - x + 3}{\sqrt{x}}$, $y'(1) = ?$
10. $y(t) = (t+2)^2(t-1)$, $y'(t) = ?$
11. $f(x) = 16x^2(x-1)^{\frac{2}{3}}$, $f'(x) = ?$
12. $y(x) = (2x+3)^{105}$, $y'(x) = ?$
13. $f(x) = \sqrt{x} + 6$, $f'(x) = ?$
14. $f(x) = x^3 - 3x^2 + 3x - 1$, $f'(x) = ?$
15. $y(x) = \frac{1}{x} + \sqrt{x^2 - 1}$, $y'(x) = ?$
16. $f(x) = \frac{1}{1 + \sqrt{x}}$, $f''(x) = ?$
17. $f(x) = (x-1)^2 + (x-2)^3$, $f''(0) = ?$

18. $y(x) = (x + 0.5)^{-1.324}$, $y''(x) = ?$
19. Let f be a differentiable function for every real number x . Show that $\frac{d}{dx}f(x^2) = 2xf'(x^2)$.
20. Let g be a differentiable function for every x with $g(x) > 0$. Show that $\frac{d}{dx}\sqrt[3]{g(x)} = \frac{g'(x)}{3\sqrt[3]{g(x)^2}}$.
21. Let f be a function with the property that f is differentiable and

$$f'(x) + f(x) = 0.$$

Show that $y = f(x^2)$ satisfies the differential equation

$$y'(x) + 2xy(x) = 0.$$

22. Let $y = f(x)$ and assume f is differentiable for each x in $(0, 1)$. Assume that f has an inverse function, F , defined on its range, so that $f(F(x)) = x$ for every $x, 0 < x < 1$. Show that F has a derivative satisfying the equation $F'(x) = \frac{1}{f'(F(x))}$ at each $x, 0 < x < 1$.

(Hint: Differentiate both sides of $f(F(x)) = x$.)

23. Let $y = t^3$ and $t = \sqrt{u} + 6$. Find $\frac{dy}{du}$ when $u = 9$.
24. Find the equation of the tangent line to the curve $y = (x^2 - 3)^8$ at the point $(x, y) = (2, 1)$.
25. Given $y(x) = f(g(x))$ and that $g'(2) = 1, g(2) = 0$ and $f'(0) = 1$. What is the value of $y'(2)$?
26. Let $y = r + \frac{2}{r}$ and $r = 3t - 2\sqrt{t}$. Use the Chain Rule to find an expression for $\frac{dy}{dt}$.

27. **Hard.** Let $f(x) = \sqrt{x + \sqrt{x}}$. Evaluate $f'(9)$. If $x = t^2$, what is $\frac{df}{dt}$?

28. Use the definition of $\sqrt{x^2}$ as $\sqrt{x^2} = |x|$ for each x , to show that the function $y = |x|$ has a derivative whenever $x \neq 0$ and $y'(x) = \frac{x}{|x|}$ for $x \neq 0$.

29. **Hard** Show that if f is differentiable at the point $x = x_0$ then f is continuous at $x = x_0$. (Hint: You can try a proof by 'contradiction', that is you assume the conclusion is false and, using a sequence of logically correct arguments, you deduce that the original claim is false as well. Since, generally speaking, a statement in mathematics cannot be both true **and** false, (aside from **undecidable statements**) it follows that the conclusion has to be true. So, assume f is not continuous at $x = x_0$, and look at each case where f is discontinuous (unequal one-sided limits, function value is infinite, etc.) and, **in each case**, derive a contradiction.)

Alternately, you can prove this directly using the methods in the *Advanced Topics* chapter. See the Solution Manual for yet another method of showing this.

Suggested Homework Set 7. Do problems 4, 10, 16, 23, 25, 27

Web Links

For more information and applications of the Chain Rule see:

people.hofstra.edu/Stefan_Waner/tutorials3/unit4-2.html

www.math.hmc.edu/calculus/tutorials/chainrule/

www.ugrad.math.ubc.ca/coursedoc/math100/notes/derivative/chainap.html

math.ucdavis.edu/~kouba/CalcOneDIRECTORY/chainrulesoldirectory/
(contains more than 20 solved examples)

NOTES:

SAMPLE

3.4 Implicit Functions and Their Derivatives

You can imagine the variety of different functions in mathematics. So far, all the functions we've encountered had this one thing in common: You could write them as $y = f(x)$ (or $x = F(y)$) in which case you know that x is the independent variable and y is the dependent variable. We know which variable is which. Sometimes it is not so easy to see "which variable is which" especially if the function is written as, say,

$$x^2 - 2xy + \tan(xy) - 2 = 0.$$

What do we do? Can we solve for either one of these variables at all? And if we can, do we solve for x in terms of y , or y in terms of x ? Well, we don't always "have to" solve for any variable here, and we'll still be able to find the derivative so long as we agree on which variable x , or y is the independent one. Actually, Newton was the first person to perform an *implicit differentiation*. Implicit functions appear very often in the study of **general solutions** of differential equations. We'll see later on that the general solution of a *separable* differential equation is usually given by an implicit function. Other examples of implicit functions include the equation of closed curves in the xy -plane (circles, squares, ellipses, etc. to mention a few of the common ones).

In his *Method of Fluxions*, (1736), Isaac Newton was one of the first to use the procedure of this section, namely, *implicit differentiation*. He used his brand of derivatives though, things he called *fluxions* and he got into big trouble because they weren't well defined. In England, one famous philosopher by the name of **Bishop Berkeley** criticized Newton severely for his inability to actually explain what these fluxions really were. Nevertheless, Newton obtained the right answers (according to our calculations). *What about Leibniz?* Well, even Leibniz got into trouble with his, so-called, *differentials* because he really couldn't explain this stuff well, either! His nemesis in this case was one **Bernard Nieuwentijt** of Amsterdam (1694). Apparently, neither Berkeley nor Nieuwentijt could put the Calculus on a rigorous foundation either, so, eventually the matter was dropped. It would take another 150 years until Weierstrass and others like him would come along and make sense out of all this Calculus business with rigorous definitions (like those in the Advanced Topics chapter).

Review

You should review the **Chain Rule** and the **Generalized Power Rule** in the preceding section. A mastery of these concepts and the usual rules for differentiation will make this section much easier to learn.

We can call our usual functions **explicit** because their values are given explicitly (i.e., we can write them down) by solving one of the variables in terms of the other. This means that for each value of x there is only one value of y . But this is the same as saying that y is a function of x , right? An equation involving two variables, say, x, y , is said to be an **explicit relation** if one can solve for y (or x) uniquely in terms of x (or y).

Example 95. For example, the equation $2y = 2x^6 - 4x$ is an explicit relation because we can easily solve for y in terms of x . In fact, it reduces to the rule $y = x^6 - 2x$ which defines a function $y = f(x)$ where $f(x) = x^6 - 2x$. Another example is given by the function y whose values are given by $y(x) = x + \sqrt{x}$ whose values are easily calculated: Each value of x gives a value of $y = x + \sqrt{x}$ and so on, and y can be found directly using a calculator. Finally, $3x + 6 - 9y^2 = 0$ also defines an explicit relation because now we can solve for x in terms of y and find $x = 3y^2 - 2$.

In the same spirit we say that an equation involving two variables, say, x, y , is said to be an **implicit relation** if it is *not explicit*.

Example 96. For example, the relation defined by the rule $y^5 + 7y = x^3 \cos x$ is implicit. Okay, you can isolate the y , but what's left still involves y and x , right? The equation defined by $x^2 y^2 + 4 \sin(xy) = 0$ also defines an implicit relation.

Such implicit relations are useful because they usually define a *curve* in the xy -plane, a curve which is not, generally speaking, the graph of a function. In fact, you can probably believe the statement that a "closed curve" (like a circle, ellipse, etc.) cannot be the graph of a function. Can you show why? For example, the circle defined by the implicit relation $x^2 + y^2 = 4$ is not the graph of a unique function,

(think of the, so-called, Vertical Line Test for functions).

So, if y is ‘obscured’ by some complicated expression as in, say, $x^2 - 2xy + \tan(xy) - 2 = 0$, then it is not easy to solve for ‘ y ’ given a value of ‘ x ’; in other words, it would be very difficult to isolate the y ’s on one side of the equation and group the x ’s together on the other side. In this case y is said to be defined implicitly or y is an **implicit function** of x . By the same token, x may be considered an implicit function of y and it would equally difficult to solve for x as a function of y . Still, it is possible to draw its graph by looking for those points x, y that satisfy the equation, see Figure 42.

Other examples of functions defined implicitly are given by:

$$(x - 1)^2 + y^2 = 16 \quad \text{A circle of radius 4 and center at } (1, 0).$$

$$\frac{(x-2)^2}{9} + \frac{(y-6)^2}{16} = 1 \quad \text{An ellipse ‘centered’ at } (2, 6).$$

$$(x - 3)^2 - (y - 4)^2 = 5 \quad \text{A hyperbola.}$$

OK, so how do we find the derivative of such ‘functions’ defined implicitly?

1. Assume, say y , is a differentiable function of x , (or x is a differentiable function of y).
2. Write $y = y(x)$ (or $x = x(y)$) to show the dependence of y on x , (even though we really don’t know what it ‘looks like’).
3. Differentiate the relation/expression which defines y **implicitly** with respect to x (or y - *this expression is a curve in the xy -plane.*)
4. Solve for the derivative $\frac{dy}{dx}$ **explicitly**, yes, *explicitly!*

Note: It can be shown that the 4 steps above always **produce an expression for $\frac{dy}{dx}$ which can be solved explicitly**. In other words, even though y is given implicitly, the function $\frac{dy}{dx}$ is explicit, that is, given a point $P(x, y)$ on the defining curve described in (3) *we can actually solve* for the term $\frac{dy}{dx}$.

This note is based on the assumption that we already know that y can be written as a differentiable function of x . This assumption isn’t obvious, and involves an important result called the *Implicit Function Theorem* which we won’t study here but which can be found in books on *Advanced Calculus*. One of the neat things about this implicit function theorem business is that it tells us that, under certain conditions, *we can always solve for the derivative dy/dx* even though we can’t solve for y ! Amazing, isn’t it?

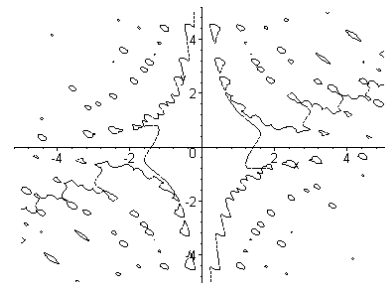
Example 97. Find the derivative of y with respect to x , that is, $\frac{dy}{dx}$, when x, y are related by the expression $xy - y^2 = 6$.

Solution We assume that y is a differentiable function of x so that we can write $y = y(x)$ and y is differentiable. Then the relation between x, y above really says that

$$xy(x) - y(x)^2 = 6.$$

OK, since this is true for all x under consideration (the x ’s were not specified, so don’t worry) it follows that we can take the derivative of both sides and still get equality, *i.e.*

$$\frac{d}{dx}(xy(x) - y(x)^2) = \frac{d}{dx}(6).$$



An approximate plot of the implicit relation

$$x^2 - 2xy + \tan(xy) - 2 = 0.$$

This plot fails the “vertical line test” and so it cannot be the graph of a function.

Figure 42.

Now, $\frac{d}{dx}(6) = 0$ since the derivative of a constant is always 0 and

$$\begin{aligned}\frac{d}{dx}(xy(x) - y(x)^2) &= \frac{d}{dx} xy(x) - \frac{d}{dx} y(x)^2 \\ &= \left[x \frac{dy}{dx} + y(x) \frac{d(x)}{dx} \right] - 2y(x) \frac{dy}{dx} \\ &= [x - 2y(x)] \frac{dy}{dx} + y(x)\end{aligned}$$

where we used a combination of the Product Rule and the Generalized Power Rule (see Example 86 for a similar argument). So, we have

$$[x - 2y(x)] \frac{dy}{dx} + y(x) = \frac{d}{dx}(6) = 0,$$

and solving for $\frac{dy}{dx}$ we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{-y(x)}{x - 2y(x)} \\ &= \frac{y(x)}{2y(x) - x}.\end{aligned}$$

OK, so we have found $\frac{dy}{dx}$ in terms of x and $y(x)$ that is, since $y = y(x)$,

$$\frac{dy}{dx} = \frac{y}{2y - x}$$

provided x and y are related by the original expression $xy - y^2 = 6$ which describes a curve in the xy -plane. This last display then describes the values of the derivative $y'(x)$ along this curve for a given point $P(x,y)$ on it.

To find the slope of the tangent line to a point $P(x_0, y_0)$ on this curve we calculate

$$\frac{dy}{dx} = \frac{y_0}{2y_0 - x_0}$$

where $x_0 y_0 - y_0^2 = 6$, that's all. So, for example the point $(7, 1)$ is on this curve because $x_0 = 7, y_0 = 1$ satisfies $x_0 y_0 - y_0^2 = 6$. You see that the derivative at this point $(7, 1)$ is given by

$$\frac{dy}{dx} = \frac{1}{2(1) - 7} = -\frac{1}{5}$$

Example 98. Let $x^3 + 7x = y^3$ define an implicit relation for x in terms of y . Find $x'(1)$.

