ZM

CARLETON UNIVERSITY

FINAL EXAMINATION
December 2016

DURATION: 3 HOURS — **SCANTRON FORMS REQUIRED**

Department Name and Course Number: School of Mathematics and Statistics, MATH 1005 H, D&P
Course Instructor(s): Dr. Z. Montazeri.

AUTHORIZED MEMORANDA
Non-programmable, non-graphic calculators

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Instructions:

1. This examination consists of 9 pages. Please report any missing pages to the proctor.
2. This examination consists of 25 multiple-choice questions, worth 4 marks each.
3. Enter your name, ID number, course and section number, and all answers on the Scantron form, one correct answer per question.
4. Submit only the Scantron form.

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1. The change variable $u = y^{1-\alpha}$ transform the Bernoulli equation $y' + p(x)y = q(x)y^\alpha$ into an equation of the type

   (a) Separable (b) Homogeneous (c) Linear (d) Exact (e) None of these

$\frac{dy}{dx} = \frac{3x^2}{2y}$

2. The solution of initial-value problem $y' = \frac{3x^2}{2y}$, $y(0) = \sqrt{3}$, satisfies $y(1) =$

   (a) $\sqrt{2}$ (b) $-\sqrt{2}$ (c) 2 (d) 4 (e) None of these

$2y\frac{dy}{dx} = 3x^2$

$\int 2y\,dy = \int 3x^2\,dx$

$y^2 = x^3 + C$

$3 = C$

$y(1) = \sqrt{1+3} = 2$

$\frac{3x^2}{2y} = \frac{dy}{dx}$ $\quad \frac{3x^2}{2y}\,dx = dy$ $\quad \sqrt{1+3} = y \Rightarrow y = 2$

$3x^2\,dx = 2y\,dy \Rightarrow x^3 + C = y^2$

$\boxed{3 = C}$

3. The orthogonal trajectories of the general solution of the equation $y' = -\frac{2y}{3x}$ satisfy the equation

$\frac{3x}{2y}$

   (a) $y' = \frac{3x}{2y}$ (b) $y' = -\frac{2y}{3x}$ (c) $y' = \frac{3y}{2x}$ (d) $y' = -\frac{2x}{3y}$ (e) None of these

4. The equation $xy + y^2 - x^2y' = 0$ can be expressed as an equation of the type

   (a) Homogeneous and Bernoulli (b) Homogeneous and linear
   (c) Exact and Bernoulli (d) Linear and Bernoulli (e) None of these

5. The general solution of the differential equation $x^2\frac{dy}{dx} = xy + y^2$ is $y =$

   (a) $\frac{1}{c + \ln|x|}$ (b) $\frac{x}{c - \ln|x|}$ (c) $\frac{-x}{\ln|x| + c}$ (d) $\frac{-1}{\ln|x| + c}$ (e) None of these

$x^2y' = xy + y^2 \Rightarrow y' = \frac{y}{x} + \frac{y^2}{x}$ $\quad u = \frac{y}{x} \Rightarrow y = ux$ $\quad y' = u + u'x$

$\cancel{u} + u'x = \cancel{u} + u^2$ $\quad u'x = u^2 \Rightarrow \frac{du}{dx}x = u^2 \Rightarrow \frac{x}{dx} = \frac{u^2}{du} \Rightarrow \int \frac{1}{x}\,dx = \int \frac{1}{u^2}$

$\ln|x| = \int u^{-2}\,du \Rightarrow \ln|x| + C = \frac{u^{-1}}{-1} \Rightarrow \ln|x| = -\left(\frac{y}{x}\right)^{-1}$

$\frac{1}{\ln|x| + C} = \frac{-y}{x} \Rightarrow \frac{-x}{\ln|x| + C}$

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6. If $3 - \frac{1}{3n} < a_n < 3 + \frac{2}{n^3}$ for $n \geqslant 1$, what is $\lim_{n\to\infty} a_n$, if it exists?

(a) 0 (b) 1 (c) 2 (d) 3 (e) ∞

$\frac{3 \cdot 3n - 1}{3n} \Rightarrow 3$

$\lim\left[3 - \frac{1}{3n}\right] < \lim_{n\to\infty} a_n < \lim 3 + \frac{2}{n^3}$

$\frac{3n^3+2}{n^3} \Rightarrow \frac{3 + \frac{2}{n^3}}{1} = 3$

7. Solve the initial-value problem $y' - 2y = -e^{2x}\sin(x)$, $y(0) = -1$

(a) $y = e^{2x}(\sin(x) + 1)$ (b) $y = e^{2x}\cos(x) - e^{2x}$ (c) $y = -e^{2x}\cos(x) + 1$
(d) $y = -e^{2x}\sin(x) + e^{2x}$ (e) None of these

