MATH 3705A Tutorial 4

1. The solution of the wave equation $u_{xx} = \frac{1}{c^2} u_{tt}$, $0 < x < L$, $t > 0$, which satisfies the boundary conditions $u(0,t) = u(L,t) = 0$, $t > 0$, has the form

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right].$$

Find the solution of $u_{xx} = \frac{1}{25} u_{tt}$, $0 < x < 3$, $t > 0$, which satisfies the boundary conditions $u(0,t) = u(3,t) = 0$ and the initial conditions $u(x,0) = 1$ and $u_t(x,0) = \sin\left(\frac{\pi x}{3}\right) + 2\sin(11\pi x)$. Write down the complete solution.

Solution: Here $c = 5$, $L = 3$, so the solution

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{3}\right)\left[a_n \cos\left(\frac{5n\pi t}{3}\right) + b_n \sin\left(\frac{5n\pi t}{3}\right)\right],$$

and its first derivative in time

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{3}\right)\frac{5n\pi}{3}\left[-a_n \sin\left(\frac{5n\pi t}{3}\right) + b_n \cos\left(\frac{5n\pi t}{3}\right)\right].$$

With the initial condition:

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{3}\right) = 1,$$

we obtain $a_n = \frac{2}{3}\int_0^3 \sin\left(\frac{n\pi x}{3}\right)dx = -\frac{2}{\pi n}\cos\left(\frac{\pi n x}{3}\right)\Big|_0^3 = \frac{2}{\pi n}[1-(-1)^n]$.
From another initial condition

$$u_t(x,0) = \sum_{n=1}^{\infty} b_n \frac{5n\pi}{3}\sin\left(\frac{n\pi x}{3}\right) = \sin\left(\frac{\pi x}{3}\right) + 2\sin(11\pi x),$$

we see that $b_1 = \frac{3}{5\pi}$ and $b_{33} = \frac{2}{55\pi}$, while $b_n = 0$ if $n \neq 1$ or $n \neq 33$.

Finally, the solution of the problem:

$$u(x,t) = \frac{3}{5\pi}\sin\left(\frac{\pi x}{3}\right)\sin\left(\frac{5\pi t}{3}\right) + \frac{2}{55\pi}\sin(11\pi x)\sin(55\pi t) + \\ + \sum_{n=1}^{\infty}\frac{2}{\pi n}[1-(-1)^n]\sin\left(\frac{n\pi x}{3}\right)\cos\left(\frac{5n\pi t}{3}\right).$$

2. Find the polynomial solution of Laplace's equation $u_{xx}+u_{yy}=0$, $0<x<2$, $0<y<3$ subject to the boundary conditions: $u(x,0)=3x+2$, $u(x,3)=5-\frac{3}{2}x$, $u(0,y)=2+y$, $u(2,y)=8-2y$.

   Solution: Notice that all the boundary conditions are in the linear form, so now we need to check the corners:

$$\begin{aligned}
u(0,0) &= (3x+2)\Big|_{x=0} = (2+y)\Big|_{y=0} = 2, \\
u(2,0) &= (3x+2)\Big|_{x=2} = (8-2y)\Big|_{y=0} = 8, \\
u(0,3) &= (5-\frac{3x}{2})\Big|_{x=0} = (2+y)\Big|_{y=3} = 5, \\
u(2,3) &= (5-\frac{3x}{2})\Big|_{x=2} = (8-2y)\Big|_{y=3} = 2.
\end{aligned}$$

Thus, the polynomial solution $u(x,y)=\alpha x+\beta y+\gamma xy+\delta$ does exist. From BCs:

$$\begin{aligned}
u(x,0) &= \alpha x+\delta = 3x+2 \Rightarrow \alpha=3, \delta=2 \\
u(0,y) &= \beta y+2 = 2+y \Rightarrow \beta = 1
\end{aligned}$$

Also we know, that at the right upper corner $u(2,3)=3\cdot 2+3+6\gamma+2=2 \Rightarrow \gamma=-\frac{3}{2}$. Thus, the polynomial solution $u(x,y)=3x+y-\frac{3}{2}xy+2$.

3. The solution of Laplace's equation $u_{xx} + u_{yy} = 0$ within the rectangular region $0 < x < L$, $0 < y < M$, which satisfies the boundary conditions $u(0, y) = 0$, $u(L, y) = 0$, $u(x, 0) = 0$ and $u(x, M) = f(x)$, has the form

$$u(x, y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi y}{L}\right) \sin\left(\frac{n\pi x}{L}\right).$$

Find the solution within the region $0 < x < 3$, $0 < y < 2$, which satisfies the boundary conditions $u(0, y) = 0$, $u(3, y) = 0$, $u(x, 0) = 0$ and $u(x, 2) = x(x - 3)$. Write down the complete solution $u(x, y)$.

Solution: With $L = 3$ on the edge $y = 2$:

$$u(x, 2) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{2n\pi}{3}\right) \sin\left(\frac{n\pi x}{3}\right) = x(x - 3).$$

Then from the Fourier sine series theory

$$\begin{aligned} a_n \sinh\left(\frac{2n\pi}{3}\right) &= \int_0^3 x(x-3) \sin\left(\frac{n\pi x}{3}\right) dx = -\frac{3}{\pi n} x(x-3) \cos\left(\frac{n\pi x}{3}\right)\Big|_0^3 + \\ &+ \frac{3}{\pi n} \int_0^3 (2x-3) \cos\left(\frac{n\pi x}{3}\right) dx = \frac{9}{(\pi n)^2}(2x-3) \sin\left(\frac{n\pi x}{3}\right)\Big|_0^3 - \\ &- \frac{18}{(\pi n)^2} \int_0^3 \sin\left(\frac{n\pi x}{3}\right) dx = \frac{54}{(\pi n)^3} \cos\left(\frac{n\pi x}{3}\right)\Big|_0^3 = \frac{54}{(\pi n)^3}[(-1)^n - 1]. \end{aligned}$$

The complete solution

$$u(x, y) = \sum_{n=1}^{\infty} \frac{54}{\sinh\left(\frac{2n\pi}{3}\right)(\pi n)^3}[(-1)^n - 1] \sinh\left(\frac{n\pi y}{3}\right) \sin\left(\frac{n\pi x}{3}\right).$$

4. The solution of Laplace's equation $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$ inside the circle $r = 3$ is given by

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n [a_n \cos(n\theta) + b_n \sin(n\theta)].$$

Find the solution $u(r, \theta)$ subject to $u(3, \theta) = 4 \sin^2 \theta + 3 \sin(4\theta)$.

Solution: First, notice that $4\sin^2\theta = 2 - 2\cos(2\theta)$.

Solution on the boundary $r = 3$:

$$u(3,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} 3^n [a_n \cos(n\theta) + b_n \sin(n\theta)] = 2 - 2\cos(2\theta) + 3\sin(4\theta).$$

From here one by one:

$$\begin{aligned} &\frac{a_0}{2} = 2 \Rightarrow a_0 = 4, \\ &n = 2: \quad 3^2 a_2 = -2 \Rightarrow a_2 = -\frac{2}{9}, \quad a_n = 0 \text{ if } n \neq 2, \\ &n = 4: \quad 3^4 b_2 = 3 \Rightarrow b_4 = \frac{1}{27}, \quad b_n = 0 \text{ if } n \neq 4. \end{aligned}$$

The solution

$$u(r,\theta) = 2 - 2\left(\frac{r}{3}\right)^2 \cos(2\theta) + 3\left(\frac{r}{3}\right)^4 \sin(4\theta).$$