Solution We'll assume that x can be written as a differentiable function of y . We take the derivative of both sides (with respect to y this time!). We see that

$$3x^2 \frac{dx}{dy} + 7 \frac{dx}{dy} = 3y^2,$$

since

$$\frac{d}{dy} x^3 = 3x^2 \frac{dx}{dy}$$

by the Generalized Power Rule. We can now solve for the expression dx/dy and find a formula for the derivative, namely,

$$\frac{dx}{dy} = \frac{3y^2}{3x^2 + 7}.$$



Now we can find the derivative easily at any point (x, y) on the curve $x^3 + 7x = y^3$. For instance, the derivative at the point $(1, 2)$ on this curve is given by substituting the values $x = 1, y = 2$ in the formula for the derivative just found, so that

$$\frac{dx}{dy} = \frac{(3)(2)^2}{(3)(1)^2 + 7} = \frac{12}{10} = \frac{6}{5}.$$

For a geometrical interpretation of this derivative, see Figure 43 on the next page.

Example 99. Find the slope of the tangent line to the curve $y = y(x)$ given implicitly by the relation $x^2 + 4y^2 = 5$ at the point $(-1, 1)$.

Solution First, you should always check that the given point $(-1, 1)$ is **on** this curve, otherwise, there is nothing to do! Let $x_0 = -1, y_0 = 1$ and $P(x_0, y_0) = P(-1, 1)$. We see that $(-1)^2 + 4(1)^2 = 5$ and so the point $P(-1, 1)$ is on the curve.

Since we want the slope of a tangent line to the curve $y = y(x)$ at $x = x_0$, we need to find its derivative $y'(x)$ and evaluate it at $x = x_0$.

OK, now

$$\begin{aligned} \frac{d}{dx}(x^2 + 4y(x)^2) &= \frac{d}{dx}(5) \\ 2x + 4 \underbrace{\frac{d}{dx}(y(x)^2)} &= 0 \\ 2x + 4 \cdot 2y(x) \cdot y'(x) &= 0 \\ y'(x) &= -\frac{2x}{8y(x)} = -\frac{x}{4y(x)}, \quad (\text{if } y(x) \neq 0) \end{aligned}$$

and this gives the value of the derivative, $y'(x)$ at any point (x, y) on the curve, that is $y' = -\frac{x}{4y}$ where (x, y) is on the curve (remember $y = y(x)$). It follows that at $(-1, 1)$, this derivative is equal to

$$y'(-1) = (-1) \frac{(-1)}{4(1)} = \frac{1}{4}.$$

Example 100. A curve in the xy -plane is given by the set of all points (x, y) satisfying the equation $y^5 + x^2y^3 = 10$. Find $\frac{dx}{dy}$ at the point $(x, y) = (-3, 1)$.

Solution Verify that $(-3, 1)$ is, indeed, on the curve. This is true since $1^5 + (-3)^2(1)^3 = 10$, as required. Next, we assume that $x = x(y)$ is a differentiable function of y . Then

$$\frac{d}{dy}(y^5 + x^2y^3) = \frac{d}{dy}(10)$$

$$5y^4 + 2x(y) x'(y) y^3 + x(y)^2(3y^2) = 0,$$

(where we used the Power Rule and the Product Rule). Isolating the term $x'(y) = \frac{dx}{dy}$ gives us the required derivative,

$$\frac{dx}{dy} = \frac{-3x^2y^2 - 5y^4}{2xy^3}.$$

When $x = -3, y = 1$ so,

$$\frac{dx}{dy} = \frac{-27 - 5}{-6} = \frac{16}{3}.$$



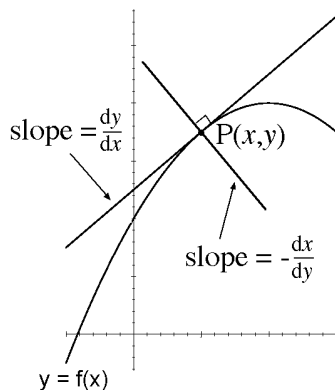


Figure 43. Geometric meaning of $\frac{dx}{dy}$

Remark If we were to find $\frac{dy}{dx}$ at $(-3, 1)$ we would obtain $\frac{3}{16}$; the **reciprocal** of $\frac{dx}{dy}$. Is this a coincidence? No. It turns out that if y is a differentiable function of x and x is a differentiable function of y then their derivatives are related by the relation

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

if $\frac{dx}{dy} \neq 0$, at the point $P(x, y)$ under investigation. This is another consequence of the Implicit Function Theorem and a result on Inverse Functions.

O.K., we know what $\frac{dy}{dx}$ means geometrically, right? Is there some geometric meaning for $\frac{dx}{dy}$? Yes, the value of $\frac{dx}{dy}$ at $P(x, y)$ on the given curve is equal to the **negative of the slope of the line perpendicular to the tangent line through P**. For example, the equation of the tangent line through $P(-3, 1)$ in Example 100 is given by $y = (3x + 25)/16$, while the equation of the line perpendicular to this tangent line and through P is given by $y = (16x + 51)/3$. This last (perpendicular) line is called the **normal line** through P . See Figure 43.

Exercise Set 13.

Use implicit differentiation to find the required derivative.

1. $x^2 + xy + y^2 = 1$, $\frac{dy}{dx}$ at $(1, 0)$
2. $2xy^2 - y^4 = x^3$, $\frac{dy}{dx}$ and $\frac{dx}{dy}$
3. $\sqrt{x+y} + xy = 4$, $\frac{dy}{dx}$ at $(16, 0)$
4. $x - y^2 = 4$, $\frac{dy}{dx}$
5. $x^2 + y^2 = 9$, $\frac{dy}{dx}$ at $(0, 3)$

Find the equation of the tangent line to the given curve at the given point.

6. $2y^2 - x^2 = 1$, at $(-1, -1)$
7. $2x = xy + y^2$, at $(1, 1)$
8. $x^2 + 2x + y^2 - 4y - 24 = 0$, at $(4, 0)$
9. $(x + y)^3 - x^3 - y^3 = 0$, at $(1, -1)$

Suggested Homework Set 8. Problems 1, 2, 4, 7, 9

Web Links

For more examples on implicit differentiation see:

www.math.ucdavis.edu/~kouba/CalcOneDIRECTORY/implicitdiffdirectory/
www.ugrad.math.ubc.ca/coursedoc/math100/notes/derivative/implicit.html
 (the above site requires a Java-enabled browser)

NOTES:

3.5 Derivatives of Trigonometric Functions

Our modern world runs on electricity. In these days of computers, space travel and robots we need to have a secure understanding of the basic laws of electricity and its uses. In this realm, electric currents both alternating (as in households), and direct (as in a flashlight battery), lead one to the study of sine and cosine functions and their interaction. For example, how does an electric current vary over time? We need its ‘rate of change’ with respect to time, and this can be modeled using its derivative.



An integrated circuits board

In another vein, so far we’ve encountered the derivatives of many different types of functions; polynomials, rational functions, roots of every kind, and combinations of such functions. In many applications of mathematics to physics and other physical and natural sciences we need to study combinations of trigonometric functions and other ‘changes’, the shapes of their graphs and other relevant data. In the simplest of these applications we can mention the study of **wave phenomena**. In this area we model incoming or outgoing waves in a fluid (such as a lake, tea, coffee, etc.) as a combination of sine and cosine waves, and then study how these waves change over time. Well, to study how these waves change over time we need to study their derivatives, right? This, in turn, means that we need to be able to find the derivatives of the sine and cosine functions and that’s what this section is all about.

There are two fundamental limits that we need to recall here from an earlier chapter, namely

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1, \quad (3.2)$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0, \quad (3.3)$$

Let’s also recall some fundamental trigonometric identities in Table 3.4.

All angles, A , B and x are in **radians** in the Table above, and this is customary in calculus.

Recall that 1 radian = $\frac{180}{\pi}$ degrees.

$$I1 \quad \sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

$$I2 \quad \cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

$$I3 \quad \sin^2 x + \cos^2 x = 1$$

$$I4 \quad \sec^2 x - \tan^2 x = 1$$

$$I5 \quad \csc^2 x - \cot^2 x = 1$$

$$I6 \quad \cos 2x = \cos^2 x - \sin^2 x$$

$$I7 \quad \sin 2x = 2 \sin x \cos x$$

$$I8 \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

$$I9 \quad \sin^2 x = \frac{1 - \cos 2x}{2}$$

Table 3.4: Useful Trigonometric Identities

The first result is that the derivative of the sine function is the cosine function, that is,

$$\boxed{\frac{d}{dx} \sin x = \cos x.}$$

This is not too hard to show; for example, assume that $h \neq 0$. Then

$$\begin{aligned} \frac{\sin(x+h) - \sin x}{h} &= \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}, \quad (\text{by I1}) \\ &= \sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h}, \quad (\text{re-arranging terms}) \end{aligned}$$

Now we use a limit theorem from Chapter 2: Since the last equation is valid for each $h \neq 0$ we can pass to the limit and find

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} &= \sin x \lim_{h \rightarrow 0} \left(\frac{\cos h - 1}{h} \right) + \cos x \lim_{h \rightarrow 0} \left(\frac{\sin h}{h} \right) \\ &= (\sin x) \cdot (0) + (\cos x) \cdot (1), \quad (\text{by (3.3) and (3.2)}) \\ &= \cos x. \end{aligned}$$



A similar derivation applies to the next result;

$$\boxed{\frac{d}{dx} \cos x = -\sin x}$$

For example,

$$\begin{aligned} \frac{\cos(x+h) - \cos x}{h} &= \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}, \quad (\text{by I2}) \\ &= \cos x \frac{\cos h - 1}{h} - \sin x \frac{\sin h}{h}, \quad (\text{re-arranging terms}) \end{aligned}$$

As before, since this last equation is valid for each $h \neq 0$ we can pass to the limit and find

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} &= \cos x \lim_{h \rightarrow 0} \left(\frac{\cos h - 1}{h} \right) - \sin x \lim_{h \rightarrow 0} \left(\frac{\sin h}{h} \right) \\ &= (\cos x) \cdot (0) - (\sin x)(1), \quad (\text{by (3.3) and (3.2)}) \\ &= -\sin x. \end{aligned}$$

Since these two limits define the derivative of each trigonometric function we get the boxed results, above.

OK, now that we know these two fundamental derivative formulae for the sine and cosine functions we can derive all the other such formulae (for tan, cot, sec, and csc) using basic properties of derivatives.

For example, let's show that

$$\frac{d}{dx} \tan x = \frac{1}{\cos^2 x}$$

or, since $\frac{1}{\cos^2 x} = \sec^2 x$, we get

$$\boxed{\frac{d}{dx} \tan x = \sec^2 x}$$

as well. To see this we use the Quotient Rule and recall that since $\tan x = \frac{\sin x}{\cos x}$,

$$\begin{aligned} \frac{d}{dx} \tan x &= \frac{d}{dx} \left(\frac{\sin x}{\cos x} \right) \quad (\text{by definition}) \\ &= \frac{\cos x \frac{d}{dx}(\sin x) - \left(\frac{d}{dx} \cos x \right) \sin x}{\cos^2 x} \quad (\text{Quotient Rule}) \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \quad (\text{just derived above}) \\ &= \frac{1}{\cos^2 x} \quad (\text{by I3}). \end{aligned}$$



By imitating this argument it's not hard to show that

$$\frac{d}{dx} \cot x = -\frac{1}{\sin^2 x},$$

or, equivalently,

$$\frac{d}{dx} \cot x = -\csc^2 x$$

a formula which we leave to the reader as an **exercise**, as well.

There are two more formulae which need to be addressed, namely, those involving the derivative of the secant and cosecant functions. These are:

$$\begin{aligned} \frac{d}{dx} \sec x &= \sec x \tan x \\ \frac{d}{dx} \csc x &= -\csc x \cot x \end{aligned}$$

Each can be derived using the Quotient Rule. Now armed with these formulae and the Chain Rule we can derive formulae for derivatives of very complicated looking functions, see Table 3.5.

Example 101. Find the derivative of f where $f(x) = \sin^2 x + 6x$.

Solution The derivative of a sum is the sum of the derivatives. So

$$\begin{aligned} f'(x) &= \frac{d}{dx}(\sin x)^2 + \frac{d}{dx}(6x) \\ &= \frac{d}{dx}(\sin x)^2 + 6 \end{aligned}$$

Now let $\square = \sin x$. We want $\frac{d}{dx} \square^2$ so we'll need to use the Generalized Power Rule here... So,

$$\begin{aligned} \frac{d}{dx}(\sin x)^2 &= \frac{d}{dx} \square^2 \\ &= 2 \square^1 \frac{d \square}{dx} \quad (\text{Power Rule}) \\ &= 2 \square \square' \\ &= 2(\sin x)(\cos x), \quad (\text{since } \square' = \cos x) \\ &= \sin 2x, \quad (\text{by I7}) \end{aligned}$$

The final result is $f'(x) = 6 + \sin 2x$.