$r - 2 = 0$

$y - y^2 =$

$e^{\int 2\,dx} \Rightarrow \boxed{e^{2x}} = I(x)$

$y = \frac{1}{e^{2x}} \int e^{2x} \cdot -e^{2x} \sin(x) = \frac{1}{e^{2x}} \int -e^{4x} \sin(x)\,dx$

(+) $-e^{4x}$	$\sin(x)$
(−) $-4e^{4x}$	$-\cos(x)$
$-16e^{4x}$	$-\sin(x)$

$\frac{e^{4x}\cos(x) + 4e^{4x}}{e^{2x}} \Rightarrow e^{2x}\cos(x) + 4e^{2x}$

8. The general solution of the equation $\underbrace{(3x^2y^2 + y^3 + 2)}_{} + \underbrace{(2x^3y + 3xy^2 - 3)}_{} \frac{dy}{dx} = 0$ is

P_y

(a) $x^3y^2+xy^3+2x = c$ (b) $x^3y^2+xy^3+2x+3y = c$ (c) $x^3y^2+xy^3+2x-3y = c$
(d) $x^3y^2 + xy^3 - 3y = c$ (e) None of these

$P_y = 6x^2y + 3y^2 = Q_x = 6x^2y + 3y^2 \to$ exact

$f_x = \int (3x^2y^2 + y^3 + 2)\,dx = x^3y^2 + y^3x + 2x + g(y)$

$f_y = \cancel{2x^3y} + \cancel{3y^2x} + g'(y) = \cancel{2x^3y} + \cancel{3xy^2} - 3 \Rightarrow g'(y) = -3 \quad g(y) = -3y$

$x^3y^2 + y^3x + 2x - 3y = C$

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9. An integrating factor which makes the equation $y+y^2e^{-x}+(1+2xye^{-x})\dfrac{dy}{dx}=0$ exact is

(a) $I(x)=e^{x}$ (b) $I(x)=xe^{-x}$ (c) $I(x)=e^{-x}$ (d) $I(y)=ye^{y}$
(e) None of these

$P_y = 1+2ye^{-x}$ $\quad Q_x = 2ye^{-x} - 2xye^{+x}$

$$\frac{I'(x)}{I(x)} = \frac{P_y - Q_x}{Q} = \frac{1+\cancel{2ye^{-x}}-\cancel{2ye^{-x}}+2xye^{-x}}{1+2xye^{-x}} = 1$$

$\ln|I(x)| = \int 1\,dx = x \Rightarrow$

$\underbrace{ye^{x}+y^2}_{e^x+2y} + \underbrace{(e^{x}+2xy)}_{e^x+2y}y' = 0$

10. The solution of the initial-value problem $y''-2y'+y=0,\ y(0)=0,\ y'(0)=1$, is

(a) $y=xe^{x}$ (b) $y=e^{x}$ (c) $y=e^{x}+xe^{x}$ (d) $y=e^{x}-xe^{x}$
(e) $y=e^{x}-1$

$r^2-2r+1=0$ $\quad \boxed{y=C_1e^{x}+C_2xe^{x}}$ G. S.

$r=\dfrac{2\pm\sqrt{4-4}}{2} \Rightarrow r=1$ $\quad \boxed{0=C_1}$

$\boxed{0=C_1}$ $\quad y' = \cancel{C_1e^{x}} + C_2e^{x} + \cancel{C_2xe^{x}}$

$y'(0) = \boxed{1 = C_2}$ $\quad y = xe^{x}$

11. The general solution of $y''+2y'+5y=0.\ x>0$ is given by

(a) $e^{-x}[c_1\cos(2x)+c_2\sin(2x)]$ (b) $e^{-2x}[c_1\cos(x)+c_2\sin(x)]$
(c) $(-x)[c_1\cos(2x)+c_2\sin(2x)]$ (d) $(-2x)[c_1\cos(x)+c_2\sin(x)]$
(e) None of these

$r^2+2r+5=0$ $\quad r=\dfrac{-2\pm\sqrt{4-20}}{2} \Rightarrow \dfrac{-2\pm 4i}{2}$ $\quad \alpha=-1 \quad \beta=2$

$y=c_1e^{-x}\cos(2x)+C_2e^{-x}\sin(2x) \Rightarrow$

$y=e^{-x}[c_1\cos(2x)+c_2\sin(2x)]$

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12. The general solution of $x^2y'' - 3xy' + 13y = 0, \quad x > 0$ is given by

