# Discrete Mathematics for Computing MAT1348B

## **Second Test** — Version $\alpha$

10 February 2017

You must **sign below** to confirm that you have read, understand, and will follow these **instructions**:

- This is an 75-minute **closed-book** exam; no notes are allowed. **Calculators are not permitted.**
- The exam consists of 10 questions on 10 pages. Page 10 gives you additional work space. *Please do not detach it.*
- Questions 1-4 are **short-answer**. Write the final answer in the appropriate answer box. You need not show any other work.
- Questions 5-7 are **multiple-choice**. In each part, you must choose the correct response. You need not justify your answers.
- Questions 8-10 are **long-answer**. To receive full marks, your solution/proof must be complete, correct, and show all relevant details.
- Be sure to read carefully and follow the instructions for the individual problems.
- For rough work or additional work space, you may use the back pages. **Do not use scrap paper of your own**.
- You must use **proper mathematical notation and terminology**.
- You may ask for clarification. Your professor will visit the room to answer questions.
- **Cellular phones** and other unauthorized electronic devices **are not permitted** during this exam. Phones and other devices must be turned off completely and stored out of students' reach. Do not keep them in your possession, such as in your pockets. If you are caught with such a device, the following may occur: academic fraud allegations will be filed which may result in your obtaining a 0 (zero) for the exam.

LAST NAME: SOLUTIONS

First name:

Signature:

DGD (circle): SMD VNR STE

LAST NAME: ______________________

First name: ______________________

Student number: ______________________

| Question | 1 | 2 | 3 | 4 | 5 – 7 | 8 | 9 | 10 | Total |
|---|---|---|---|---|---|---|---|---|---|
| Max | 4 | 4 | 4 | 4 | $1+2+3$ | 4 | 6 | 5 | 37 |
| Marks | | | | | | | | | |

*Short-answer questions — write your final answer in the answer box. No justification is needed.*

[4pts] 1. Complete the following **definitions**:

An argument with premises $P_1, P_2, \ldots, P_k$ and conclusion $C$ is called **valid** if

$P_1 \wedge P_2 \wedge \ldots \wedge P_k \rightarrow C$ is a tautology.

Let $P_1, P_2, \ldots, P_k$ be propositions. The set $\{P_1, P_2, \ldots, P_k\}$ is called **consistent** if

there exists truth assignment such that $P_1, P_2, \ldots, P_k$ are all true.

OR: $P_1 \wedge \ldots \wedge P_k$ is not a contradiction

[4pts] 2. Define the following propositions in propositional variables $a, b, c$:

$$\begin{aligned} P_1 &: \quad \neg(a \wedge c) \\ P_2 &: \quad b \to a \\ P_3 &: \quad \neg a \wedge (b \vee c) \\ C &: \quad \neg a \wedge b \end{aligned}$$

Circle the correct response below. When applicable, give the additional information.

Is the set $\{P_1, P_2, P_3\}$ consistent? [✓ YES] NO

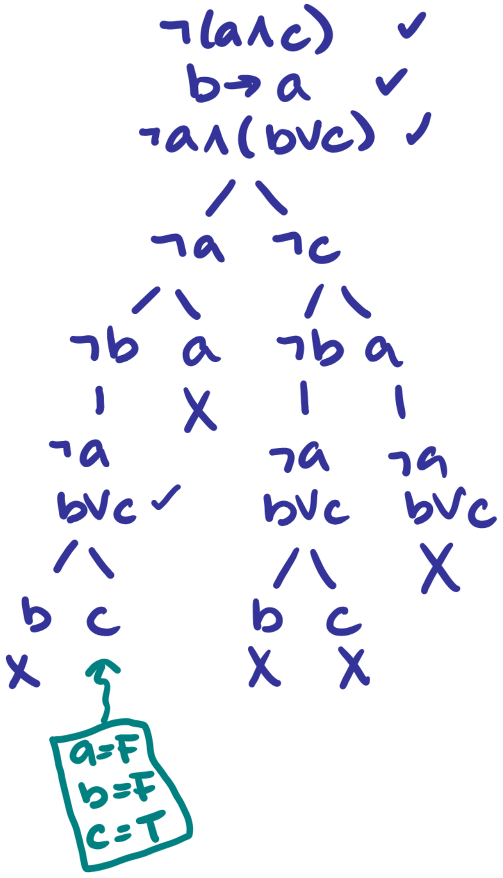
If you circled YES, give **all truth assignments** that justify your answer:

a=F, b=F, c=T

Is the argument $P_1 \wedge P_2 \wedge P_3 \to C$ **valid**? YES [✓ NO]