Example 102.Evaluate $\frac{d}{dx}\sqrt{1+\cos x}$ at $x=0$.

Joseph Louis (Comte de) Lagrange, 1736 -1813, was born in Torino, Italy and died in Paris, France. His main contributions to mathematics were in the fields of analysis where he studied analytical and celestial mechanics, although he excelled in everything that he studied. In 1766, Lagrange became the successor of Euler in the Berlin Academy of Science, and during the next year he was awarded the first of his many prizes for his studies on the irregularities of the motion of the moon. He helped to found the Academy of Science in Torino in 1757, and the École Polytechnique in 1795. He also helped to create the first commission on *Weights and Measures* and was named to the *Légion d'Honneur* by Napoleon and elevated to Count in 1808.

Solution We write $f(x) = \sqrt{1+\cos x}$ and convert the root to a power (always do this so you can use the Generalized Power Rule).

We get $f(x) = (1+\cos x)^{\frac{1}{2}} = \square^{\frac{1}{2}}$ if we set $\square = 1+\cos x$ so that we can put the original function into a more recognizable form. So far we know that

$$f(x) = \sqrt{1+\cos x} = \square^{\frac{1}{2}}$$

So, by the Power Rule, we get

$$f'(x) = \frac{1}{2}\square^{-\frac{1}{2}}\square'$$

where \square' is the derivative of $1+\cos x$ (**without** the root), *i.e.*

$$\begin{aligned}\square' &= \frac{d}{dx}(1+\cos x) \\ &= \frac{d}{dx}(1) + \frac{d}{dx}(\cos x) \\ &= 0 - \sin x \\ &= -\sin x\end{aligned}$$

Combining these results we find

$$\begin{aligned}f'(x) &= \frac{1}{2}(1+\cos x)^{-\frac{1}{2}}(-\sin x) \\ &= -\frac{\sin x}{2\sqrt{1+\cos x}}\end{aligned}$$

after simplification. At $x=0$ we see that

$$\begin{aligned}f'(0) &= -\frac{\sin 0}{2\sqrt{1+\cos 0}} = -\frac{0}{2\sqrt{2}} \\ &= 0\end{aligned}$$

which is what we are looking for.

Example 103.Evaluate $\frac{d}{dx}\left(\frac{\cos x}{1+\sin x}\right)$.

Solution Write $f(x) = \cos x$, $g(x) = 1+\sin x$. We need to find the derivative of the quotient $\frac{f}{g}$ and so we can think about using the Quotient Rule.

Now, recall that

$$\frac{d}{dx}(f/g) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

In our case,

$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
$\cos x$	$-\sin x$	$1+\sin x$	$\cos x$

Combining these results we find, (provided $1 + \sin x \neq 0$),

$$\begin{aligned} \frac{d}{dx} \left(\frac{\cos x}{1 + \sin x} \right) &= \frac{(-\sin x)(1 + \sin x) - (\cos x)(\cos x)}{(1 + \sin x)^2} \\ &= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1 + \sin x)^2} \\ &= -\frac{1 + \sin x}{(1 + \sin x)^2} \quad (\text{by I3}) \\ &= -\frac{1}{1 + \sin x}. \end{aligned}$$

Example 104.

Let the function be defined by $f(t) = \frac{3}{\sin(t)}$. Evaluate $f'(\frac{\pi}{4})$.

Solution OK, we have a constant divided by a function so it looks like we should use the Power Rule (or the Quotient Rule, either way you'll get the same answer). So, let's write $f(t) = 3\Box^{-1}$ where $\Box = \sin(t)$ then

$$f'(t) = (-1) \cdot 3 \cdot \Box^{-2} \Box'$$

by the Generalized Power Rule. But we still need \Box' , right? Now $\Box = \sin(t)$, so $\Box' = \cos t$. Combining these results we find

$$\begin{aligned} f'(t) &= -3(\sin t)^{-2}(\cos t) \\ &= -3\frac{\cos t}{(\sin t)^2}. \end{aligned}$$

Note that this last expression is also equal to $-3\csc t \cot t$. At $t = (\frac{\pi}{4})$, (which is 45 degrees expressed in radians), $\cos(\frac{\pi}{4}) = \sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$ and so

$$\begin{aligned} f'(\frac{\pi}{4}) &= -3\frac{(\frac{1}{\sqrt{2}})}{(\frac{1}{\sqrt{2}})^2} = -3\frac{1}{\sqrt{2}} \cdot \frac{(\sqrt{2})^2}{1}, \\ &= -3\sqrt{2}. \end{aligned}$$

Note: Notice that we could have written $f(t) = \frac{3}{\sin(t)}$ as $f(t) = 3\csc t$ and use the derivative formula for $\csc t$ mentioned above. This would give $f'(t) = -3\csc t \cot t$ and we could then continue as we did above.

Example 105.

Let's look at an example which can be solved in two different ways. Consider the implicit relation

$$y + \sin^2 y + \cos^2 y = x.$$

We want $y'(x)$.

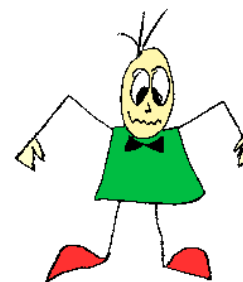
The easy way to do this is to note that, by trigonometry (I3), $\sin^2 y + \cos^2 y = 1$ regardless of the value of y . So, the original relation is really identical to $y + 1 = x$. From this we observe that $dy/dx = 1$.

But what if you didn't notice this identity? Well, we differentiate both sides as is the case whenever we use implicit differentiation. The original equation really means

$$y + \{\sin(y)\}^2 + \{\cos(y)\}^2 = x.$$

Use of the Generalized Power Rule then gives us,

$$\frac{dy}{dx} + 2\{\sin(y)\}^1 \frac{d}{dx} \sin(y) + 2\{\cos(y)\}^1 \frac{d}{dx} \cos(y) = 1,$$



Derivatives of Trigonometric Functions: Summary

Let \square denote any differentiable function, and D denote the operation of differentiation. Then

$$\begin{array}{ll} D \sin \square = \cos \square \cdot D \square & D \cos \square = -\sin \square \cdot D \square \\ D \tan \square = \sec^2 \square \cdot D \square & D \cot \square = -\csc^2 \square \cdot D \square \\ D \sec \square = \sec \square \cdot \tan \square \cdot D \square & D \csc \square = -\csc \square \cdot \cot \square \cdot D \square \end{array}$$

Table 3.5: Derivatives of Trigonometric Functions

or

$$\frac{dy}{dx} + 2\{\sin(y)\}^1 \cos(y) \frac{dy}{dx} + 2\{\cos(y)\}^1 (-\sin(y)) \frac{dy}{dx} = 1.$$

But the second and third terms cancel out, and we are left with

$$\frac{dy}{dx} = 1,$$

as before. Both methods do give the same answer as they should.

Example 106.

Evaluate the following derivatives using the rules of this Chapter and Table 3.5.

- a) $f(x) = \sin(2x^2 + 1)$
- b) $f(x) = \cos 3x \sin \sqrt{x}$
- c) $f(t) = (\cos 2t)^2$, at $t = 0$
- d) $f(x) = \cos(\sin x)$ at $x = 0$
- e) $h(t) = \frac{t}{\sin 2t}$ at $t = \pi/4$

Solution **a)** Replace the stuff between the outermost brackets by a box, \square . We want $D \sin \square$, right? Now, since $\square = 2x^2 + 1$, we know that $D \square = 4x$, and so Table 3.5 gives

$$\begin{aligned} D \sin \square &= \cos \square \cdot D \square \\ &= \cos(2x^2 + 1) \cdot 4x \\ &= 4x \cos(2x^2 + 1). \end{aligned}$$

b) We use a combination of the Product Rule and Table 3.5. So,

$$\begin{aligned} f'(x) &= D(\cos 3x) \cdot \sin \sqrt{x} + \cos(3x) \cdot D \sin \sqrt{x} \\ &= (-3 \cdot \sin(3x)) \cdot (\sin \sqrt{x}) + \cos(3x) \cdot \frac{\cos(\sqrt{x})}{2\sqrt{x}}, \end{aligned}$$

since $D \sin \sqrt{x} = \cos(\sqrt{x}) \cdot D(\sqrt{x}) = \cos(\sqrt{x}) \cdot ((1/2) x^{-1/2}) = \cos(\sqrt{x})/(2\sqrt{x})$.

c) Let $\square = \cos 2t$. The Generalized Power Rule comes to mind, so, use of Table 3.5 shows that

$$\begin{aligned} f'(t) &= 2 \square \cdot D \square, \\ &= (2 \cdot \cos 2t) \cdot (-2 \cdot \sin 2t) \\ &= -4 \cos 2t \sin 2t \\ &= -2 \sin 4t, \quad (\text{where we use Table 3.4, (I7), with } x = 2t). \end{aligned}$$

So $f'(0) = -2\sin(0) = 0$.

d) We need to find the derivative of something that looks like $\cos \square$. So, let $\square = \sin x$. We know that $D\square = D\sin x = \cos x$, and once again, Table 3.5 shows that

$$\begin{aligned} f'(x) &= -\sin \square \cdot D\square, \\ &= -\sin(\sin x) \cdot \cos x, \\ &= -\cos x \cdot \sin(\sin x). \end{aligned}$$

So $f'(0) = -\cos(0) \cdot \sin(\sin(0)) = -1 \cdot \sin(0) = 0$.

e) We see something that looks like a quotient so we should be using the Quotient Rule, right? Write $f(t) = t$, $g(t) = \sin 2t$. We need to find the derivative of the quotient $\frac{f}{g}$. Now, recall that this Rule says that (replace the x 's by t 's)

$$\frac{d}{dt}(f(g(t))) = \frac{f'(t)g(t) - f(t)g'(t)}{g(t)^2}.$$

In our case,

$f(t)$	$f'(t)$	$g(t)$	$g'(t)$
t	1	$\sin 2t$	$2 \cos 2t$

Substituting these values into the Quotient Rule we get

$$\begin{aligned} \frac{d}{dt}(f(g(t))) &= \frac{1 \cdot \sin 2t - t \cdot 2 \cos 2t}{(\sin 2t)^2}, \\ &= \frac{\sin 2t - 2t \cos 2t}{(\sin 2t)^2}. \end{aligned}$$

At $\pi/4$, $\sin(\pi/4) = \sqrt{2}/2$, so, $\sin(2 \cdot \pi/4) = 1$, $\cos(2 \cdot \pi/4) = 0$, and the required derivative is equal to 1.

Exercise Set 14.

Evaluate the derivative of the functions whose values are given below, at the indi-

cated point.

- | | |
|-------------------------------------|---|
| 1. $\sin \sqrt{x}$, at $x = 1$ | 11. $\frac{x+1}{\sin x}$, at $x = \pi/2$ |
| 2. $\sec(2x) \cdot \sin x$ | 12. $\sin(2x^2)$ |
| 3. $\sin x \cos x$, at $x = 0$ | 13. $\sin^2 x$, at $x = \pi/4$ |
| 4. $\frac{\cos x}{1 - \sin x}$ | 14. $\cot(3x - 2)$ |
| 5. $\sqrt{1 + \sin t}$, at $t = 0$ | 15. $\frac{2x+3}{\sin x}$ |
| 6. $\sin(\cos(x^2))$ | 16. $\cos(x \cdot \sin x)$ |
| 7. $x^2 \cdot \cos 3x$ | 17. $\sqrt{x} \cdot \sec(\sqrt{x})$ |
| 8. $x^{2/3} \cdot \tan(x^{1/3})$ | 18. $\csc(x^2 - 2) \cdot \sin(x^2 - 2)$ |
| 9. $\cot(2 + x + \sin x)$ | 19. $\cos^2(x - 6) + \csc(2x)$ |
| 10. $(\sin 3x)^{-1}$ | 20. $(\cos 2x)^{-2}$ |

21. Let y be defined by

$$y(x) = \begin{cases} \frac{\sin x}{\tan x} & x \neq 0 \\ 1 & x = 0 \end{cases},$$

- a) Show that y is continuous at $x = 0$,
 b) Show that y is differentiable at $x = 0$ and,
 c) Conclude that $y'(0) = 0$.

Suggested Homework Set 9. Do problems 1, 4, 6, 13, 20

3.6 Important Results About Derivatives

This section is about things we call **theorems**. Theorems are truths about things mathematical ... They are statements which can be substantiated (or proved) using the language of mathematics and its underlying logic. It's not always easy to prove something, whether it be mathematical or not. The point of a 'proof' is that it makes everything you've learned 'come together', so to speak, in a more logical, coherent fashion.

The results here form part of the cornerstones of basic Calculus. One of them, the **Mean Value Theorem** will be used later when we define the, so-called, **anti-derivative** of a function and the **Riemann integral**.

We will motivate this first theorem by looking at a sample real life situation.

