Carleton UNIVERSITY
Canada's Capital University

ECON4020 B

Advanced Microeconomic Theory

Midterm Examination II

March 16, 2014

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Duration: 1 hour and 20 minutes
Calculators are permitted.

30 1. 10 (a) State and define the three assumptions on preference relation $\succcurlyeq$ that ensure the existence of a continuous utility function.

10 (b) State the five properties of the indirect utility function. Be precise.

10 (c) State and prove the Roy's identity.

35 2. The expenditure function of a consumer is given by

$$e(p_1, p_2, u) = 10 p_1^{0.6} p_2^{0.4} u^{1.2}.$$

8 (a) Derive the Hicksian demand functions for goods 1 and 2.

9 (b) Find the matrix of substitution terms. Show that the matrix is negative semi-definite.

9 (c) Find the Marshallian demand function for good 1.

9 (d) Verify the Slutsky equation for $\partial x_1 / \partial p_1$.

35 3. A consumer's preference is represented by the utility function

$$u(x_1, x_2) = \ln x_1 + 2 \ln x_2.$$

15 (a) Derive the Hicksian demand functions for goods 1 and 2.

4 (b) Determine whether goods 1 and 2 are substitutes or complements.

8 (c) Derive the expenditure function of this consumer, and verify that it is increasing in p_1 and p_2.

8 (d) Derive the money metric utility function. Explain the economic meaning of this function.

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1(a) The three assumptions on $\succsim$:

(A1) Completeness: For all $\vec{x}, \vec{y} \in X$, either $\vec{x} \succsim \vec{y}$, or $\vec{y} \succsim \vec{x}$, or both.

(A2) Transitivity: For all $\vec{x}, \vec{y}, \vec{z} \in X$, if $\vec{x} \succsim \vec{y}$ and $\vec{y} \succsim \vec{z}$, then $\vec{x} \succsim \vec{z}$.

(A3) Continuity: For all $\vec{y} \in X$, the sets $\{\vec{x} \in X \mid \vec{x} \succsim \vec{y}\}$ and $\{\vec{x} \in X \mid \vec{y} \succsim \vec{x}\}$ are closed sets.

(b) The five properties of the indirect utility function:

(1) $v(\vec{p}, m)$ is non-increasing in $\vec{p}$.
(2) $v(\vec{p}, m)$ is increasing in m.
(3) $v(\vec{p}, m)$ is homogeneous of degree 0 in $(\vec{p}, m)$
(4) $v(\vec{p}, m)$ is quasi-convex in $(\vec{p}, m)$.
(5) $v(\vec{p}, m)$ is continuous in $(\vec{p}, m)$ at all $\vec{p} \gg \vec{0}$ and $m > 0$.

(c) Roy's identity:

$$x_i(\vec{p}, m) = -\frac{\frac{\partial v(\vec{p}, m)}{\partial p_i}}{\frac{\partial v(\vec{p}, m)}{\partial m}} \quad \text{for any good } i.$$

Proof: Let $\vec{x}^* = \vec{x}(\vec{p}, m)$ and $\lambda^* = \lambda(\vec{p}, m)$ be the solution to the Lagrange function

$$\mathcal{L} = u(\vec{x}) + \lambda(m - \vec{p} \cdot \vec{x}).$$

Then $v(\vec{p}, m) = \mathcal{L}(\vec{p}, m) = u(\vec{x}(\vec{p}, m)) + \lambda(\vec{p}, m)[m - \vec{p} \cdot \vec{x}(\vec{p}, m)]$

By the envelope theorem:

$$\frac{\partial v(\vec{p}, m)}{\partial p_i} = \frac{\partial \mathcal{L}}{\partial p_i}\Bigg|_{\lambda = \lambda^*, \vec{x} = \vec{x}^*} = -x_i^* \lambda^* \qquad (1)$$

$$\frac{\partial V(\vec{p},m)}{\partial m} = \left.\frac{\partial L}{\partial m}\right|_{\lambda=\lambda^*,\ \vec{x}=\vec{x}^*} = \lambda^* \qquad (2)$$

Plug (2) into (1):

$$\frac{\partial V(\vec{p},m)}{\partial p_i} = -x_i^* \frac{\partial V(\vec{p},m)}{\partial m}$$

Hence,

$$x_i(\vec{p},m) = x_i^* = -\frac{\partial V(\vec{p},m)/\partial p_i}{\partial V(\vec{p},m)/\partial m}$$

2. (a) By Shephard's lemma:

$$h_1(p_1,p_2,u)=\frac{\partial e(p_1,p_2,u)}{\partial p_1}=6p_1^{-0.4}p_2^{0.4}u^{1.2}$$

$$h_2(p_1,p_2,u)=\frac{\partial e(p_1,p_2,u)}{\partial p_2}=4p_1^{0.6}p_2^{-0.6}u^{1.2}$$

