

CONCORDIA UNIVERSITY

COMPUTER SCIENCE AND SOFTWARE ENGINEERING

COMP232

MATHEMATICS FOR COMPUTER SCIENCE

ASSIGNMENT 2 - SOLUTIONS

FALL 2016

1. Consider the implication

$$p \wedge (\neg q \rightarrow \neg p) \Rightarrow q$$

- (a) Prove the implication by a direct proof, *i.e.* assume that the left hand side of the implication is True, and then show that the right hand side must be True also.

SOLUTION: Assume the LHS is True. Then p is True and $\neg q \rightarrow \neg p$ is True. Now $\neg q \rightarrow \neg p$ is equivalent to $p \rightarrow q$, and since p is True and $p \rightarrow q$ is True it follows that q is True, *i.e.*, the RHS is True.

- (b) Prove the implication by contradiction.

SOLUTION: Suppose the LHS is True but the RHS is False. Thus p and $\neg q \rightarrow \neg p$ are True, but q is False. Equivalently, by contrapositive, p and $p \rightarrow q$ are True, but q is False. Since q is False and $p \rightarrow q$ is True it follows that p is False. But this contradicts the earlier assumption that p is True.

2. For each of the statements below state whether it is True or False. If True then give a proof. If False then give a counterexample.

- (a) If $a, b \in \mathbb{Z}^+$ are odd then the product ab is odd.

SOLUTION: True. We can write $a = 2k_a + 1$ and $b = 2k_b + 1$, for nonnegative integers k_a and k_b . Then $ab = (2k_a + 1)(2k_b + 1) = 4k_a k_b + 2k_a + 2k_b + 1 = 2(2k_a k_b + k_a + k_b) + 1$, which shows that ab is odd.

- (b) If $a, b \in \mathbb{Z}^+$, where a is even and b is odd then $a + b$ is odd.

SOLUTION: True. We can write $a = 2k_a$ and $b = 2k_b + 1$, for positive integer k_a and nonnegative integer k_b . Then $a + b = 2k_a + (2k_b + 1) = 2(k_a + k_b) + 1$, which shows that $a + b$ is odd.

- (c) Let $a, b \in \mathbb{Z}^+$. If $a + b$ is even, then a^2 or b^2 is even.

SOLUTION: False. Let $a = b = 3$. Then $a + b = 6$ is even, but both a^2 and b^2 are odd, namely both equal 9.

- (d) For all $a \in \mathbb{Z}^+$, if $a > 3$ then $a^2 - 4$ is composite.

SOLUTION: True: $a^2 - 4 = (a - 2)(a + 2)$, where both factors are greater than 1.

3. Let $n \in \mathbb{Z}^+$. Prove that the following statements are equivalent:

$$n^3 \text{ is odd, } \quad n^2 \text{ is odd, } \quad 1 - n \text{ is even, } \quad n^2 + 1 \text{ is even.}$$

PROOF: Re-order the above as

$$(1) \ 1 - n \text{ is even, } \quad (2) \ n^2 \text{ is odd, } \quad (3) \ n^2 + 1 \text{ is even, } \quad (4) \ n^3 \text{ is odd.}$$

To establish equivalence it suffices to prove that

$$(1) \Rightarrow (2), \quad (2) \Rightarrow (3), \quad (3) \Rightarrow (4), \quad \text{and} \quad (4) \Rightarrow (1).$$

- (a) $(1) \Rightarrow (2)$: If $1 - n$ is even then $1 - n = 2k_1$ for some integer k_1 . Hence $n = -2k_1 + 1$ so that n is odd. Then $n^2 = (-2k_1 + 1)^2 = 2(2k_1^2 - 2k_1) + 1$, so n^2 is odd.
- (b) $(2) \Rightarrow (3)$: Since n^2 is odd we have $n^2 = 2k_2 + 1$. Thus $n^2 + 1 = 2k_2 + 2 = 2(k_2 + 1)$, so that $n^2 + 1$ is even.
- (c) $(3) \Rightarrow (4)$: If $n^2 + 1$ is even then (easily) n^2 is odd, $n^2 = 2k_3 + 1$. Now if n^2 is odd then n is odd (which follows easily from the contrapositive: if n is even then n^2 is even). Thus $n = 2k_4 + 1$. Hence $n^3 = n^2 n = (2k_3 + 1)(2k_4 + 1) = 2(2k_3k_4 + k_3 + k_4) + 1$, so that n^3 is odd.
- (d) $(4) \Rightarrow (1)$: We prove the contrapositive: If $1 - n$ is odd then n^3 is even. Now if $1 - n$ is odd then (easily) n is even, say, $n = 2k_5$. Hence $n^3 = 2(4k_5^3)$ is even.