A ball is thrown upwards by an outfielder during a baseball game. It is clear to everyone that the ball will reach a maximum height and then begin to fall again, hopefully in the hands of an infielder. Since the motion of the ball is 'smooth' (not 'jerky') we expect the trajectory produced by the ball to be that of a differentiable function (remember, there are no 'sharp corners' on this flight path). OK, now since the trajectory is differentiable (as a function's graph) there must be a (two-sided) derivative at the point where the ball reaches its maximum, right? What do you think is the value of this derivative? Well, look at an idealized trajectory. . . it has to be mainly 'parabolic' (because of gravity) and it looks like the path in the margin.

Tangent lines to the left (respectively, right) of the point where the maximum height is reached have positive (respectively, negative) slope and so we expect the tangent line to be horizontal at M (the point where the maximum value is reached). This is the key point, **a horizontal tangent line means a 'zero derivative' mathematically**. Why? Well, you recall that the derivative of f at a point x is the slope of the tangent line at the point $P(x, f(x))$ on the graph of f . Since a horizontal line has zero slope, it follows that the derivative is also zero.

OK, now let's translate all this into the language of mathematics. The curve has an equation $y = f(x)$ and the ball leaves the hand of the outfielder at a point a with a height of $f(a)$ (meters, feet, . . . we won't worry about units here). Let's say that the ball needs to reach ' b ' at a height $f(b)$, where $f(b) = f(a)$, above the ground. The fastest way of doing this, of course, is by throwing the ball in a straight line path from point to point (see Figure 45), but this is not realistic! If it were, the tangent line along this flight path would still be horizontal since $f(a) = f(b)$, right!?

So, the ball can't really travel in a 'straight line' from a to b , and will always reach a 'maximum' in our case, a maximum where necessarily $y'(M) = 0$, as there is a horizontal tangent line there, see Figure 45. OK, now let's look at all possible (differentiable) curves from $x = a$ to $x = b$, starting at height $f(a)$ and ending at height $f(b) = f(a)$, (as in Figure 46). We want to know "Is there always a point between a and b at which the curve reaches its maximum value?"

A straight line from $(a, f(a))$ to $(b, f(b))$, where $f(b) = f(a)$, is one curve whose maximum value is the same everywhere, okay? And, as we said above, this is necessarily horizontal, so this line is the same as its tangent line (for each point x between a and b). As can be seen in Figure 46, all the 'other' curves seem to have a maximum value at some point between a and b and, when that happens, there is a horizontal tangent line there.

It looks like we have discovered something here: If f is a differentiable function on

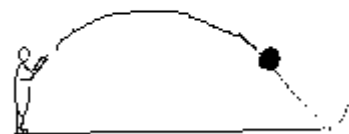


Figure 44.

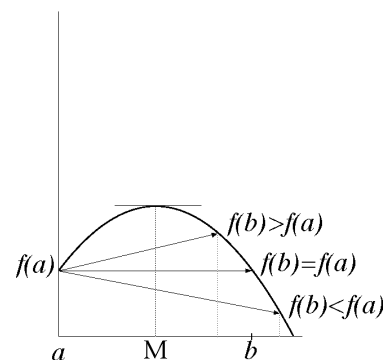


Figure 45.

an interval $I = (a, b)$ (recall $(a, b) = \{x : a < x < b\}$) and $f(a) = f(b)$ then $f'(c) = 0$ for some c between a and b ; *at least one* c , but there may be more than one. Actually, this mathematical statement is true! The result is called **Rolle's Theorem** and it is named after **Michel Rolle**, (1652-1719), a French mathematician.

Of course we haven't 'proved' this theorem of Rolle but it is believable! Its proof can be found in more advanced books in **Analysis**, a field of mathematics which includes Calculus.)

We will state it here for future reference, though:

Rolle's Theorem (1691)

Let f be a continuous function on $[a, b]$ and let f be differentiable at each point in (a, b) . If $f(a) = f(b)$, then there is at least one point c between a and b at which $f'(c) = 0$.

Remark

1. Remember that the point c , whose existence is guaranteed by the theorem, is not necessarily unique. There may be lots of them... but there is always **at least one**. Unfortunately, **the theorem doesn't tell us where it is** so we need to rely on graphs and other techniques to find it.
2. Note that whenever the derivative is zero it seems that the graph of the function has a 'peak' or a 'sink' at that point. In other words, such points appear to be related to where the graph of the function has a **maximum** or **minimum** value. This observation is very important and will be very useful later when we study the general problem of sketching the graph of a general function.

Example 107.

Show that the function whose values are given by $f(x) = \sin(x)$ on the interval $[0, \pi]$ satisfies the assumptions of Rolle's Theorem. Find the required value of c explicitly.

Solution We know that 'sin' as well as its derivative, 'cos', are continuous everywhere. Also, $\sin(0) = 0 = \sin(\pi)$. So, if we let $a = 0, b = \pi$, we see that we can apply Rolle's Theorem and find that $y'(c) = 0$ where c is somewhere in between 0 and π . So, this means that $\cos c = 0$ for some value of c . This is true! We can choose $c = \pi/2$ and see this c exactly.

We have seen Rolle's Theorem in action. Now, let's return to the case where the baseball goes from $(a, f(a))$ to $(b, f(b))$ but where $f(a) \neq f(b)$ (players of different heights!). **What can we say in this case?**

Well, we know that there is the straight line path from $(a, f(a))$ to $(b, f(b))$ which, unfortunately, does not have a zero derivative *anywhere* as a curve (see Figure 47). But look at all possible curves going from $(a, f(a))$ to $(b, f(b))$. This is only a thought experiment, OK? They are differentiable (let's assume this) and they bend this way and that as they proceed from their point of origin to their destination. Look at **how** they turn and compare this to the straight line joining the origin and destination. It looks like **you can always find a tangent line** to any one of these curves **which is parallel to the line joining $(a, f(a))$ to $(b, f(b))$** ! (see Figure 48). It's *almost* like Rolle's Theorem (graphically) but it is **not** Rolle's Theorem because $f(a) \neq f(b)$. Actually, if you think about it a little, you'll see that it's more general than Rolle's Theorem. It has a different name ... and it too is a

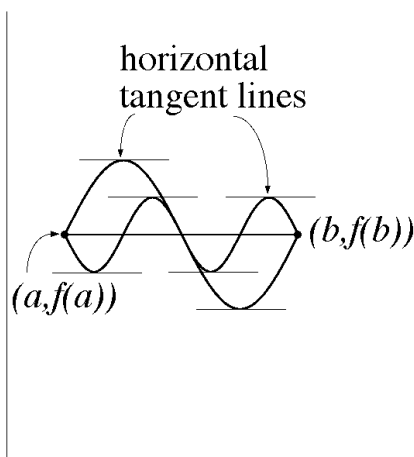


Figure 46.

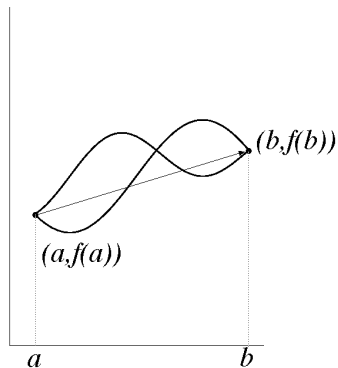


Figure 47.

true mathematical statement! We call it the **Mean Value Theorem** and it says the following:

Mean Value Theorem

Let f be continuous on the interval $a \leq x \leq b$ and differentiable on the interval $a < x < b$. Then there is a point c between a and b at which

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Remark The number on the left of the equation, namely, $\frac{f(b)-f(a)}{b-a}$ is really the slope of the line pointing from $(a, f(a))$ to $(b, f(b))$. Moreover, $f'(c)$ is the slope of the tangent line through some point $(c, f(c))$ on the graph of f . Since **these slopes are equal**, the corresponding lines must be parallel, which is what we noticed above.

Example 108.

Show that the function whose values are given by $f(x) = \cos 2x$ on the interval $[0, \pi/2]$ satisfies the assumptions of the Mean Value Theorem. Show that there is a value of c such that $\sin 2c = 2/\pi$.

Solution Here, ' $\cos 2x$ ' as well as its derivative, ' $-2 \sin 2x$ ', are continuous everywhere. Also, $\cos(0) = 1$ and $\cos(\pi) = -1$. So, if we let $a = 0, b = \pi/2$, we see that we can apply the Mean Value Theorem and find that $y'(c) = 0$ where c is somewhere in between 0 and $\pi/2$. This means that $-2 \sin 2c = -4/\pi$, or for some value of c , we must have $\sin 2c = 2/\pi$. We may not know what this value of c is,

exactly, but it does exist! In fact, in the next section we'll show you how to find this value of c using **inverse trigonometric functions**.

Applications

Example 109.

Let y be continuous in the interval $a \leq x \leq b$ and a differentiable function on an interval (a, b) whose derivative is equal to zero at each point x , $a < x < b$. Show that $y(x) = \text{constant}$ for each x , $a < x < b$. *i.e.* If $y'(x) = 0$ for all x then the values $y(x)$ are equal to one and the same number (or, y is said to be a **constant function**).

Solution This is one very nice application of the Mean Value Theorem. OK, let t be any point in (a, b) . Since y is continuous at $x = a$, $y(a)$ is finite. Re-reading the statement of this example shows that all the assumptions of the Mean Value Theorem are satisfied. So, the quotient

$$\frac{y(t) - y(a)}{t - a} = y'(c)$$

where $a < c < t$ is the conclusion. But whatever c is, we know that $y'(c) = 0$ (by hypothesis, *i.e.* at each point x the derivative at x is equal to 0). It follows that $y'(c) = 0$ and this gives $y(t) = y(a)$. But now look, t can be changed to some **other** number, say, t^* . We do the same calculation once again and we get

$$\frac{y(t^*) - y(a)}{t^* - a} = y'(c^*)$$

where now $a < c^* < t^*$, and c^* is generally different from c . Since $y'(c^*) = 0$ (again, by hypothesis) it follows that $y(t^*) = y(a)$ as well. OK, but all this means that $y(t) = y(t^*) = y(a)$. So, we can continue like this and repeat this argument for



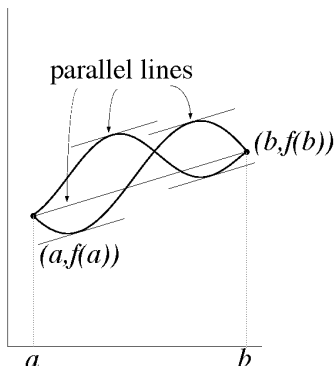


Figure 48.

Example 110.

The function defined by $y = |x|$ has $y(-1) = y(1)$ but yet $y'(c) \neq 0$ for any value of c . **Explain why this doesn't contradict Rolle's Theorem.**

Solution All the assumptions of a theorem need to be verified **before** using the theorem's conclusion. In this case, the function f defined by $f(x) = |x|$ has no derivative at $x = 0$ as we saw earlier, and so the assumption that f be differentiable over $(-1, 1)$ is not true since it is not differentiable at $x = 0$. So, we can't use the Theorem at all. This just happens to be one of the many functions that doesn't satisfy the conclusion of this theorem. You can see that there's no contradiction to Rolle's Theorem since it doesn't apply.

Example 111.

The function f is defined by

$$f(x) = \begin{cases} -x, & -2 \leq x \leq 0 \\ 1 - x, & 0 < x \leq 3. \end{cases}$$

In this example, the function f is defined on $[-2, 3]$ by the 2 curves in the graph and $\frac{f(3)-f(-2)}{3-(-2)} = -\frac{4}{5}$ but there is no value of c , $-2 < c < 3$ such that $f'(c) = -\frac{4}{5}$, because $f'(c) = -1$ at every c except $c = 0$ where $f'(0)$ is not defined. **Does this contradict the Mean Value Theorem?**

Solution No, there is no contradiction here. Once again, all the assumptions of the Mean Value Theorem must be verified before proceeding to its conclusion. In this example, the function f defined above is not continuous at $x = 0$ because its left-hand limit at $x = 0$ is $f_-(0) = 0$, while its right-hand limit, $f_+(0) = 1$. Since these limits are different f is not continuous at $x = 0$. Since f is not continuous at $x = 0$, it cannot be continuous on **all** of $[-2, 3]$. So, we can't apply the conclusion. So, there's nothing wrong with this function or Rolle's Theorem.

Example 112.

Another very useful application of the Mean Value Theorem/Rolle's Theorem is in the theory of differential equations which we spoke of earlier.

Let y be a differentiable function for each x in $(a, b) = \{x : a < x < b\}$ and continuous in $[a, b] = \{x : a \leq x \leq b\}$. Assume that y has the property that for every number x in (a, b) ,

$$\frac{dy}{dx} + y(x)^2 + 1 = 0.$$

Show that this function y cannot have two zeros (or roots) in the interval $[a, b]$.

Solution Use Rolle's Theorem and show this result by assuming the contrary. This is called a **a proof by contradiction**, remember? Assume that, if possible, there are two points A, B in the interval $[a, b]$ where $y(A) = y(B) = 0$. Then, by Rolle's Theorem, there exists a point c in (A, B) with $y'(c) = 0$. Use this value of c in the equation above. This means that

$$\frac{dy}{dx}(c) + y(c)^2 + 1 = 0,$$

Summary**Rolle's Theorem**

Let f be continuous at each point of a closed interval $[a, b]$ and differentiable at each point of (a, b) . If $f(a) = f(b)$, then there is a point c between a and b at which $f'(c) = 0$.