If you circled NO, give **all counterexamples**:

a=F, b=F, c=T



| $a$ | $b$ | $c$ | $\neg a \wedge b$ |
|---|---|---|---|
| F | F | T | F |

[4pts] 3. Consider the following argument:

**The ant bites the dog only if the dog barks.**
**The dog does not bark unless the dog chases the cat.**
**Therefore, the dog ~~does not~~ chases the cat if the ant ~~does not~~ bites the dog.**

Use the following propositional variables to translate the above argument into propositional logic:

$a$ : "The ant bites the dog."
$b$ : "The dog barks."
$c$ : "The dog chases the cat."

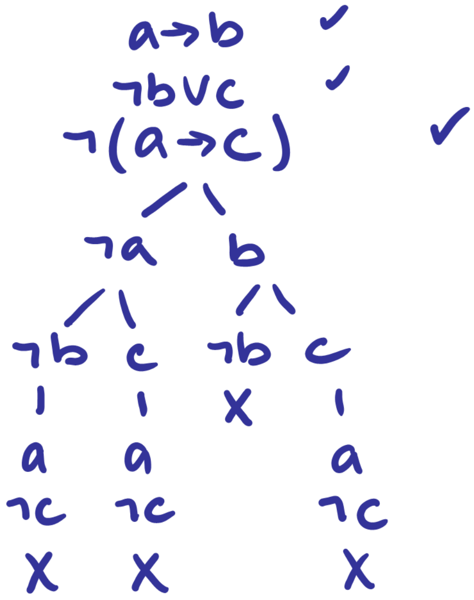
*Your answer must be in the form of an argument in propositional logic, consisting of compound propositions that use appropriate logical connectives and the variables a, b, c.*

Translation of the argument into propositional logic:

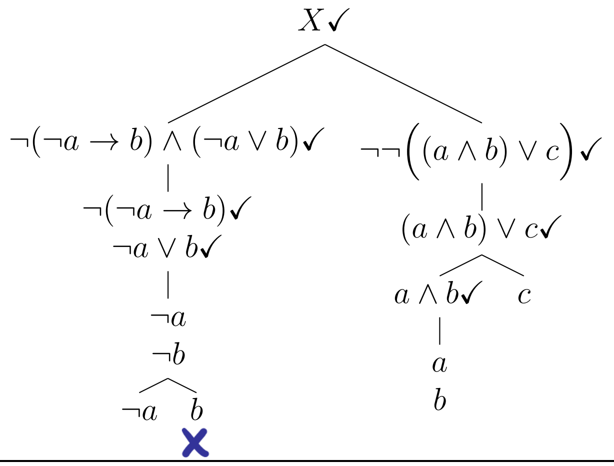
$$\begin{array}{l} a \to b \\ \neg b \lor c \\ \hline \therefore a \to c \end{array}$$

Is the given argument **valid**? [✓ YES] NO

If you circled NO, give **one counterexample**:



[4pts] 4. Let $X$ be a mystery compound proposition in propositional variables $a$, $b$, and $c$. Consider the following complete (and correct) truth tree, with $X$ at its root.



Is $X$ a contradiction? YES [✓ NO]

If you circled NO, give **one counterexample**:

a=F, b=F, c=T

Based on the above tree, give a DNF (disjunctive normal form) of $X$ (*do not simplify*):

$(\neg a \wedge \neg b) \vee (a \wedge b) \vee c$

All counterexamples:

TTT
TTF
TFT
FTT
FFT
FFF

*Multiple-choice questions. Write the correct response in the answer box. No justification is required.*

[1pts] 5. Exactly **one** of the following arguments is **not valid**. Which one?