(b) $$\frac{\partial h_1}{\partial p_1}=-2.4p_1^{-1.4}p_2^{0.4}u^{1.2}$$

$$\frac{\partial h_2}{\partial p_2}=-2.4p_1^{0.6}p_2^{-1.6}u^{1.2}$$

$$\frac{\partial h_1}{\partial p_2}=\frac{\partial h_2}{\partial p_1}=2.4p_1^{-0.4}p_2^{-0.6}u^{1.2}$$

The matrix of substitution terms.

$$\sigma=\begin{bmatrix}-2.4p_1^{-1.4}p_2^{0.4}u^{1.2} & 2.4p_1^{-0.4}p_2^{-0.6}u^{1.2}\\ 2.4p_1^{-0.4}p_2^{-0.6}u^{1.2} & -2.4p_1^{0.6}p_2^{-1.6}u^{1.2}\end{bmatrix}$$

$\because$ $\frac{\partial h_1}{\partial p_1}<0$

$\frac{\partial h_2}{\partial p_2}<0$

$$|\sigma|=(-2.4)^2p_1^{-0.8}p_2^{-1.2}u^{2.4}-(2.4)^2p_1^{-0.8}p_2^{-1.2}u^{2.4}=0$$

$\therefore$ This matrix is negative semidefinite.

(c) First, use the expenditure function to find the indirect utility function.

$$e=10p_1^{0.6}p_2^{0.4}u^{1.2}$$

$$\Rightarrow u^{1.2}=\frac{e}{10p_1^{0.6}p_2^{0.4}}$$

$$\Rightarrow u = \frac{e^{\frac{1}{1.2}}}{10^{\frac{1}{1.2}} p_1^{0.6/1.2} p_2^{0.4/1.2}} = \frac{e^{5/6}}{10^{5/6} p^{1/2} p_2^{1/3}}$$

Hence, the indirect utility function is:

$$v(p_1, p_2, m) = \frac{m^{5/6}}{10^{5/6} p_1^{1/2} p_2^{1/3}}$$

Use Roy's identity to find the demand for good 1:

$$\frac{\partial v}{\partial p_1} = -\frac{1}{2} \frac{m^{5/6}}{10^{5/6} p_1^{3/2} p_2^{1/3}}$$

$$\frac{\partial v}{\partial m} = \frac{5}{6} \frac{m^{5/6-1}}{10^{5/6} p_1^{1/2} p_2^{1/3}}$$

$\therefore$ The Marshallian demand function for good 1:

$$x_1(p_1, p_2, m) = -\frac{\partial v/\partial p_1}{\partial v/\partial m} = \frac{\frac{1}{2}}{5/6} \cdot \frac{m}{p_1} = \frac{3m}{5p_1}$$

(d) The Slutsky equation for good 1.

$$\frac{\partial x_1}{\partial p_1} = \frac{\partial h_1}{\partial p_1} - \frac{\partial x_1}{\partial m} x_1$$

From (c): $\dfrac{\partial x_1}{\partial p_1} = -\dfrac{3m}{5p_1^2} = -0.6\dfrac{m}{p_1^2}$

$$\frac{\partial x_1}{\partial m} = \frac{3}{5p_1}$$

$$\frac{\partial h_1}{\partial p_1} - \frac{\partial x_1}{\partial m} x_1 = -2.4 p_1^{-1.4} p_2^{0.4} \left(v(p_1, p_2, m)\right)^{1.2} - \frac{3}{5p_1} \cdot x_1(p_1, p_2, m)$$

$$= -2.4 p_1^{-1.4} p_2^{0.4} \left[\frac{m^{5/6}}{10^{5/6} p_1^{1/2} p_2^{1/3}}\right]^{1.2} - \frac{3}{5p_1} \cdot \frac{3m}{5p_1}$$

$$\frac{\partial h_1}{\partial p_1} - \frac{\partial x_1}{\partial m} x_1 = -2.4\, p_1^{-1.4} p_2^{0.4} \frac{m}{10\, p_1^{0.6} p_2^{0.4}} - \frac{9m}{25 p_1^2}$$

$$= -\frac{24}{100} \frac{m}{p_1^2} - \frac{9m}{25 p_1^2}$$

$$= -\frac{(6+9)}{25} \cdot \frac{m}{p_1^2}$$

$$= -\frac{3}{5} \frac{m}{p_1^2}$$

$$\therefore \frac{\partial x_1}{\partial p_1} = \frac{\partial h_1}{\partial p_1} - \frac{\partial x_1}{\partial m} x_1$$