Remark Don't confuse this result with Bolzano's Theorem (Chapter 2). Bolzano's Theorem deals with the existence of a **root** of a continuous function f while Rolle's Theorem deals with the existence of a root of the *derivative* of a function.

Mean Value Theorem

Let f be continuous on the interval $a \leq x \leq b$ and differentiable on the interval $a < x < b$. Then there is a point c between a and b at which $\frac{f(b)-f(a)}{b-a} = f'(c)$.

Table 3.6: Rolle's Theorem and the Mean Value Theorem

right? Now, since $y'(c) = 0$, it follows that $y(c)^2 + 1 = 0$. But $y(c)^2 \geq 0$. So, this is an impossibility, it can never happen. This last statement is the contradiction. The original assumption that *there are two points A, B in the interval $[a, b]$ where $y(A) = y(B) = 0$* must be false. So, there can't be 'two' such points. It follows that y cannot have two zeros in $a \leq x \leq b$.

Remark This is a really interesting aspect of most differential equations: We **really don't know what ' $y(x)$ ' looks like** either explicitly or implicitly but still, we can get some information about its graph! In the preceding example we showed that $y(x)$ could not have two zeros, for example. This sort of analysis is part of an area of differential equations called "**qualitative analysis**".

Note The function y defined by $y(x) = \tan(c - x)$ where c is any fixed number, has the property that $\frac{dy}{dx} + y(x)^2 + 1 = 0$. If $c = \pi$, say, then $y(x) = \tan(\pi - x)$ is such a function whose graph is reproduced in Figure 49.

Note that this function has 'lots' of zeros! Why does this graph not contradict the result of Example 112? It's because on this interval, $[0, \pi]$ the function f is not defined at $x = \pi/2$ (so it not continuous on $[0, \pi]$), and so Example 112 does not apply.

Example 113.

In a previous example we saw that if y is a differentiable function on $[a, b]$ and $y'(x) = 0$ for all x in (a, b) then $y(x)$ must be a constant function.

The same ideas may be employed to show that if y is a twice differentiable function on (a, b) (i.e. the derivative itself has a derivative), y and y' are each continuous on $[a, b]$ and if $y''(x) = 0$ for each x in (a, b) then $y(x) = mx + b$, for each $a < x < b$, for some constants m and b . That is, y must be a linear function.

Solution We apply the Mean Value Theorem to the function y' first. Look at the interval $[a, x]$. Then $\frac{y'(x)-y'(a)}{x-a} = y''(c)$ where $a < c < x$. But $y''(c) = 0$ (regardless of the value of c) so this means $y'(x) = y'(a) = \text{constant}$. Since x can be any

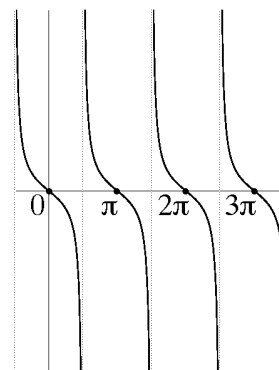


Figure 49.

Intermediate Value Theorem

Let f be continuous at each point of a closed interval $[a, b] = \{x : a \leq x \leq b\}$. Assume,

1. $f(a) \neq f(b)$

2. z is a point between the numbers $f(a)$ and $f(b)$.

Then there is at least one value of c between a and b such that $f(c) = z$

Remark This result is very useful in finding the **root** of certain equations, or the points of intersection of two or more curves in the plane.

Bolzano's Theorem

Let f be **continuous on a closed interval** $[a, b]$ (i.e., at each point x in $[a, b]$).

If $f(a)f(b) < 0$, then there is at least one point c between a and b such that $f(c) = 0$. In other words there is at least one root of f in the interval (a, b) .



Table 3.7: Main Theorems about Continuous Functions

number, $x > a$, and the *constant* above does not 'change' (it is equal to $y'(a)$), it follows that $y'(x) = y'(a)$ for **any** x in (a, b) .

Now, apply the Mean Value Theorem to y , NOT y' ... Then

$$\frac{y(x) - y(a)}{x - a} = y'(c)$$

where $a < c < x$ (**not** the same c as before, though). But we know that $y'(c) = y'(a)$ (from what we just proved) so this means that $y(x) - y(a) = y'(a)(x - a)$ or

$$\begin{aligned} y(x) &= y'(a)(x - a) + y(a) \\ &= mx + b \end{aligned}$$

if we chose $m = y'(a)$ and $b = y(a) - ay'(a)$. That's all!

Two other big theorems of elementary Calculus are the Intermediate Value Theorem and a special case of it called Bolzano's Theorem, both of which we saw in our chapter on Limits and Continuity. We recall them here.

Example 114.

Show that there is a root of the equation $f(x) = 0$ in the interval $[0, \pi]$, where $f(x) = x \sin x + \cos x$.

Solution OK, what's this question about? The key words are 'root' and 'function' and at this point, basing ourselves on the big theorems above, we must be dealing with an application of **Bolzano's Theorem**, you see? (Since it deals with roots of functions, see Table 3.7.) So, let $[a, b] = [0, \pi]$ which means that $a = 0$ and $b = \pi$. Now the function whose values are given by $x \sin x$ is continuous (as it is the product of two continuous functions) and since ' $\cos x$ ' is continuous it follows that $x \sin x + \cos x$ is continuous (as the sum of continuous functions is, once again continuous). Thus f is continuous on $[a, b] = [0, \pi]$.

What about $f(0)$? Well, $f(0) = 1$ (since $0 \sin 0 + \cos 0 = 0 + (+1) = +1$).

Bernhard Bolzano, 1781-1848, Czechoslovakian priest and mathematician who specialized in Analysis where he made many contributions to the areas of limits and continuity and, like Weierstrass, he produced a method (1850) for constructing a continuous function which has no derivative anywhere! He helped to establish the tenet that mathematical truth should rest on rigorous proofs rather than intuition.

And $f(\pi)$? Here $f(\pi) = -1$ (since $\pi \sin \pi + \cos \pi = 0 + (-1) = -1$).

So, $f(\pi) = -1 < 0 < f(0) = 1$ which means that $f(a) \cdot f(b) = f(0) \cdot f(\pi) < 0$. So, **all** the hypotheses of Bolzano's theorem are satisfied. This means that the conclusion follows, that is, there is a point c between 0 and π so that $f(c) = 0$.

Remark Okay, but 'where' is the root of the last example?

Well, we need more techniques to solve this problem, and there is one, very useful method, called **Newton's method** which we'll see soon, (named after the same Newton mentioned in Chapter 1, one of its discoverers.)

Exercise Set 15.

Use Bolzano's theorem to show that each of the given functions has a root in the given interval. **Don't forget to verify the assumption of continuity in each case.** You may want to use your calculator.

- $y(x) = 3x - 2$, $0 \leq x \leq 2$
- $y(x) = x^2 - 1$, $-2 \leq x \leq 0$
- $y(x) = 2x^2 - 3x - 2$, $0 \leq x \leq 3$
- $y(x) = \sin x + \cos x$, $0 \leq x \leq \pi$
- $y(x) = x \cos x + \sin x$, $0 \leq x \leq \pi$
- The function y has the property that y is three-times differentiable in (a, b) and continuous in $[a, b]$. If $y'''(x) = 0$ for all x in (a, b) show that $y(x)$ is of the form $y(x) = Ax^2 + Bx + C$ for a suitable choice of A, B , and C .
- The following function y has the property that $\frac{dy}{dx} + y(x)^4 + 2 = 0$ for x in (a, b) . Show that $y(x)$ cannot have two zeros in the interval $[a, b]$.
- Use the Mean Value Theorem to show that $\sin x \leq x$ for any x in the interval $[0, \pi]$.
- Use Rolle's Theorem applied to the sine function on $[0, \pi]$ to show that the cosine function must have a root in this interval.
- Apply the Mean Value Theorem to the sine function on $[0, \pi/2]$ to show that $x - \sin x \leq \frac{\pi}{2} - 1$. Conclude that if $0 \leq x \leq \frac{\pi}{2}$, then $0 \leq x - \sin x \leq \frac{\pi}{2} - 1$.
- Use a calculator to find that value c in the conclusion of the Mean Value Theorem for the following two functions:
 - $f(x) = x^2 + x - 1$, $[a, b] = [0, 2]$
 - $g(x) = x^2 + 3$, $[a, b] = [0, 1]$

Hint In (a) calculate the number $\frac{f(b)-f(a)}{b-a}$ explicitly. Then find $f'(c)$ as a function of c , and, finally, solve for c .

- An electron is shot through a 1 meter wide plasma field and its time of travel is recorded at 0.3×10^{-8} seconds on a timer at its destination. Show, using the Mean Value Theorem, that its velocity at some point in time had to **exceed the speed of light** in that field given approximately by 2.19×10^8 m/sec. (**Note** This effect is actually observed in nature!)

In the first few exercises show 1) that the function is continuous, and 2) that there are two points a, b inside the given interval with $f(a)f(b) < 0$. Then use Bolzano's Theorem.



Suggested Homework Set 10. Do problems 1, 3, 6, 8, 11

3.7 Inverse Functions

One of the most important topics in the theory of functions is that of the **inverse of a function**, a function which is NOT the same as the reciprocal (or 1 divided by the function). Using this new notion of an inverse we are able to ‘back-track’ in a sense, the idea being that we interchange the domain and the range of a function when defining its inverse and points in the range get associated with the point in the domain from which they arose. These inverse functions are used everywhere in Calculus especially in the topic of finding the **area between two curves**, or calculating the **volume of a solid of revolution** two topics which we will address later. The two main topics in Calculus namely, differentiation and integration of functions, are actually related. In the more general sense of an **inverse of an operator**, these operations on functions are *almost* inverses of one another. Knowing how to manipulate and find inverse functions is a necessity for a thorough understanding of the methods in Calculus. In this section we will learn what they are, how to find them, and how to sketch them.



Review

You should be completely familiar with Chapter 1, and especially how to find the composition of two functions using the ‘box’ method or any other method.

We recall the notion of the **composition of two functions** here: Given two functions, f, g where the range of g is contained in the domain of f , (i.e., $R=\text{Ran}(g) \subseteq \text{Dom}(f)=D$) we define the **composition of f and g** , denoted by the symbol $f \circ g$, a new function whose values are given by $(f \circ g)(x) = f(g(x))$ where x is in the domain of g (denoted briefly by D).

Example 115.

Let $f(x) = x^2 + 1$, $g(x) = x - 1$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

Solution Recall the box methods of Chapter 1. By definition, since $f(x) = x^2 + 1$ we know that $f(\square) = \square^2 + 1$. So,

$$f(\boxed{g(x)}) = \boxed{g(x)}^2 + 1 = \boxed{(x-1)}^2 + 1 = (x-1)^2 + 1 = x^2 - 2x + 2.$$

On the other hand, when the same idea is applied to $(g \circ f)(x)$, we get $(g \circ f)(x) = x^2$.

Note: This shows that the operation of **composition is not commutative**, that is, $(g \circ f)(x) \neq (f \circ g)(x)$, in general. The point is that **composition is not the same as multiplication**.

Let f be a given function with domain, $D=\text{Dom}(f)$, and range, $R=\text{Ran}(f)$. We say that the function **F is the inverse of f** if all these four conditions hold:

$$\text{Dom}(F) = \text{Ran}(f)$$

$$\text{Dom}(f) = \text{Ran}(F)$$

$$(F \circ f)(x) = x, \text{ for every } x \text{ in } \text{Dom}(f)$$

$$(f \circ F)(x) = x, \text{ for every } x \text{ in } \text{Dom}(F)$$



Thus, the inverse function’s domain is R . The inverse function of f is usually written f^{-1} whereas the reciprocal function of f is written as $\frac{1}{f}$ so that $(\frac{1}{f})(x) = \frac{1}{f(x)} \neq f^{-1}(x)$. This is the source of much confusion!

Example 116.

Find the composition of the functions f, g where $f(x) = 2x + 3$, $g(x) = x^2$, and show that $(f \circ g)(x) \neq (g \circ f)(x)$.

Solution Using the box method or any other method we find

$$(f \circ g)(x) = f(g(x)) = 2g(x) + 3 = 2x^2 + 3$$

while

$$(g \circ f)(x) = g(f(x)) = (f(x))^2 = (2x + 3)^2 = 4x^2 + 12x + 9$$

So we see that

$$(f \circ g)(x) \neq (g \circ f)(x)$$

as the two expressions need to be exactly the same for equality.

Example 117.

Show that the functions f, F defined by $f(x) = 2x + 3$ and $F(x) = \frac{x-3}{2}$ are inverse of one another. That is, show that F is the inverse of f and f is the inverse of F .