**A.** $$\begin{array}{l} P \lor Q \\ \underline{\neg P} \\ \therefore \neg Q \end{array}$$

**B.** $$\begin{array}{l} P \\ \underline{P \to Q} \\ \therefore Q \end{array}$$

**C.** $$\begin{array}{l} \neg Q \\ \underline{P \to Q} \\ \therefore \neg P \end{array}$$

**D.** $$\begin{array}{l} \underline{P} \\ \therefore P \lor Q \end{array}$$

Answer: A

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6. Consider the complete truth tree A with compound proposition $P$ at its root. Which **two** of the following statements are **true**? [2pts]

**A.** If $A$ has **no** complete active (open) paths, then $P$ is a contradiction.

**B.** If $A$ has **no** complete active (open) paths, then $\neg P$ is a contradiction.

**C.** If $A$ has **no** complete active (open) paths, then $P$ is a tautology.

**D.** If $A$ has **no** complete active (open) paths, then $\neg P$ is a tautology.

Answer: A and D

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7. Suppose you want to prove a theorem of the form $P \to Q$ using a particular type of a proof. What should be the **first step** in your proof strategy? *Enter the corresponding letter in each box below.* [3pts]

**A.** Assume $P$ is true.

**B.** Assume $P$ is false.

**C.** Assume $Q$ is false.

**D.** Assume $Q$ is true.

**E.** Assume $P \land \neg Q$ is true.

First step in an **indirect proof (proof by contraposition):** C

First step in a **direct proof:** A

First step in an **proof by contradiction:** E

*Long-answer questions. Detailed solutions are required.*

[4pts] 8. Let $n$ be an integer. Give an **indirect proof (proof by contraposition)** of the following theorem:

**Theorem.** If $n^2 - 2$ is odd, then $n + 3$ is even.

*Your proof should use only the definition of odd and even integers, and should use no similar results from class. Clearly explain your proof strategy.*

The proposition is of the form $p \to q$, where

$p$: " $n^2-2$ is odd "
$q$: " $n+3$ is even "

For an indirect proof, we assume $\neg q$ and show $\neg p$ follows. Note:

$\neg p$: " $n^2-2$ is even "
$\neg q$: " $n+3$ is odd "

Proof Assume $n+3$ is odd. Then $n+3 = 2k+1$ for some integer $k$, and $n = 2k-2$.

$$n^2-2 = (2k-2)^2-2 = 4k^2-8k+4-2 = 2(2k^2-4k+1)$$
$$= 2m$$

for $m = 2k^2-4k+1$
Since $k$ is an integer, so is $m$.
Hence $n^2-2 = 2m$ for an integer $m$, showing that $n^2-2$ is even.

We proved: if $n+3$ is odd, then $n^2-2$ is even.