3. (a) The expenditure minimization problem:

$$\min_{x_1, x_2} \; p_1 x_1 + p_2 x_2$$

$$\text{s.t.} \quad \ln x_1 + 2\ln x_2 = u$$

The Lagrange function:

$$\mathcal{L} = p_1 x_1 + p_2 x_2 + \lambda\,[u - \ln x_1 - 2\ln x_2]$$

The first-order conditions:

$$\frac{\partial \mathcal{L}}{\partial x_1} = p_1 - \frac{\lambda}{x_1} = 0 \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = p_2 - \frac{2\lambda}{x_2} = 0 \qquad (2)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = u - \ln x_1 - 2\ln x_2 = 0 \qquad (3)$$

From (1) and (2), we obtain:

$$\frac{p_1}{p_2} = \frac{x_2}{2x_1}$$

$$\Rightarrow \quad x_2 = \frac{2p_1 x_1}{p_2} \qquad (4)$$

Substitute (4) into (3):

$$u = \ln x_1 + 2\ln x_2 = \ln x_1 + 2\ln\left[\frac{2p_1 x_1}{p_2}\right]$$

$$= \ln x_1 + 2\ln 2 + 2\ln p_1 + 2\ln x_1 - 2\ln p_2$$

$$= 3\ln x_1 + 2\ln 2 + 2\ln p_1 - 2\ln p_2$$

$$\Rightarrow 3\ln x_1 = u - 2\ln 2 - 2\ln p_1 + 2\ln p_2$$

$$\Rightarrow \ln x_1 = \frac{u}{3} - \frac{2}{3}\ln 2 - \frac{2}{3}\ln p_1 + \frac{2}{3}\ln p_2$$

$$= \ln\left[e^{u/3} \cdot \left(\frac{p_2}{2p_1}\right)^{2/3}\right]$$

Hence, the Hicksian demand function for good 1 is

$$\therefore \; h_1(p_1, p_2; u) = \left(\frac{p_2}{2p_1}\right)^{2/3} e^{u/3}$$

The Hicksian demand function for good 2 is

$$h_2(p_1, p_2, u) = \frac{2p_1 h_1}{p_2} = \frac{2p_1}{p_2} \cdot \left(\frac{p_2}{2p_1}\right)^{2/3} e^{u/3}$$

$$= \left(\frac{2p_1}{p_2}\right)^{1/3} e^{u/3}$$

(b) $$\frac{\partial h_1}{\partial p_2} = \frac{2}{3} \frac{p_2^{-1/3}}{(2p_1)^{2/3}} e^{u/3} > 0$$

Hence, good 1 and good 2 are substitutes.

(c) The expenditure function

$$e(p_1, p_2, u) = p_1 h_1(p_1, p_2, u) + p_2 h_2(p_1, p_2, u)$$

$$= p_1 \cdot \left(\frac{p_2}{2p_1}\right)^{2/3} e^{u/3} + p_2 \cdot \left(\frac{2p_1}{p_2}\right)^{1/3} e^{u/3}$$

$$= \left(\frac{1}{2}\right)^{2/3} \cdot p_1^{1/3} p_2^{2/3} e^{u/3} + 2^{1/3} p_1^{1/3} p_2^{2/3} e^{u/3}$$

$$= \left[\left(\frac{1}{2}\right)^{2/3} + 2^{1/3}\right] p_1^{1/3} p_2^{2/3} e^{u/3}$$

$$\frac{\partial e(p_1, p_2, u)}{\partial p_1} = \frac{1}{3}\left[\left(\frac{1}{2}\right)^{2/3} + 2^{1/3}\right] p_1^{-2/3} p_2^{2/3} e^{u/3} > 0$$

$$\frac{\partial e(p_1, p_2, u)}{\partial p_2} = \frac{2}{3}\left[\left(\frac{1}{2}\right)^{2/3} + 2^{1/3}\right] p_1^{1/3} p_2^{-1/3} e^{u/3} > 0$$

(d) The money metric utility function:

$$m(p_1, p_2, x_1, x_2) = e(p_1, p_2, u(x_1, x_2))$$

$$= \left[\left(\tfrac{1}{2}\right)^{2/3} + 2^{1/3}\right] p_1^{1/3} p_2^{2/3} e^{\frac{1}{3}u(x_1, x_2)}$$

$$= \left[\left(\tfrac{1}{2}\right)^{2/3} + 2^{1/3}\right] p_1^{1/3} p_2^{2/3} e^{\frac{1}{3}(\ln x_1 + 2\ln x_2)}$$

(Note: $e^{\frac{1}{3}(\ln x_1 + 2\ln x_2)} = e^{\ln(x_1^{1/3} x_2^{2/3})} = x_1^{1/3} x_2^{2/3}$.
But do not impose a penalty if the student does not take this additional step).

Economic meaning of the money metric function:
It gives the minimum expenditure at prices (p_1, p_2) necessary to purchase a bundle at least as good as (x_1, x_2).