Solution As a check we note that $\text{Dom}(F) = \text{Ran}(f) = (-\infty, \infty)$ and

$$f(F(x)) = 2F(x) + 3 = 2\left(\frac{x-3}{2}\right) + 3 = x,$$

which means that $(f \circ F)(x) = x$. On the other hand, $\text{Dom}(f) = \text{Ran}(F) = (-\infty, \infty)$ and

$$F(f(x)) = \frac{f(x) - 3}{2} = \frac{(2x + 3) - 3}{2} = x,$$

which now means that $(F \circ f)(x) = x$. So, by definition, these two functions are inverse functions of one another.

How can we tell if a given function has an inverse function? In order that two functions f, F be inverses of one another it is necessary that each function be **one-to-one** on their respective domains. This means that the **only** solution of the equation $f(x) = f(y)$ (resp. $F(x) = F(y)$) is the solution $x = y$, whenever x, y are in $\text{Dom}(f)$, (resp. $\text{Dom}(F)$). The simplest geometrical test for deciding whether a given function is one-to-one is the so-called *Horizontal Line Test*. Basically, one looks at the graph of the given function on the xy -plane, and if every horizontal line through the range of the function intersects the graph at only one point, then the function is one-to-one and so it has an inverse function, see Figure 50. The moral here is “Not every function has an inverse function, only those that are one-to-one!”

Example 118.

Show that the function $f(x) = x^2$ has no inverse function if we take its domain to be the interval $[-1, 1]$.

Solution This is because the Horizontal Line Test shows that every horizontal line through the range of f intersects the curve at two points (except at $(0, 0)$, see Figure 51). Since the Test fails, f is not one-to-one and this means that f cannot have an inverse. Can you show that this function **does have an inverse** if its domain is restricted to the smaller interval $[0, 1]$?

Example 119.

Find the form of the inverse function of the function f defined by $f(x) = 2x + 3$, where x is real.

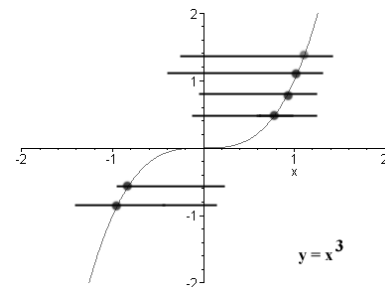


Figure 50.

EXAMPLES

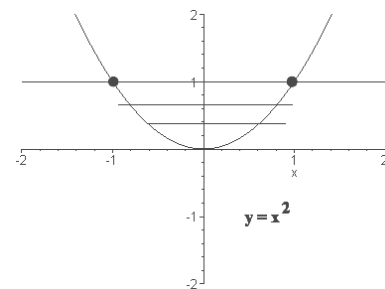


Figure 51.

How to find the inverse of a function

1.
 - Write $y = f(x)$
 - Solve for x in terms of y
 - Then $x = F(y)$ where F is the inverse.
2.
 - Interchange the x 's and y 's.
 - Solve for the symbol y in terms of x .
 - This gives $y = F(x)$ where F is the inverse.

It follows that the graph of the inverse function, F , is obtained by reflecting the graph of f about the line $y = x$. More on this later.

Table 3.8: How to Find the Inverse of a Function

Solution Use Table 3.8. Write $y = f(x) = 2x + 3$. We solve for x in terms of y . Then

$$y = 2x + 3 \text{ means } x = \frac{y - 3}{2} = F(y).$$

Now interchange x and y . So the inverse of f is given by F where $F(x) = \frac{x - 3}{2}$.

Example 120. $f(x) = x^4, x \geq 0$, what is its inverse function $F(x)$ (also denoted by $f^{-1}(x)$) ?

Solution Let's use Table 3.8, once again. Write $y = x^4$. From the graph of f (Figure 52) we see that it is one-to-one if $x \geq 0$. Solving for x in terms of y , we get $x = \sqrt[4]{y}$ since x is real, and $y \geq 0$. So $f^{-1}(y) = F(y) = \sqrt[4]{y}$ or $f^{-1}(x) = F(x) = \sqrt[4]{x}$ is the inverse function of f .

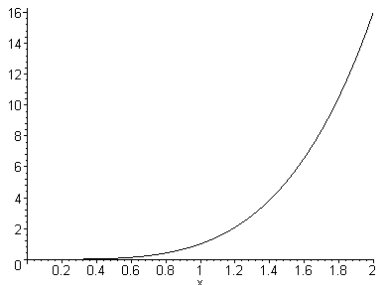
Example 121. If $f(x) = x^3 + 1$, what is its inverse function, $f^{-1}(x)$?

Solution We solve for x in terms of y , as usual. Since $y = x^3 + 1$ we know $y - 1 = x^3$ or $x = \sqrt[3]{y - 1}$ (and y can be *any* real number here). Interchanging x and y we get $y = \sqrt[3]{x - 1}$, or $f^{-1}(x) = \sqrt[3]{x - 1}$, or $F(x) = \sqrt[3]{x - 1}$

The **derivative of the inverse function** f^{-1} of a given function f is related to the derivative of f by means of the next formula

$$\frac{dF}{dx}(x) = \frac{df^{-1}}{dx}(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(F(x))} \tag{3.4}$$

where the symbol $f'(f^{-1}(x))$ means that the derivative of f is evaluated at the point $f^{-1}(x)$, where x is given. **Why?**



The graph of $f(x) = x^4$. If $x \geq 0$ this function is one-to-one. It is not true that f is one-to-one if the domain of f contains negative points, since in this case there are horizontal lines that intersect the graph in TWO points.

Figure 52.

The simplest reason is that the Chain Rule tells us that since $f(F(x)) = x$ we can differentiate the composition on the left using the Box Method (with $F(x)$ in the box...). By the Chain Rule we know that

$$Df(\square) = f'(\square) \cdot D(\square).$$

Applying this to our definition of the inverse of f we get

$$\begin{aligned} x &= f(F(x)) \\ Dx &= Df(F(x)) = Df(\square) \\ 1 &= f'(\square) \cdot D(\square) \\ &= f'(F(x)) \cdot F'(x). \end{aligned}$$

Now solving for the symbol $F'(x)$ in the last display (because this is what we want) we obtain

$$F'(x) = \frac{1}{f'(F(x))},$$

where $F(x) = f^{-1}(x)$ is the inverse of the original function $f(x)$. This proves our claim.

Another, more geometrical, argument proceeds like this: Referring to Figure 53 in the margin let (x, y) be a point on the graph of $y = f^{-1}(x)$. We can see that the tangent line to the graph of f has equation $y = mx + b$ where m , its slope, is also the derivative of f at the point in question (i.e., $f'(y)$). On the other hand, its reflection is obtained by interchanging x, y , and so the equation of its counterpart (on the other side of $y = x$) is $x = my + b$. Solving for y in terms of x in this one we get $y = \frac{x}{m} - \frac{b}{m}$. This means that it has slope equal to the reciprocal of the first one. Since these slopes are actually derivatives this means that

$$(f^{-1})'(x) = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$$

since our point (x, y) lies on the graph of the inverse function, $y = f^{-1}(x)$.

Example 122. For Example 120 above, what is $\frac{df^{-1}}{dx}(16)$? i.e., the derivative of the inverse function of f at $x = 16$?

Solution Using Equation 3.4, we have

$$(f^{-1})'(16) = \frac{1}{f'(f^{-1}(16))}$$

But $f^{-1}(x) = \sqrt[4]{x}$ means that $f^{-1}(16) = \sqrt[4]{16} = 2$. So, $(f^{-1})'(16) = \frac{1}{f'(2)}$ and now we need $f'(2)$. But $f(x) = x^4$, so $f'(2) = 4(2)^3 = 32$. Finally, we find that $(f^{-1})'(16) = \frac{1}{32}$.

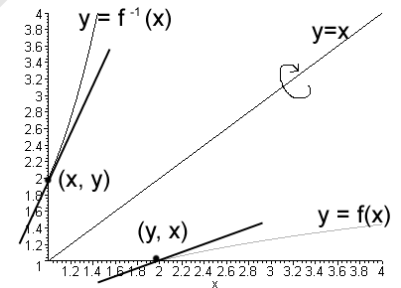
Example 123. A function f with an inverse function denoted by F has the property that $F(0) = 1$ and $f'(1) = 0.2$. Calculate the value of $F'(0)$.

Solution We don't have much given here but yet we can actually find the answer as follows: Since

$$F'(0) = \frac{1}{f'(F(0))}$$

by (3.4) with $x = 0$, we set $F(0) = 1$, and note that $f'(F(0)) = f'(1)$. But since $f'(1) = 0.2$ we see that

$$F'(0) = \frac{1}{f'(F(0))} = \frac{1}{f'(1)} = \frac{1}{0.2} = 5.$$



The two tangents are reflections of one another, and so their slopes must be the reciprocal of one another (see the text).

Figure 53.



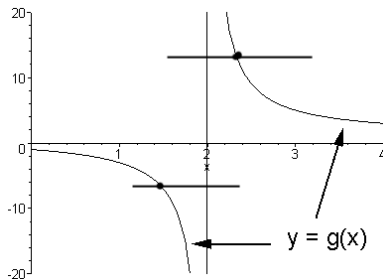
Example 124.

Let g be a function defined by

$$g(x) = \frac{x+2}{x-2}$$

with $\text{Dom}(g) = \{x : x \neq 2\}$. Show that g has an inverse function, G , find its form, and describe its Domain and Range.

Solution Let's denote its inverse function by G . The first question you should be asking yourself is: "How do we know that there *is* an inverse function at all?" In other words, we have to show that the graph of g satisfies the *Horizontal Line Test* mentioned above (see Fig. 50), or, in other words, g is one-to-one. To do this we can do one of two things: We can either draw the graph as in Fig. 54 (if you have that much patience), or check the condition algebraically by showing that if $g(x) = g(y)$ then $x = y$ must be true (for any points x, y in the domain of g). Since the graph is already given in the margin we are done, but let's look at this using the algebraic test mentioned here.



The graph of the function g in Example 124. Note that any horizontal line intersects the graph of g in only one point! This means that g is one-to-one on its domain. The vertical line across the point $x = 2$ is called a vertical asymptote (a line on which the function becomes infinite). More on this in Chapter 5.

Figure 54.

In order to prove that g is one-to-one algebraically, we have to show that if $g(x) = g(y)$ then $x = y$. Basically, we use the definitions, perform some algebra, simplify and see if we get $x = y$ at the end. If we do, we're done. Let's see.

We assume that $g(x) = g(y)$ (here y is thought of as an independent variable, just like x). Then, by definition, this means that

$$\frac{x+2}{x-2} = \frac{y+2}{y-2}$$

for $x, y \neq 2$. Multiplying both sides by $(x-2)(y-2)$ we get $(x+2)(y-2) = (y+2)(x-2)$. Expanding these expressions we get

$$xy + 2y - 2x - 4 = yx + 2x - 2y - 4$$

from which we easily see that $x = y$. That's all. So g is one-to-one. Thus, its inverse function G exists.

Next, to find its values, $G(x)$, we replace all the x 's by y 's and solve for y in terms of x , (cf., Table 3.8). Replacing all the x 's by y 's (and the only y by x) we get

$$x = \frac{y+2}{y-2}$$

Multiplying both sides by $(y-2)$ and simplifying we get

$$y = \frac{2x+2}{x-1}$$

This is $G(x)$.

Its domain is $\text{Dom}(G) = \{x : x \neq 1\} = \text{Ran}(g)$ while its range is given by $\text{Ran}(G) = \text{Dom}(g) = \{x : x \neq 2\}$ by definition of the inverse.

Now that **we know how to find the form of the inverse** of a given (one-to-one) function, the natural question is: "What does it look like?". Of course, it is simply another one of those graphs whose shape may be predicted by means of existing computer software or by the old and labor intensive method of finding the critical points of the function, the asymptotes, etc. So, *why worry about the graph of an inverse function?* Well, one reason is that the **graph of an inverse function is related to the graph of the original function** (that is, the one for which it is the inverse). How? Let's have a look at an example.

Example 125.

Let's look at the graph of the function $f(x) = \sqrt{x}$, for x in $(0, 4)$, and its inverse, the function, $F(x) = x^2$, for x in $(0, 2)$, Figure 55.

When we study these graphs carefully, we note, by definition of the inverse function, that the **domain and the range are interchanged**. So, this means that if we interchanged the x -axis (on which lies the domain of f) and the y -axis, (on which we find the range of f) we would be in a position to graph the inverse function of f . This graph of the inverse function is simply the reflection of the graph for $y = \sqrt{x}$, about the line $y = x$. Try it out! Better still, check out the following experiment!