Hence: if $n^2-2$ is odd, then $n+3$ is even. □

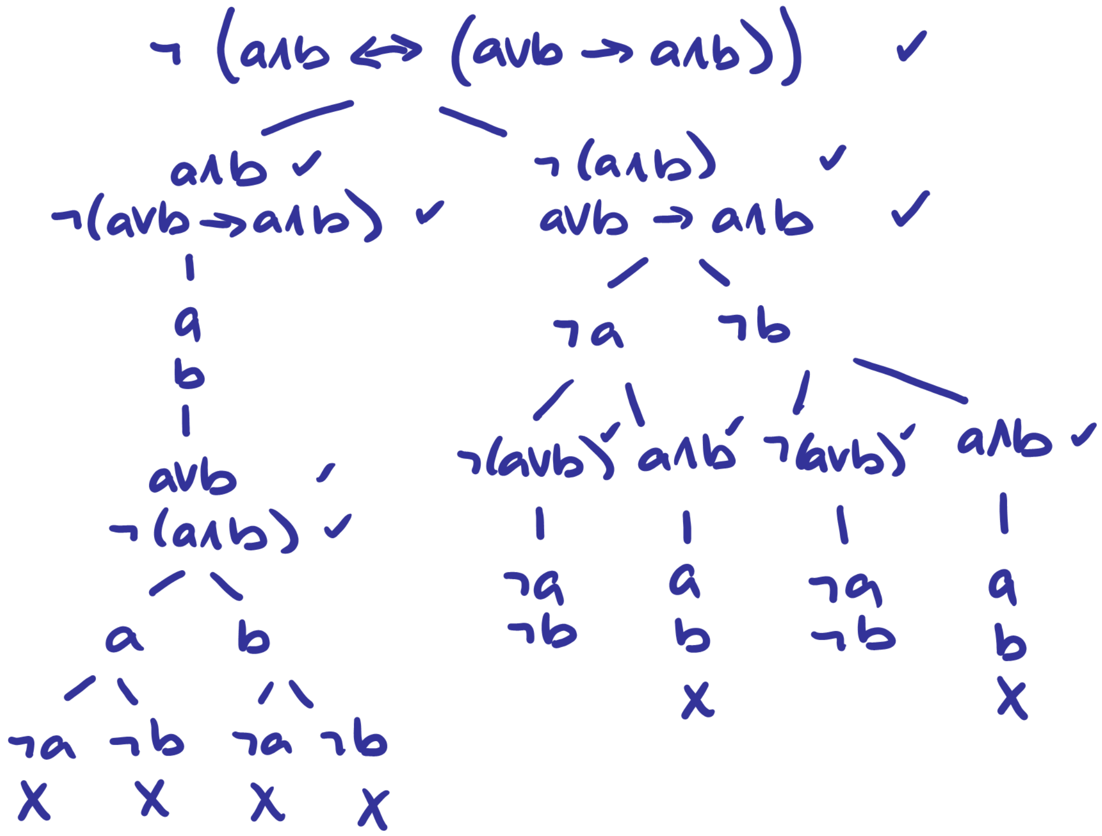
9. Let $a$ and $b$ be propositional variables. Define compound propositions $P$ and $Q$ as follows:

$$P: \quad a \wedge b \qquad \text{and} \qquad Q: \quad (a \vee b) \rightarrow (a \wedge b)$$

[6pts] Using an appropriate **truth tree**, determine whether $P$ is **logically equivalent** to $Q$.

*Clearly indicate the root of the tree. Use the Branching Algorithm precisely as taught in class. Do not use equivalences. Do not skip steps or combine branching rules.*

Root: $\neg(P \leftrightarrow Q)$



Are $P$ and $Q$ logically equivalent? *Circle:* YES [✓ NO]

If you circled NO, give **all counterexamples**.

a=F b=F

10. On the Island of Knights and Knaves you meet two inhabitants, $A$ and $B$. Suppose person $A$ says: "If there is treasure on the island, then $B$ is a knave." To which $B$ adds: "There is no treasure on the island and we are not both knights."

[5pts]

Use a **proof by contradiction** to show that there is indeed treasure on the island. *Clearly explain your proof strategy. Do not use truth tables.*

Define propositional variables:

$a$: "A is a knight"
$b$: "B is a knight"
$t$: "There is treasure on the island."

Then A said $t \to \neg b$ and B said $\neg t \wedge \neg(a \wedge b)$.

We are to prove $t = T$. For a proof by contradiction, we assume $t = F$ and show contradiction follows.

Proof. Suppose $t = F$. Then $t \to \neg b$ is T, so $a = T$ (A is a knight).

Case 1: B is a knight. Then $b = T$ and $\neg t \wedge \neg(a \wedge b)$ is F, which means B is lying – a contradiction.

Case 2: B is a knave. Then $b = F$ and $\neg t \wedge \neg(a \wedge b)$ is T, so B is telling the truth – a contradiction.

Thus $t = F$ leads to a contradiction (in both cases). We conclude $t = T$, i.e. there is treasure on the island.

*Additional work space. Please do not detach this page.*