MATH 209/4 all sections except EC: - Fundamental Mathematics II

Midterm - February, 2014, 2pm (1h30min)
Only approved calculators are permitted.

<u>*MARKS*</u>

[7] **1.** (a) Find $\lim_{x\to 3} \dfrac{x^2 - 7x + 12}{x^2 - x - 6}$.

[7] (b) Give examples of functions $g(x)$ and $h(x)$ with the following properties:

(i) $\lim_{x\to 7} g(x) = 0$ (ii) $\lim_{x\to 7} h(x) = 0$ (iii) $\lim_{x\to 7} \dfrac{[g(x)]^2}{h(x)} = 2$

[7] **2.** Let $k(x) = x^3 - 5$. Work out the following in detail:

$$\lim_{s\to 0} \frac{k(x+s) - k(x)}{s}$$

[12] **3.** (a) If $f(x) = 4x^{\frac{3}{4}} - x^{-6}$, find $f'(1)$. You need not simplify.

(b) If $g(x) = [4x^3 + 7][3 - \ln(x^2)]$, find $g'(2)$. You need not simplify.

(c) Find $h'(x)$ if $h(x) = \dfrac{x^2 - \frac{1}{x}}{e^x - x^2}$. You need not simplify.

(d) Find the value of dy if $y = \ln(x+1)$, $x = 3$, and the change in x is 0.2.

[7] **4.** A stock grew from \$35 to \$120,000 in 43 years. Assuming continuous compounding, what is the associated annual rate of growth?

[10] **5.** The total profit (in dollars) from the sale of x lawn mowers is $P(x) = 30x - 0.03x^2 - 750$, $0 \le x \le 1{,}000$.

(a) Find the average profit per mower if 50 mowers are produced.

(b) Find the marginal average profit at a production level of 50 mowers, and interpret the results.

(c) Use the results from parts (a) and (b) to estimate the average profit per mower if 51 mowers are produced.

[10] **6.** Find x' for the function $x = x(t)$ defined implicitly by $1 + x\ln t = te^x$ and evaluate x' at $(t,\ x) = (1,\ 0)$.

[10] **7.** A person who is new on an assembly line performs an operation in T minutes after x performances of the operation, as given by

$$T = 6\left(1 + \frac{1}{\sqrt{x}}\right)$$

If $dx/dt = 6$ operations per hour, where t is time in hours, find dT/dt after 36 performances of the operation.

Lesson 11 Tuesday February 14, 201[illegible]

Midterm Review - Feb 2014

1 (a) Find $\lim_{x \to 3} \dfrac{x^2 - 7x + 12}{x^2 - x - 6}$

$= \lim_{x \to 3} \dfrac{(x-3)(x-4)}{(x-3)(x+2)}$

$\lim_{x \to 3} \dfrac{x-4}{x+2}$

$= \dfrac{3-4}{3+5}$

$= -1/5$

(b) Give examples of function $g(x)$ and $h(x)$ with the following properties.

(i) $\lim_{x \to 7} g(x) = 0$ (ii) $\lim_{x \to 7} h(x) = 0$ (iii) $\lim_{x \to 7} \dfrac{[g(x)]^2}{h(x)} = 2$

Find $g(x)$ and $h(x)$

→ THERE ARE MANY ANSWERS

① Let $g(x) = (x-7)(x-5) = x^2 - 12x + 35$

$h(x) = 2(x-7)^2$

$\dfrac{[g(x)]^2}{h(x)} = \dfrac{[x^2 - 12x + 35]^2}{2(x-7)^2}$

$\lim_{x \to 7} \dfrac{[g(x)]^2}{h(x)}$

$\lim_{x \to 7} \dfrac{[x^2 - 12x + 35]^2}{2(x-7)^2} \left(\dfrac{0}{0}\right)$

$\lim_{x \to 7} \dfrac{\cancel{(x-7)}(x-5)\cancel{(x-7)}(x-5)}{2\cancel{(x-7)^2}}$

$\lim_{x \to 7} \dfrac{(x-5)(x-5)}{2} = \dfrac{(7-5)(7-5)}{2} = \dfrac{2 \cdot 2}{2} = 2$

(2) The more obvious answer:

Let $g(x) = \sqrt{2(x-7)}$

$h(x) = x-7$

$$\lim_{x \to 7} \frac{[g(x)]^2}{h(x)}$$

$$= \lim_{x \to 7} \frac{\left[\sqrt{2(x-7)}\right]^2}{x-7}$$

$$= \lim_{x \to 7} \frac{2(x-7)}{(x-7)}$$

$$= 2$$

On some exams where such a question appears, it read:

Find two functions $f(x)$ and $g(x)$ such that

$$\lim_{x \to 4} f(x) = 0$$

$$\lim_{x \to 4} g(x) = 0$$

$$\lim_{x \to 4} \frac{f(x)}{g(x)} = 5$$

Let $f(x) = 5(x-4)$

$g(x) = (x-4)$

$$\lim_{x \to 4} \frac{f(x)}{g(x)} = \lim_{x \to 4} \frac{5(x-4)}{(x-4)} = 5$$

2. Let $k(x) = x^3 - 5$

$$\lim_{s \to 0} \frac{k(x+s) - k(x)}{s}$$

$$\lim_{s \to 0} \frac{(x+s)^3 - 5 - (x^3 - 5)}{s}$$

$$\to \underbrace{(x+s)(x+s)}(x+s)$$

$$(x^2 + 2xs + s^2)(x+s)$$

$$= x^3 + 2x^2s + xs^2 + x^2s + 2xs^2 + s^3$$

$$= x^3 + 3x^2s + 3xs^2 + s^3$$

$$= \lim_{s \to 0} \frac{x^3 + 3x^2s + 3xs^2 + s^3 - 5 - x^3 + 5}{s}$$

$$= \lim_{s \to 0} \frac{3x^2s + 3xs^2 + s^3}{s}$$

$$= \lim_{s \to 0} \frac{s(3x^2 + 3xs + s^2)}{s}$$

$$= \lim_{s \to 0} 3x^2 + 3xs + s^2$$

$$= 3x^2$$

3\. (a) If $f(x) = 4x^{3/4} - x^{-6}$, find $f'(1)$.