4. (a) Give a proof by cases to show that the equation

$$5x^2 + 4y^3 = 51$$

does not have solutions $x, y \in \mathbb{Z}^+$.

SOLUTION: It is clear that we only need to consider $x \leq 3$ and $y \leq 2$. This leads to six cases: $(x, y) = (1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2)$. The cases $(x, y) = (2, 1), (2, 2)$ can be excluded immediately since they would give an even sum. Thus we need only consider $(x, y) = (1, 1), (1, 2), (3, 1), (3, 2)$. However, these give the sums 9, 37, 49, 77, respectively, that is, not 51.

- (b) For $n \in \mathbb{Z}^+$ prove by contraposition that if $2n^3 + 3n^2 + 4n + 5$ is odd then n is even.
 SOLUTION: The contrapositive is: if n is odd then $2n^3 + 3n^2 + 4n + 5$ is even. So let n be odd. Then $2n^3$ and $4n$ are even, while both $3n^2$ and 5 are odd, so the overall sum is even.

5. For each of the statements below state whether it is True or False. If True then give a proof. If False then explain why, *e.g.*, by giving a counterexample.

- (a) The sum of the squares of any two rational numbers is a rational number.

SOLUTION: True: Suppose x and y are rational numbers :

$$x = \frac{p_1}{q_1} \quad \text{and} \quad y = \frac{p_2}{q_2}.$$

where p_1, q_1 and p_2, q_2 are integers, with q_1 and q_2 nonzero. Then

$$x^2 + y^2 = \frac{p_1^2}{q_1^2} + \frac{p_2^2}{q_2^2} = \frac{p_1^2 q_2^2 + p_2^2 q_1^2}{q_1^2 q_2^2}, \text{ which is rational.}$$

- (b) For all positive $x \in \mathbb{R}$, if x is irrational then \sqrt{x} is irrational.

SOLUTION: True. By contrapositive: suppose \sqrt{x} is not irrational, *i.e.*, it is rational. Then we can write $\sqrt{x} = \frac{p}{q}$, for certain $p, q \in \mathbb{Z}^+$. It follows that $x = \left(\frac{p}{q}\right)^2 = \frac{p^2}{q^2}$, which is rational, *i.e.*, not irrational.

- (c) For all $x, y \in \mathbb{R}$, if x and y are irrational then $x^2 + y$ is irrational.

SOLUTION: False. Let $x = 2^{\frac{1}{4}}$ and $y = -2^{\frac{1}{2}}$. Then both x and y are irrational, but $x^2 + y = (2^{\frac{1}{4}})^2 - 2^{\frac{1}{2}} = 2^{\frac{1}{2}} - 2^{\frac{1}{2}} = 0$, where 0 is considered to be a rational number, as it can be written as $\frac{0}{1}$.

In the above proof we used the well-known fact that $2^{\frac{1}{2}}$ is irrational. For completeness we now show that $2^{\frac{1}{4}}$ is also irrational. Suppose on the contrary that $2^{\frac{1}{4}}$ is rational, *i.e.*, $2^{\frac{1}{4}} = \frac{p}{q}$, for certain $p, q \in \mathbb{Z}^+$. Then $2^{\frac{1}{2}} = (2^{\frac{1}{4}})^2 = \left(\frac{p}{q}\right)^2 = \frac{p^2}{q^2}$, which contradicts the fact that $2^{\frac{1}{2}}$ is irrational.

- (d) $\log_{10}(2)$ is irrational.

SOLUTION: True: This is most easily proved by contradiction: Suppose $\log_{10}(2)$ is rational. (We know that $\log_{10}(2)$ is positive.) Then $\log_{10}(2) = \frac{p}{q}$ for positive integers p and q . By definition of the logarithm function this means that $2 = 10^{p/q}$, or equivalently, $2^q = 10^p$. Now 2 is the only prime divisor of 2^q , while $10^p = 2^p 5^p$ has 5 as a prime divisor, so that 2^q and 10^p cannot be equal. Hence we have a contradiction.

6. Give a proof of each of the following:

- (a) If the integers 1, 2, 3, \dots , 7, are placed around a circle, in any order, then there exist two adjacent integers that have a sum greater than or equal to 9.

SOLUTION: The number 7 must be somewhere along the circle. The smallest neighbors that 7 can have are the integers 1 and 2. Thus 7 must have a neighbor that is greater than or equal to 2. Hence the sum of 7 and one of its neighbors is at least 9.

- (b) If the integers 1, 2, 3, \dots , 16 are placed around a circle, in any order, then there exist three integers in consecutive locations around the circle that have a sum greater than or equal to 27.

SOLUTION: Suppose the sum of any three adjacent numbers is less than 27, *i.e.*, less than or equal to 26. The number 1 must be somewhere along the circle. The remaining 15 positions can be grouped in five non-intersecting groups of three adjacent positions. Then the total sum will be less than or equal to $5 \cdot 26 + 1 = 131$. However, we know that $1 + 2 + \dots + 16 = 16 \cdot 17/2 = 136$, so that we have a contradiction.

7. For each of the following, determine whether it is valid or invalid. If valid then give a proof. If invalid then give a counter example.

(a) $(A \cap B) \cup (C \cap D) = (A \cap D) \cup (C \cap B)$

SOLUTION: False. Take A and D be non-empty, with $A = D$, and let both B and C be empty. Then $(A \cap B) \cup (C \cap D)$ is empty, but $(A \cap D) \cup (C \cap B) = A$ is not empty, so that they cannot be equal.

(b) $A - (B \cup C) = (A - B) \cap (A - C)$

SOLUTION: True:

$$\begin{aligned} x \in A - (B \cup C) &\iff x \in A \wedge x \notin (B \cup C) \\ &\iff x \in A \wedge x \notin B \wedge x \notin C \\ &\iff x \in A \wedge x \notin B \wedge x \in A \wedge x \notin C \\ &\iff x \in (A - B) \wedge x \in (A - C) \\ &\iff x \in (A - B) \cap (A - C) \end{aligned}$$

(c) $B \cap C \subseteq A \Rightarrow (C - A) \cap (B - A)$ is empty

SOLUTION: This statement is valid. We prove the contrapositive:

If $(C - A) \cap (B - A)$ is not empty. then $B \cap C$ is not a subset of A .

PROOF: Since $(C - A) \cap (B - A)$ is not empty, there must be an x such that $x \in (C - A) \cap (B - A)$, *i.e.*, $x \in C$ and $x \in B$ and $x \notin A$. Thus $x \in B \cap C$ and $x \notin A$. Thus $B \cap C$ is not a subset of A .

(d) $(A \cup B) - (A \cap B) = A \Rightarrow B$ is empty

SOLUTION: This statement is also valid. We prove the contrapositive:

If B is not empty then $(A \cup B) - (A \cap B) \neq A$.

To prove the contrapositive we use a proof by contradiction: Suppose B is not empty, but $(A \cup B) - (A \cap B) = A$. Then there exists an element $b \in B$. There are two cases to consider:

Case 1: $b \in A$: Then $b \in A \cup B$ and $b \in A \cap B$. Hence $b \notin (A \cup B) - (A \cap B)$. Thus, using our assumption that $(A \cup B) - (A \cap B) = A$ it follows that $b \notin A$, which is a contradiction.

Case 2: $b \notin A$: In this case $b \in A \cup B$, but $b \notin A \cap B$. Hence $b \in (A \cup B) - (A \cap B)$. Using our assumption that $(A \cup B) - (A \cap B) = A$, it follows that $b \in A$, which is a contradiction.