(a) $e^{2x}[c_1\cos(3\ln x) + c_2\sin(3\ln x)]$ (b) $x^2[c_1\cos(3\ln x) + c_2\sin(3\ln x)]$
(c) $x^2[c_1\cos(3x) + c_2\sin(3x)]$ (d) $e^{2x}[c_1\cos(3x) + c_2\sin(3x)]$
(e) $c_1\cos(3x) + c_2\sin(3x)$

$r^2 + (-3-1)r + 13 = 0 \Rightarrow r^2 - 4r + 13 = 0$

$r = \frac{4 \pm \sqrt{16-52}}{2} \Rightarrow \frac{4 \pm 6i}{2} \quad \alpha = 2 \quad \beta = 3$

$13 \times 4 = 52$, $52 - 16 = 36$

$y = x^{2x}[c_1\cos(3\ln(x)) + c_2\sin(3\ln(x))]$

13. A particular solution of $y'' + 2y' + y = 2e^{-x}$ is given by $y_p =$?

(a) e^{-x} (b) xe^{-x} (c) x^2e^{-x} (d) x^3e^{-x} (e) None of these

$r^2 + 2r + 1 = 0 \quad r = \frac{-2 \pm \sqrt{4-4}}{2} \Rightarrow r = -1$

$y_h = c_1e^{-x} + c_2xe^{-x}$

$y_p = Ae^{-x}$ ✗ $y_p = Axe^{-x}$ ✗ $y_p = Ax^2e^{-x}$

$y' = 2Axe^{-x} - Ax^2e^{-x}$

$y'' = 2e^{-x}A - 2xe^{-x}A + x^2e^{-x}A - 2Axe^{-x}$

$\cancel{2Ae^{-x} - 4Axe^{-x} + x^2e^{-x} + 4xe^{-x} - 2xe^{-x} + x^2e^{-x} = 2e^{-x}}$

$2Ae^{-x} - 2Axe^{-x} + \cancel{Ax^2e^{-x}A} + 4Axe^{-x} - \cancel{2Ax^2e^{-x}} + \cancel{Ax^2e^{-x}} = 2e^{-x}$

$2Ae^{-x} + 2Axe^{-x} = 2e^{-x} \Rightarrow A = 1$

$$\begin{pmatrix} 8 & -18 \\ 3 & -7 \end{pmatrix} \Rightarrow \begin{pmatrix} \lambda-8 & 18 \\ -3 & \lambda+7 \end{pmatrix} \Rightarrow (\lambda-8)(\lambda+7)+54$$

$$\lambda^2 - 8\lambda + 7\lambda - 56 + 54 = 0 \qquad \lambda^2 - \lambda - 2 = 0$$

$$\lambda = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} \qquad \lambda_1 = 2, \quad \lambda_2 = -1$$

$\lambda = 2$

$$\begin{pmatrix} -6 & 18 \\ -3 & 9 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \quad -6a + 18b = 0 \Rightarrow 6a = 18b \quad a = 3b$$

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$\lambda = -1$

$$\begin{pmatrix} -9 & 18 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \quad -9a + 18b = 0 \Rightarrow 9a = 18b \Rightarrow a = 2b$$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$c_1 e^{2t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad c_2 e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

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14. The general solution of the system $\begin{cases} x' = 8x - 18y \\ y' = 3x - 7y \end{cases}$ is

(a) $c_1e^t\begin{pmatrix}1\\-1\end{pmatrix} + c_2e^{2t}\begin{pmatrix}1\\1\end{pmatrix}$ (b) $c_1e^{2t}\begin{pmatrix}2\\1\end{pmatrix} + c_2e^t\begin{pmatrix}1\\3\end{pmatrix}$

(c) $c_1\begin{pmatrix}2\\1\end{pmatrix} + c_2\begin{pmatrix}3\\1\end{pmatrix}$ (d) $c_1e^{-t}\begin{pmatrix}2\\1\end{pmatrix} + c_2e^{2t}\begin{pmatrix}3\\1\end{pmatrix}$ (e) None of these

$\begin{pmatrix}8 & -18\\3 & -7\end{pmatrix} \Rightarrow \begin{pmatrix}\lambda-8 & 18\\-3 & \lambda+7\end{pmatrix} \Rightarrow (\lambda-8)(\lambda+7)+54$

$18 \times 3 = 54$, $54 - 48 = 6$

$\lambda^2+7\lambda-8\lambda-56+54=0 \Rightarrow \lambda^2-\lambda-2=0 \quad \lambda=\frac{1\pm\sqrt{1+8}}{2}$

$\lambda=\frac{1\pm3}{2} \quad \underline{\lambda_1=2} \quad \underline{\lambda_2=-1}$