$f'(x) = 3x^{-1/4} + 6x^{-7}$

$f'(1) = 3(1)^{-1/4} + 6(1)^{-7}$

(b) If $g(x) = [4x^3 + 7][3 - \ln(x^2)]$, find $g'(2)$

$g'(x) = [12x^2][3 - \ln(x^2)] + [4x^3 + 7][-\frac{1}{x^2}(2x)]$

$g'(2) = [12(2)^2][3 - \ln(2^2)] + [4(2)^3 + 7][-\frac{1}{(2)^2}(2)(2)]$

(c) Find $h'(x)$ if $h(x) = \dfrac{x^2 - \frac{1}{x}}{e^x - x^2}$

$h(x) = \dfrac{x^2 - x^{-1}}{e^x - x^2}$

$h'(x) = \dfrac{[2x + x^{-2}][e^x - x^2] - [x^2 - x^{-1}][e^x - 2x]}{(e^x - x^2)^2}$

(d) Find the value of dy if $y = \ln(x+1)$, $x = 3$, and the change in x is 0.2 ($dx = 0.2$)

$\frac{dy}{dx} = \frac{1}{x+1}(1)$

$dy = \frac{1}{x+1}(dx)$

$dy = \frac{1}{3+1}(0.2)$

$= 0.05$

4\. A stock grew from \$35 to \$120,000 in 43 years. What is the associated annual rate of growth?

$A = Pe^{rt}$

$\left.\begin{array}{l} P = \$35 \\ A = \$120{,}000 \\ t = 43 \end{array}\right\}$ find r

$120{,}000 = 35e^{43r}$

$\frac{120{,}000}{35} = e^{43r}$

$e^{43r} = 3428.571429$

$\ln(e^{43r}) = \ln\left(\frac{120{,}000}{35}\right)$

$43r(\ln e) = \ln 3428.571429$

$r = \frac{\ln 3428.571429}{43}$

$r = 0.18929$ or 18.93%

5. The total profit (in dollars) from the sale of x lawn mowers is $P(x) = 30x - 0.03x^2 - 750$, $0 \le x \le 1000$

(a) Find the average profit per mower if 50 mowers are produced.

$$\bar{P}(x) = \frac{P(x)}{x} = \frac{-0.03x^2 + 30x - 750}{x}$$

rewrite to avoid quotient rule

$$\bar{P}(50) = \frac{-0.03(50)^2 + 30(50) - 750}{50}$$

$$\bar{P}(50) = 13.5$$

(b) Find the marginal average profit at a production level of 50 mowers, and interpret the results

derivative

$$\bar{P}(x) = -0.03x + 30 - 750x^{-1}$$

$$\bar{P}'(x) = -0.03 + 750x^{-2}$$

$$\bar{P}'(x) = -0.03 + \frac{750}{x^2}$$

$$\bar{P}'(50) = -0.03 + \frac{750}{50^2}$$

$$\bar{P}(50) = 0.27.$$

The marginal average profit at a production level of 50 mowers is \$0.27. This means the average profit will increase by \$0.27 if 51 mowers are sold

(c) Estimate average profit per mower if 51 mowers are produced

The average profit when 51 mowers are produced would be \$13.50 + \$0.27 = \$13.77

6. Find x' for the function $x = x(t)$ defined implicitly by $1 + x\ln t = te^x$ and evaluate x' at $(t, x) = (1, 0)$

Find $x' = \frac{dx}{dt}$ $\rightarrow$ $\frac{d}{dt}(1) + \frac{d}{dt}(x\ln t) = \frac{d}{dt}(te^x)$

$$0 + x' \cdot \ln t + x \cdot \frac{1}{t} = 1 \cdot e^x + t \cdot e^x \cdot x'$$

$$x'\ln t - te^x x' = e^x - \frac{x}{t}$$

$$x'(\ln t - te^x) = e^x - \frac{x}{t}$$

$$x' = \frac{e^x - \frac{x}{t}}{\ln t - te^x}$$

$$x'\Big|_{(1,0)} = \frac{e^0 - \frac{0}{1}}{\ln 1 - 1e^0} = \frac{1 - 0}{0 - 1} = -1$$

7. A person who is new on an assembly line performs an operation in T min after x performances of the operation, as given by:

$$T = 6\left(1 + \frac{1}{\sqrt{x}}\right)$$

If dx/dt = 6 operations/hour, where t = time in hours, find dT/dt after 36 performances of the operation

$T = 6 + \frac{6}{\sqrt{x}}$

$T = 6 + 6x^{-1/2}$

$T' = -3x^{-3/2} x'$

Find $T' = \frac{dT}{dt}$

Given $x' = \frac{dx}{dt} = 6$

$T' = -3(36)^{-3/2}(6)$

$= -0.083$

$x = 36$