EXPERIMENT:

1. Make a copy of the graph of $f(x) = \sqrt{x}$, below, by tracing it onto some **tracing paper** (so that you can see the graph from both sides). Label the axes, and fill in the domain and the range of f by thickening or thinning the line segment containing them, or, if you prefer, by colouring them in.
2. Now, **turn the traced image around, clockwise, by 90 degrees** so that the x -axis is vertical (but pointing down) and the y -axis is horizontal (and pointing to the right).
3. Next, **flip the paper over onto its back** without rotating the paper! **What do you see? The graph of the inverse function of $f(x) = \sqrt{x}$, that is, $F(x) = x^2$.**

The graph of a function and its inverse

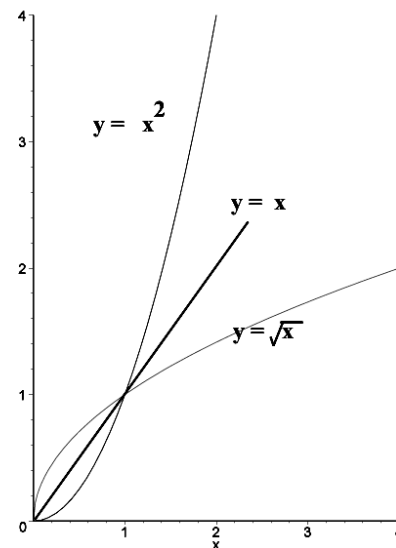
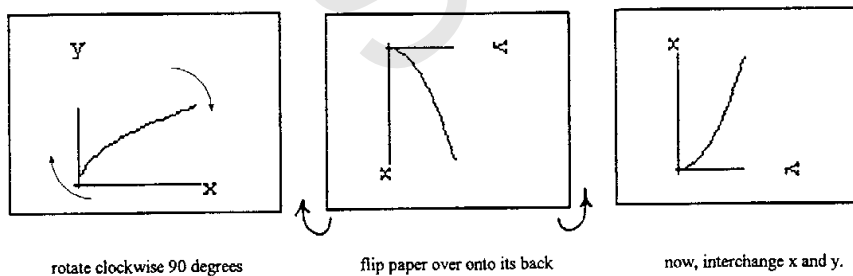


Figure 55. The graphs of $y = x^2$ and its inverse, $y = \sqrt{x}$ superimposed on one another

REMARK This technique of making the graph of the inverse function by rotating the original graph clockwise by 90 degrees and then flipping it over always works! You will **always** get the graph of the inverse function on the back side (verso), as if it had been sketched on the x and y axes as usual (once you interchange x and y). Here's a visual summary of the construction ...



Why does this work? Well, there's some *Linear Algebra* involved. (The author's module entitled *The ABC's of Calculus: Module on Inverse Functions* has a thorough explanation!)

We summarize the above in this

RULE OF THUMB. We can always find the graph of the inverse function by applying the above construction to the original graph

or, equivalently,

by reflecting the original graph of f about the line $y = x$ and eliminating the original graph.

Example 126.

We sketch the graphs of the function f , and its inverse, F , given by $f(x) = 7x + 4$ and $F(x) = \frac{x-4}{7}$, where $\text{Dom}(f) = \mathfrak{R}$, where $\mathfrak{R} = (-\infty, +\infty)$. The graphs of f and its inverse superimposed on the same axes are shown in Figure 56.

NOTE THAT if you are given the graph of the inverse function, $F(x)$, of a function $f(x)$, you can find the graph of $f(x)$ by applying the preceding “rule of thumb” with f and F interchanged. Furthermore, the inverse of the inverse function of a function f (so, we’re looking for the inverse) is f itself. Why? Use the definition of the inverse! We know that $F(f(x)) = x$, for each x in $\text{Dom}(f)$, and $x = f(F(x))$ for each x in $\text{Dom}(F)$; together, these relations say that “ F is the inverse of f ”. If we interchange the symbols ‘ F ’ and ‘ f ’ in this equation we get the **same** equation with the interpretation “ f is the inverse of F ”, which is what we wanted!

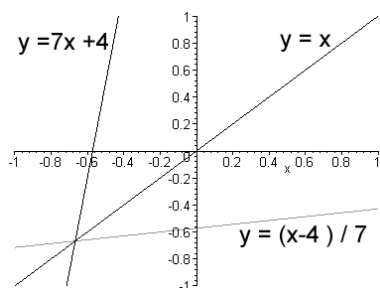


Figure 56. The graphs of $f(x) = 7x + 4$ and its inverse $F(x) = \frac{x-4}{7}$ superimposed on one another.

Exercise Set 16.

Sketch the graphs of the following functions and their inverses. Don’t forget to indicate the domain and the range of each function.

- $f(x) = 4 - x^2$, $0 \leq x \leq 2$
- $g(x) = (x - 1)^{-1}$, $1 < x < \infty$
- $f(z) = 2 - z^3$, $-\infty < z < \infty$
- $h(x) = \sqrt{5 + 2x}$, $-\frac{5}{2} \leq x < \infty$
- $f(y) = (2 + y)^{\frac{1}{3}}$, $-2 < y < \infty$
- Let f be a function with domain $D = \mathfrak{R}$. Assume that f has an inverse function, F , defined on \mathfrak{R} (another symbol for the real line) also.
 - Given that $f(2) = 0$, what is $F(0)$?
 - If $F(6) = -1$, what is $f(-1)$?
 - Conclude that the only solution of $f(x) = 0$ is $x = 2$.
 - Given that $f(-2) = 8$, what is the solution y , of $F(y) = -2$? Are there any other solutions ??
 - We know that $f(-1) = 6$. Are there any other points, x , such that $f(x) = 6$?
- Given that f is such that its inverse F exists, $f'(-2.1) = 4$, $F(-1) = -2.1$, find the value of the derivative of F at $x = -1$.

Find the form of the inverse of the given functions on the given domain and determine the Domain and the Range of the inverse function. Don't forget to show that each is one-to-one first.

8. $f(x) = x, \quad -\infty < x < +\infty$

9. $f(x) = \frac{1}{x}, \quad x \neq 0$

10. $f(x) = x^3, \quad -\infty < x < +\infty$

11. $f(t) = 7t + 4, \quad 0 \leq t \leq 1$

12. $g(x) = \sqrt{2x + 1}, \quad x \geq -\frac{1}{2}$

13. $g(t) = \sqrt{1 - 4t^2}, \quad 0 \leq t \leq \frac{1}{2}$

14. $f(x) = \frac{2 + 3x}{3 - 2x}, \quad x \neq \frac{3}{2}$

15. $g(y) = y^2 + y, \quad -\frac{1}{2} \leq y < +\infty$



Suggested Homework Set 11. Work out problems 3, 5, 6, 8, 12, 15.

Web Links

More on Inverse Functions at:

library.thinkquest.org/2647/algebra/ftinvers.htm

(requires a Java-enabled browser)

www.sosmath.com/algebra/invfunc/fnc1.html

www.math.wpi.edu/Course_Materials/MA1022B95/lab3/node5.html

(The above site uses the software "Maple")

www.math.duke.edu/education/ccp/materials/intcalc/inverse/index.html

NOTES:

3.8 Inverse Trigonometric Functions

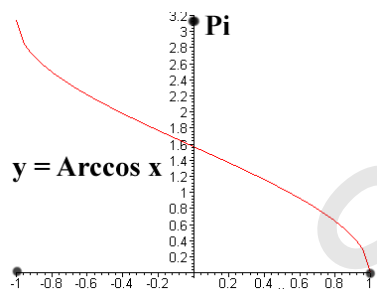
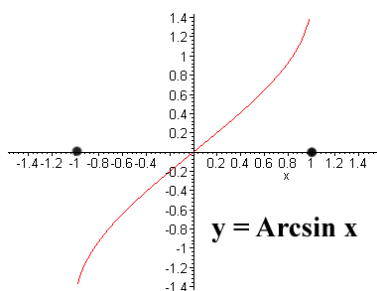
When you think of the graph of a trigonometric function you may have the general feeling that it's very *wavy*. In this case, the Horizontal Line Test should fail as horizontal lines through the range will intersect the graph quite a lot! So, how can they have an inverse? The only way this can happen is by making the domain 'small enough'. It shouldn't be surprising if it has an inverse on a suitable interval. So, every trigonometric function has an inverse on a suitably defined interval.

At this point we introduce the notion of the **inverse of a trigonometric function**. The graphical properties of the sine function indicate that it has an inverse when $\text{Dom}(\sin) = [-\pi/2, \pi/2]$. Its inverse is called the **Arcsine function** and it is defined for $-1 \leq x \leq 1$ by the rule that

$$y = \text{Arcsin}(x) \text{ means that } y \text{ is an angle whose sine is } x.$$

Since $\sin(\pi/2) = 1$, it follows that $\text{Arcsin}(1) = \pi/2$. The cosine function with $\text{Dom}(\cos) = [0, \pi]$ also has an inverse and it's called the **Arccosine function**. This Arccosine function is defined for $-1 \leq x \leq 1$, and its rule is given by $y = \text{Arccos}(x)$ which means that y is an angle whose cosine is x . Thus, $\text{Arccos}(1) = 0$, since $\cos(0) = 1$. Finally, the tangent function defined on $(-\pi/2, \pi/2)$ has an inverse called the **Arctangent function** and it's defined on the interval $(-\infty, +\infty)$ by the statement that $y = \text{Arctan}(x)$ only when y is an angle in $(-\pi/2, \pi/2)$ whose tangent is x . In particular, since $\tan(\pi/4) = 1$, $\text{Arctan}(1) = \pi/4$. The remaining inverse trigonometric functions can be defined by the relations $y = \text{Arccot}(x)$, the **Arccotangent function**, which is defined only when y is an angle in $(0, \pi)$ whose cotangent is x (and x is in $(-\infty, +\infty)$). In particular, since $\cot(\pi/2) = 0$, we see that $\text{Arccot}(0) = \pi/2$. Furthermore, $y = \text{Arcsec}(x)$, the **Arcsecant function**, only when y is an angle in $[0, \pi]$, different from $\pi/2$, whose secant is x (and x is outside the open interval $(-1, 1)$). In particular, $\text{Arcsec}(1) = 0$, since $\sec(0) = 1$. Finally, $y = \text{Arccsc}(x)$, the **Arccosecant function**, only when y is an angle in $(-\pi/2, \pi/2)$, different from 0, whose cosecant is x (and x is outside the open interval $(-1, 1)$). In particular, since $\csc(\pi/2) = 1$, $\text{Arccsc}(1) = \pi/2$.

NOTE: \sin , \cos are defined for all x (**in radians**) but this is **not true** for their 'inverses', \arcsin (or Arcsin), \arccos (or Arccos). Remember that the inverse of a function is always defined on the **range** of the original function.



Example 127.

Evaluate $\text{Arctan}(1)$.

Solution By definition, we are looking for an angle in radians whose tangent is 1. So $y = \text{Arctan}(1)$ means $\tan y = 1$ or $y = \frac{\pi}{4}$.

Function	Domain	Range
$y = \text{Arcsin } x$	$-1 \leq x \leq +1$	$-\frac{\pi}{2} \leq y \leq +\frac{\pi}{2}$
$y = \text{Arccos } x$	$-1 \leq x \leq +1$	$0 \leq y \leq \pi$
$y = \text{Arctan } x$	$-\infty < x < +\infty$	$-\frac{\pi}{2} < y < +\frac{\pi}{2}$
$y = \text{Arccot } x$	$-\infty < x < +\infty$	$0 < y < \pi$
$y = \text{Arcsec } x$	$ x \geq 1$	$0 \leq y \leq \pi, y \neq \frac{\pi}{2}$
$y = \text{Arccsc } x$	$ x \geq 1$	$-\frac{\pi}{2} \leq y \leq +\frac{\pi}{2}, y \neq 0$

Table 3.9: The Inverse Trigonometric Functions

Example 128. Evaluate $\text{Arcsin}(\frac{1}{2})$.

Solution By definition, we are looking for an angle in radians whose sine is $\frac{1}{2}$. So $y = \text{Arcsin}(\frac{1}{2})$ means $\sin y = \frac{1}{2}$ or $y = \frac{\pi}{6}$ (see Figure 57).

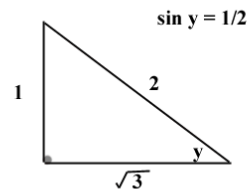


Figure 57.

Example 129. Evaluate $\text{Arccos}(\frac{1}{\sqrt{2}})$.

Solution By definition, we seek an angle in radians whose cosine is $\frac{1}{\sqrt{2}}$. So $y = \text{Arccos}(\frac{1}{\sqrt{2}})$ means $\cos y = \frac{1}{\sqrt{2}}$ or $y = \frac{\pi}{4}$.

Example 130. Evaluate $\text{Arcsec}(\sqrt{2})$.

Solution By definition, we are looking for an angle in radians whose secant is $\sqrt{2}$. So $y = \text{Arcsec}(\sqrt{2})$ means $\sec y = \sqrt{2}$ ($= \frac{\sqrt{2}}{1}$). The other side has length $s^2 = (\sqrt{2})^2 - 1^2 = 2 - 1 = 1$. So $s = 1$. Therefore, the \triangle is isosceles and $y = \frac{\pi}{4}$ (see Figure 58).

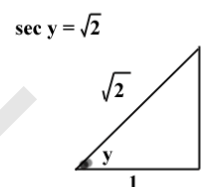


Figure 58.

Example 131. Find the value of $\sin(\text{Arccos}(\frac{\sqrt{2}}{2}))$.