$6a=18b \Rightarrow a=\frac{18}{6}b \Rightarrow a=3b$

$\underline{\lambda=2}$

$\left(\begin{array}{cc|c}-6 & 18 & a\\-3 & 9 & b\end{array}\right)=\begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \begin{array}{l}-6a+18b=0 \Rightarrow a=3b\\-3a+9b=0 \Rightarrow a=3b\end{array} \quad \begin{pmatrix}1\\3\end{pmatrix}$

$\underline{\lambda=-1}$

$\left(\begin{array}{cc|c}-9 & 18 & a\\-3 & 6 & b\end{array}\right)=\begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \begin{array}{l}-9a+18b=0 \Rightarrow a=2b\\-3a+6b=0 \Rightarrow a=2b\end{array} \quad \begin{pmatrix}1\\2\end{pmatrix}$

15. The sum of the series $\sum_{n=0}^{\infty}\frac{3.4^{n+1}}{5^n}$ is

(a) 15 (b) 60 (c) $\frac{15}{4}$ (d) 5 (e) None of these

~~$\frac{3.4\times3.4^n}{5^n}$ $3.4\left(\frac{3.4}{5}\right)^n$ $\frac{1}{1-\frac{3.4}{5}} = \frac{1}{\frac{5}{5}-\frac{3.4}{5}} = \left(\frac{1}{\frac{1.6}{5}}\right)$ $\frac{5}{1.6}\cdot 3.4$~~

$\frac{3\cdot4\cdot4^n}{5^n} \Rightarrow 12\left(\frac{1}{1-\frac{4}{5}}\right) \Rightarrow 12\cdot\frac{5}{1}$

16. The radius of convergence of the series $\sum_{n=0}^{\infty}\frac{(n+1)2^n}{3^n}x^n$ is

(a) $\frac{3}{2}$ (b) $\frac{2}{3}$ (c) 0 (d) ∞ (e) None of these

$\left(\frac{\frac{2^n}{3^n}}{\frac{2^{n+1}}{3^{n+1}}}\right) \Rightarrow \frac{2^n\,3\cdot3^n}{3^n\,2\cdot2^n} \Rightarrow \lim_{n\to\infty}\left|\frac{C_n}{C_{n+1}}\right| \quad \left|\left(\frac{\frac{2^n}{3^n}}{\frac{2^{n+1}}{3^{n+1}}}\right)\right| \Rightarrow \frac{3\cdot2^n\cdot3^n}{2\cdot3^n\cdot2^n}$

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17. Given that the radius of convergence, R, of the series $\sum_{n=0}^{\infty} \frac{(-1)^n(x-3)^n}{\sqrt{n+2}}$ is $R = 1$, the interval of convergence is

(a) $(2, 4)$ (b) $[2, 4]$ (c) $[2, 4)$ (d) $[-3, 3)$ (e) $(2, 4]$

$-1 < x - 3 < 1 \Rightarrow 2 < x < 4$ $[2, 4]$

$\frac{(-1)^n (-1)^n}{\sqrt{n+2}} \Rightarrow \frac{1}{\sqrt{n+2}}$ -> diverge

$\frac{(-1)^n (1)^n}{\sqrt{n+2}}$ -> converges

18. The series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$

(a) converges (b) diverges

p -> series $p > 1$

19. The series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n^2}{\sqrt{3n^4 + 1}}$

(a) Converges absolutely (b) Converges conditionally (c) Diverges

$\frac{n^2}{\sqrt{3n^4+1}}$

20. The series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n[\ln(n)]}$

(a) Converges absolutely (b) Converges conditionally (c) Diverges

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21. The series $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{3^n+1}$

(a) Converges absolutely (b) Converges conditionally (c) Diverges

$\frac{2^4}{3^n+1}$

22. The power-series representation of the function $f(x) = \dfrac{x^2}{1-x^2}$ about the point $a = 0$ is ?

(a) $\sum_{n=0}^{\infty} x^n$ (b) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ (c) $\sum_{n=0}^{\infty} x^{n+2}$ (d) $\sum_{n=0}^{\infty} x^{2n+2}$
(e) $\sum_{n=0}^{\infty} (-x)^{2n+2}$

$f = \frac{x^2}{1-x^2}$ $\quad f' = \frac{2x(1-x^2) - x^2(-2x)}{(1-x^2)^2}$

$(x-a)^n \quad (x)^n$

$\frac{2(1-1) - 1(-2)}{1-1} = 0$

$\frac{4(1-4) - 4(-4)}{(1-4)^2}$

$\frac{4 - \cancel{16} + \cancel{16}}{9}$ $\left(\frac{4}{9}\right)$

23. The coefficient of x^4 in the Taylor series of the function $f(x) = \cos(2x)$ about the point $a = 0$ is

(a) $\frac{2}{3}$ (b) $\frac{1}{4!}$ (c) $-\frac{1}{4!}$ (d) $-\frac{2}{3}$ (e) 16

$f(x) = \cos(2x)$ $\quad f'(x) = -2\sin(2x)$ $\quad f''(x) = -4\cos(2x)$

$f'''(x) = 8\sin(2x)$ $\quad f''''(x) = 16\cos(2x)$

$\cancel{\frac{\cos(2x)\ 16}{4!}} \Rightarrow \cancel{\frac{16}{2\cdot3\cdot4}} = \cancel{\frac{16}{24}} \Rightarrow \cancel{\frac{1}{4}}$

$\frac{16}{4!} = \frac{16}{24} \quad \frac{4}{6} = \frac{2}{3}$

$$\int_0^1 x \sin(n\pi x)\,dx = -\frac{x}{n\pi}\underbrace{\cos(n\pi x)}_{(-1)^n} + \cancel{\frac{1}{(n\pi)^2}\sin(n\pi x)} \to 0$$

F	x	$\int \sin(n\pi x)$
G	1	$-\frac{1}{n\pi}\cos(n\pi x)$
	0	$-\frac{1}{(n\pi)^2}\sin(n\pi x)$

$$-\frac{1}{n\pi}(-1)^n$$

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24. The coefficient of x^3 in the Binomial series of the function $f(x) = \frac{1}{(1+x)^3}$ about the point $a = 0$ is

(a) -10 (b) 10 (c) $\frac{1}{3!}$ (d) $-\frac{1}{3!}$ (e) 60

$f(x) = (1+x)^{-3}$ $f'(x) = -3(1+x)^{-4}$ $f''(x) = 12(1+x)^{-5}$ $f'''(x) = -60(1+x)^{-6}$

$\frac{-60}{6} = -10$

25. Let

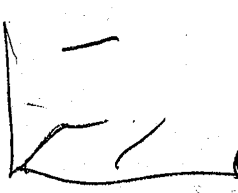
$$f(x) = \begin{cases} 0 & -1 \leqslant x \leqslant 0 \\ x & \cancel{0 < x \leqslant 1} \end{cases}$$

$f(x) = x$

$-1 < x < 1$

The Fourier series of f is of the form $\frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(n\pi x) + b_n \sin(n\pi x)]$, where $n \geqslant 1$ and

$\frac{2}{L}\left[\int_{0}^{1} \int_{-1}^{0}\right] ?$



(a) $a_0 = 1,\ a_n = \frac{2}{n^2\pi^2}[(-1)^n - 1],\ b_n = \frac{-2(-1)^n}{n\pi}$
(b) $a_0 = \frac{1}{2},\ a_n = \frac{1}{n^2\pi^2}[(-1)^n - 1],\ b_n = \frac{-(-1)^n}{n\pi}$
(c) $a_0 = 1,\ a_n = \frac{2}{n^2\pi^2}[1 - (-1)^n],\ b_n = \frac{(-1)^n}{n\pi}$
(d) $a_0 = \frac{1}{2},\ a_n = \frac{1}{n^2\pi^2}[1 - (-1)^n],\ b_n = \frac{(-1)^n}{n\pi}$

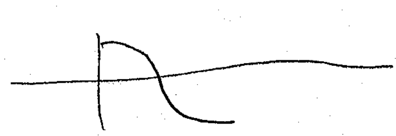
$\frac{2}{L}\int_0^1$

$a_0 = \frac{1}{1}\int_0^1 x\,dx \Rightarrow \frac{x^2}{2}\Big]_0^1 \Rightarrow \frac{1}{2}$

$a_n = 1\int_0^1 x\cos(n\pi x)\,dx = \frac{x}{n\pi}\sin(n\pi x) + \frac{1}{(n\pi)^2}\cos(n\pi x)\Big]_0^1$

$$\begin{array}{c|c} + x & \cos(n\pi x) \\ - 1 & \frac{1}{n\pi}\sin(n\pi x) \\ 0 & -\frac{1}{(n\pi)^2}\cos(n\pi x) \end{array}$$

$\frac{1}{n\pi}\cancel{\sin(n\pi)}^{0} + \frac{1}{(n\pi)^2}\cos(n\pi)^{(-1)^n} - 0 - \frac{1}{(n\pi)^2}$



$\frac{(-1)^n + 1}{(n\pi)^2}$