Solution Let $y = \text{Arccos}(\frac{\sqrt{2}}{2})$ then $\cos y = \frac{\sqrt{2}}{2}$. But we want $\sin y$. So, since $\cos^2 y + \sin^2 y = 1$, we get

$$\sin y = \pm \sqrt{1 - \cos^2 y} = \pm \sqrt{1 - \frac{1}{2}} = \pm \frac{1}{\sqrt{2}}.$$

Hence

$$\sin(\text{Arccos}(\frac{\sqrt{2}}{2})) = \frac{1}{\sqrt{2}} (= \frac{\sqrt{2}}{2}).$$

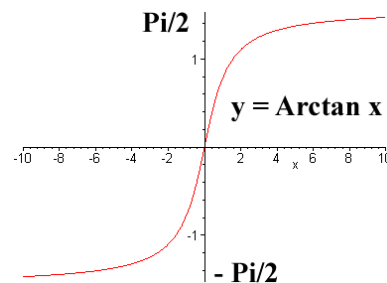
Example 132. Find $\sec(\text{Arctan}(-\frac{1}{2}))$.

Solution Now $y = \text{Arctan}(-\frac{1}{2})$ means $\tan y = -\frac{1}{2}$ but we want $\sec y$. Since $\sec^2 y - \tan^2 y = 1$ this means $\sec y = \pm \sqrt{1 + \tan^2 y} = \pm \sqrt{1 + \frac{1}{4}} = \pm \sqrt{\frac{5}{4}} = \pm \frac{\sqrt{5}}{2}$. Now we use Table 3.10, above.

Now, if we have an angle whose tangent is $-\frac{1}{2}$ then the angle is either in II or IV. But the angle must be in the interval $(-\frac{\pi}{2}, 0)$ of the domain of definition $(-\frac{\pi}{2}, \frac{\pi}{2})$ of tangent. Hence it is in IV and so its secant is > 0 . Thus, $\sec y = \sqrt{5}/2$ and we're done.

Example 133. Find the sign of $\sec(\text{Arccos}(\frac{1}{2}))$.

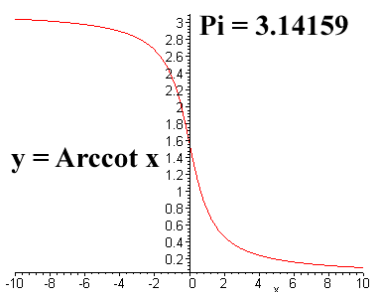
Solution Let $y = \text{Arccos}(\frac{1}{2}) \Rightarrow \cos y = \frac{1}{2} > 0$, therefore y is in I or IV. By definition,



Other Method: Signs of Trigonometric Functions

Quadrant	<u>sin</u>	<u>cos</u>	<u>tan</u>	<u>cot</u>	<u>sec</u>	<u>csc</u>
I	+	+	+	+	+	+
II	+	-	-	-	-	+
III	-	-	+	+	-	-
IV	-	+	-	-	+	-

Table 3.10: Signs of Trigonometric Functions



$\text{Arccos}(\frac{1}{2})$ is in $[0, \pi]$. Therefore y is in I or II, but this means that y must be in I. So, $\sec y > 0$ by Table 3.10, and this forces $\sec\left(\text{Arccos}\left(\frac{1}{2}\right)\right) = \sec y > 0$.

Example 134.

Determine the sign of the number $\csc(\text{Arcsec}(2))$.

Solution Let $y = \text{Arcsec}(2)$. Then $\sec y = 2 > 0$. Therefore y is in I or IV. By definition, $\text{Arcsec}(2)$ is in I or II. Therefore y is in I, and from the cosecant property, $\csc y > 0$ if y is in $[0, \frac{\pi}{2})$. So, $\csc(\text{Arcsec}(2)) = \csc y > 0$.

Example 135.

Find the sign of $\tan(\text{Arcsin}(-\frac{1}{2}))$.

Solution Let $y = \text{Arcsin}(-\frac{1}{2})$. Then $\sin y = -\frac{1}{2} \Rightarrow y$ in III or IV. By definition of Arcsin ; But, $y = \text{Arcsin}(-\frac{1}{2})$ must be in I or IV. Therefore y is in IV. So $\tan(\text{Arcsin}(-\frac{1}{2})) < 0$ (because $\tan < 0$ in IV).

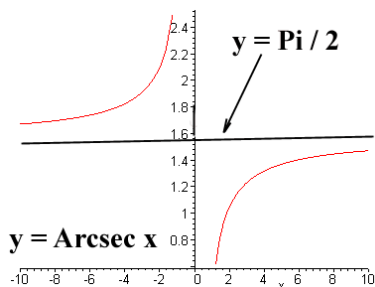


CAREFUL!!

Many authors of Calculus books use the following notations for the inverse trigonometric functions:

- $\text{Arcsin } x \iff \sin^{-1} x$
- $\text{Arccos } x \iff \cos^{-1} x$
- $\text{Arctan } x \iff \tan^{-1} x$
- $\text{Arccot } x \iff \cot^{-1} x$
- $\text{Arcsec } x \iff \sec^{-1} x$
- $\text{Arccsc } x \iff \csc^{-1} x$

The reason we try to avoid this notation is because it makes too many readers associate it with the reciprocal of those trigonometric functions and not their inverses. The reciprocal and the inverse are really different! Still, **you should be able to use both notations interchangeably**. It's best to know what the notation means, first.



NOTE: The inverse trigonometric functions we defined here in Table 3.9, are called the **principal branch** of the inverse trigonometric function, and we use the notation with an upper case letter 'A' for **Ar**csin, etc. to emphasize this. Just about every-

thing you ever wanted know about the basic theory of principal and non-principal branches of the inverse trigonometric functions may be found in the author's *Module on Inverse Functions* in the series *The ABC's of Calculus*, The Nolan Company, Ottawa, 1994.

Finally, we emphasize that since these functions are *inverses* then for any symbol, \square , representing some point in the domain of the corresponding inverse function (see Table 3.9), we always have

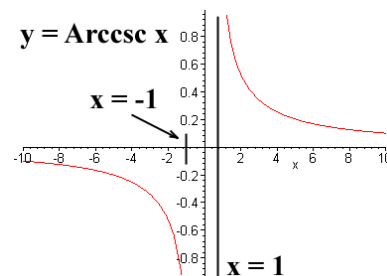
$$\begin{array}{ll} \sin(\text{Arcsin } \square) = \square & \cos(\text{Arccos } \square) = \square \\ \tan(\text{Arctan } \square) = \square & \cot(\text{Arccot } \square) = \square \\ \sec(\text{Arcsec } \square) = \square & \csc(\text{Arccsc } \square) = \square \end{array}$$

Exercise Set 17.

Evaluate the following expressions.

1. $\sin(\text{Arccos}(0.5))$ 2. $\cos(\text{Arcsin}(0))$ 3. $\sec(\sin^{-1}(\frac{1}{2}))$
4. $\csc(\tan^{-1}(-\frac{1}{2}))$ 5. $\sec(\sin^{-1} \frac{\sqrt{3}}{2})$ 6. $\text{Arcsin}(\tan(-\pi/4))$

NOTES:



3.9 Derivatives of Inverse Trigonometric Functions

Now that we know what these inverse trigonometric functions are, how do we find the derivative of the inverse of, say, the Arcsine (\sin^{-1}) function? Well, we know from equation 3.4 that

$$\frac{dF}{dx}(x) = \frac{1}{f'(F(x))}$$

where $F(x) = f^{-1}(x)$ is the more convenient notation for the inverse of f . Now let $f(x) = \sin x$, and $F(x) = \text{Arcsin } x$ be its inverse function. Since $f'(x) = \cos x$, we see that

$$\begin{aligned} \frac{d}{dx} \text{Arcsin } x &= \frac{dF}{dx}(x) \\ &= \frac{1}{f'(F(x))} \\ &= \frac{1}{\cos(F(x))} \\ &= \frac{1}{\cos(\text{Arcsin } x)} \end{aligned}$$

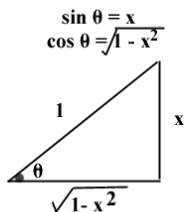


Figure 59.

Now, let $\theta = \text{Arcsin } x$, where θ is a lowercase Greek letter pronounced ‘thay-ta’. It is used to denote angles. Then, by definition, $\sin \theta = x$, and we’re looking for the value of $\cos \theta$, right? But since $\sin^2(\theta) + \cos^2(\theta) = 1$, this means that $\cos \theta = \pm\sqrt{1-x^2}$. So, which is it? There are two choices, here.

Look at the definition of the Arcsin function in Table 3.9. You’ll see that this function is defined only when the domain of the original sin function is restricted to $[-\pi/2, \pi/2]$. But, by definition, $\text{Ran}(\text{Arcsin}) = \text{Dom}(\sin) = [-\pi/2, \pi/2]$. So, $\cos \theta = \cos \text{Arcsin } x \geq 0$ because $\text{Arcsin } x$ is in the interval $[-\pi/2, \pi/2]$ and the cos function is either 0 or positive in there. So we must choose the ‘+’ sign. Good. So,

$$\cos(\text{Arcsin } x) = \sqrt{1-x^2}.$$

For another argument, see Figure 59. Finally, we see that

$$\begin{aligned} \frac{d}{dx} \text{Arcsin } x &= \frac{1}{\cos(\text{Arcsin } x)} \\ &= \frac{1}{\sqrt{1-x^2}}. \end{aligned}$$

The other derivatives are found using a similar approach.

$\frac{d}{dx} \sin^{-1}(u) = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$	$\frac{d}{dx} \cos^{-1}(u) = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx}, \quad u < 1,$
$\frac{d}{dx} \tan^{-1}(u) = \frac{1}{1+u^2} \frac{du}{dx}$	$\frac{d}{dx} \cot^{-1}(u) = \frac{-1}{1+u^2} \frac{du}{dx}$
$\frac{d}{dx} \sec^{-1}(u) = \frac{1}{ u \sqrt{u^2-1}} \frac{du}{dx},$	$\frac{d}{dx} \csc^{-1}(u) = \frac{-1}{ u \sqrt{u^2-1}} \frac{du}{dx}, \quad u > 1$

Table 3.11: Derivatives of Inverse Trigonometric Functions

If we let $u = u(x) = \square$ be any differentiable function, then we can use the basic derivative formulae and derive very general ones using the Chain Rule. In this way we can obtain Table 3.11.

Example 136. Evaluate the derivative of $y = \cos^{-1}(\frac{1}{x})$, (or $\text{Arccos}(\frac{1}{x})$).

Solution You can use any method here, but it always comes down to the Chain Rule. Let $u = \frac{1}{x}$, then $\frac{du}{dx} = -\frac{1}{x^2}$. By Table 3.11,

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \cos^{-1} u = -\frac{1}{\sqrt{1-u^2}} \frac{du}{dx} \\ &= -\frac{1}{\sqrt{1-(\frac{1}{x})^2}} \left(-\frac{1}{x^2}\right) = \frac{\frac{1}{x^2}}{\sqrt{1-\frac{1}{x^2}}} \\ &= \frac{1}{x^2 \sqrt{1-\frac{1}{x^2}}} = \frac{\sqrt{x^2}}{|x|^2 \sqrt{x^2-1}} \\ &= \frac{|x|}{|x|^2 \sqrt{x^2-1}} \\ &= \frac{1}{|x| \sqrt{x^2-1}}, \quad (|x| > 1). \end{aligned}$$

Example 137. Evaluate the derivative of $y = \cot^{-1}(\sqrt{x})$.

Solution Let $u = \sqrt{x}$, then $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$. So,

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \cot^{-1} u = -\frac{1}{1+u^2} \frac{du}{dx} \\ &= -\frac{1}{1+(\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}} = -\frac{1}{2\sqrt{x}(1+x)}. \end{aligned}$$

Example 138. If $y = \csc^{-1}(\sqrt{x+1})$, what is $y'(x)$?

Solution Let $u = \sqrt{x+1}$, then $\frac{du}{dx} = \frac{1}{2\sqrt{x+1}}$. So,

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \csc^{-1} u = \frac{-1}{|u| \sqrt{u^2-1}} \frac{du}{dx} \\ &= \frac{-1}{\sqrt{x+1}(\sqrt{x+1}-1)} \cdot \frac{1}{2\sqrt{x+1}} \\ &= \frac{-1}{2(x+1)\sqrt{x}}. \end{aligned}$$

NOTES:



Exercise Set 18.

Use Table 3.11 and the Chain Rule to find the derivatives of the functions whose values are given here.

1. $\text{Arcsin}(x^2)$, at $x = 0$
2. $x^2 \text{Arccos}(x)$
3. $\tan^{-1}(\sqrt{x})$
4. $\text{Arcsin}(\cos x)$
5. $\frac{\sin^{-1} x}{\sin x}$
6. $\sqrt{\sec^{-1} x}$
7. $\sin(2\text{Arcsin } x)$, at $x = 0$
8. $\cos(\sin^{-1}(4x))$
9. $\frac{1}{\text{Arctan } x}$
10. $x^3 \text{Arcsec}(x^3)$

Suggested Homework Set 12. Do problems 1, 4, 5, 7, 9

Web Links

On the topic of Inverse Trigonometric Functions see:

www.math.ucdavis.edu/~kouba/CalcOneDIRECTORY/invtrigderivdirectory/InvTrigDeriv.html

NOTES: