



Université d'Ottawa • University of Ottawa

Faculté des sciences / Faculty of Science
Mathématiques et de statistique / Mathematics and Statistics

Calculus III for Engineers

MAT 2322A - Fall 2016

Final Exam

Professor: Victor G. LeBlanc

Time limit: 3 hours. Closed books.

Name: Solutions ID Number: _____

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Instructions

- This is a 15 pages closed book exam. **The only calculators which are allowed are Texas Instruments TI-30, TI-34, Casio fx-260 and fx-300, scientific and non programmable.**
- Questions 1 to 10 are multiple choice questions. These questions are worth 2 points each and no partial marks are possible. **Please write your answers in the corresponding boxes in the grid below entitled "Answers to multiple choice Qs".**
- Questions 11 to 16 are long answer questions and are worth 5 marks each, so organize your time accordingly. **A correct answer requires a full, clearly-written and detailed solution.** Answer each question in the space provided, using backs of pages or the extra pages if necessary.

Answers to multiple choice Qs

1	2	3	4	5	6	7	8	9	10
E	B	B	C	F	C	A	D	C	E

Grid below is used for grading
(do not write in this grid)

MCQ	11	12	13	14	15	16	Total
	/20	/5	/5	/5	/5	/5	/50

1. A solid three dimensional region is defined by the inequalities $0 \leq x \leq 2$, $1 \leq y \leq 4$ and $0 \leq z \leq 1$. If this solid has a mass density given by $\delta(x, y, z) = x + y + z$, find the total mass of the solid.

A. 28

B. 27

C. 26

D. 25

E. 24

F. 23

$$\int_{x=0}^2 \int_{y=1}^4 \int_{z=0}^1 (x+y+z) dz dy dx = \int_0^2 \int_1^4 (xz + yz + \frac{z^2}{2}) \Big|_0^1 dy dx$$

$$= \int_0^2 \int_1^4 (x + y + \frac{1}{2}) dy dx = \int_0^2 (xy + \frac{y^2}{2} + \frac{y}{2}) \Big|_1^4 dx$$

$$= \int_0^2 ((4x + 8 + 2) - (x + \frac{1}{2} + \frac{1}{2})) dx = \int_0^2 (3x + 9) dx$$

$$= \int_0^2 (\frac{3x^2}{2} + 9x) \Big|_0^2 = 6 + 18 = 24$$

2. Which of the following is true regarding the critical points of the function

$$z = f(x, y) = x^3 + xy + y^2 ?$$

A. The function has a saddle point at $(0, 0)$ and a local maximum at $(\frac{1}{6}, -\frac{1}{12})$.

B. The function has a saddle point at $(0, 0)$ and a local minimum at $(\frac{1}{6}, -\frac{1}{12})$.

C. The function has no critical points.

D. The function has a local minimum at $(0, 0)$ and a local maximum at $(\frac{1}{6}, -\frac{1}{12})$.

E. The function has a local maximum at $(0, 0)$ and a local minimum at $(\frac{1}{6}, -\frac{1}{12})$.

F. The function has saddle points at both $(0, 0)$ and at $(\frac{1}{6}, -\frac{1}{12})$.

$$f_x = 3x^2 + y \quad f_y = x + 2y \quad f_{xx} = 6x \quad f_{xy} = f_{yx} = 1 \quad f_{yy} = 2$$

$$f_x = 0 \Rightarrow y = -3x^2 \quad x - 6x^2 = 0 \Rightarrow x(1 - 6x) = 0 \Rightarrow x = 0 \text{ or } x = \frac{1}{6}$$

$(0, 0), (\frac{1}{6}, -\frac{1}{12})$ are the critical pts. $D = f_{xx}f_{yy} - f_{xy}^2$

at $(0, 0)$: $D = -1 \rightarrow (0, 0)$ saddle

at $(\frac{1}{6}, -\frac{1}{12})$: $D = 2 - 1 = 1$, $f_{xx} = 1 > 0 \rightarrow (\frac{1}{6}, -\frac{1}{12})$ local min.

3. Compute the directional derivative of the function $f(x, y) = x^2y + 3x$ at the point $(1, 2)$ in the direction of the vector $\vec{u} = \frac{\vec{i}}{\sqrt{2}} - \frac{\vec{j}}{\sqrt{2}}$.

- A. $2\sqrt{2}$
- B. $3\sqrt{2}$**
- C. $4\sqrt{2}$
- D. $5\sqrt{2}$
- E. $6\sqrt{2}$
- F. $7\vec{i} + \vec{j}$

$$\begin{aligned}
 f_x &= 2xy + 3 & f_y &= x^2 \\
 \vec{\nabla} f(1, 2) &= 7\vec{i} + \vec{j} \\
 D_{\vec{u}} f(1, 2) &= \vec{\nabla} f(1, 2) \cdot \vec{u} \\
 &= (7\vec{i} + \vec{j}) \cdot \left(\frac{\vec{i}}{\sqrt{2}} - \frac{\vec{j}}{\sqrt{2}} \right) \\
 &= \frac{7}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \frac{6}{\sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}
 \end{aligned}$$

4. Let S be the surface which is the graph of the paraboloid $z = 4 + x^2 + y^2$ for (x, y) in the domain $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$. Consider the scalar function

$$f(x, y, z) = \sqrt{4x^2 + 4y^2 + 1}$$

What is the value of the surface integral $\iint_S f \, dS$?

- A. π
- B. 2π
- C. 3π**
- D. 4π
- E. 5π
- F. 6π

$$\begin{aligned}
 \vec{r}(a, \theta) &= a \cos \theta \vec{i} + a \sin \theta \vec{j} + (4 + a^2) \vec{k} \\
 \vec{r}_a &= \cos \theta \vec{i} + \sin \theta \vec{j} + 2a \vec{k} \\
 \vec{r}_\theta &= -a \sin \theta \vec{i} + a \cos \theta \vec{j} \\
 \vec{r}_a \times \vec{r}_\theta &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos \theta & \sin \theta & 2a \\ -a \sin \theta & a \cos \theta & 0 \end{vmatrix} \\
 &= \vec{i}(-2a^2 \cos \theta) - \vec{j}(2a^2 \sin \theta) + \vec{k} a \\
 \|\vec{r}_a \times \vec{r}_\theta\| &= \sqrt{4a^4 + a^2} = a\sqrt{4a^2 + 1} \\
 \iint_S f \, dS &= \int_0^{2\pi} \int_0^1 \sqrt{4a^2 + 1} \sqrt{4a^2 + 1} \, da \, d\theta \\
 &= \int_0^{2\pi} \int_0^1 a(4a^2 + 1) \, da \, d\theta \\
 &= \int_0^{2\pi} \left(\frac{4}{3} a^3 + a \right) \Big|_0^1 \, d\theta = \int_0^{2\pi} \left(\frac{4}{3} + 1 \right) \, d\theta = 2\pi \left(\frac{7}{3} \right) = \frac{14\pi}{3}
 \end{aligned}$$

5. Which of the following integrals, if evaluated, would give you the arc-length of the curve $\vec{r}(t) = t^6 \vec{i} + t \vec{j} + t^2 \vec{k}$, $0 \leq t \leq 4$?

A. $\int_0^4 (t^6 \vec{i} + t \vec{j} + t^2 \vec{k}) dt$

$\vec{r}'(t) = 6t^5 \vec{i} + \vec{j} + 2t \vec{k}$

B. $\int_0^4 (t^{12} + t^4 + t^2) dt$

$\|\vec{r}'(t)\| = \sqrt{36t^{10} + 1 + 4t^2}$

C. $\int_0^4 \sqrt{t^{12} + t^4 + t^2} dt$

Length = $\int_0^4 \|\vec{r}'(t)\| dt$

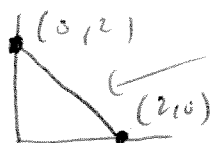
D. $\int_0^4 \sqrt{t^6 + t + t^2} dt$

E. $\int_0^4 (36t^{10} + 4t^2 + 1) dt$

F. $\int_0^4 \sqrt{36t^{10} + 4t^2 + 1} dt$

6. If C is the oriented straight line segment in the plane starting at the point $(0, 2)$ and ending at the point $(2, 0)$, and $\vec{F}(x, y) = x^2 y \vec{i} + y \vec{j}$, what is the value of the line integral $\int_C \vec{F} \cdot d\vec{r}$?

$\int_C \vec{F} \cdot d\vec{r}$?



$y = 2 - x$

$\vec{r}(x) = x \vec{i} + (2-x) \vec{j}$
 $0 \leq x \leq 2$

A. $-4/3$

B. -1

C. $-2/3$

D. $-1/3$

E. 0

F. $1/3$

$\vec{F}(\vec{r}(x)) = x^2(2-x) \vec{i} + (2-x) \vec{j}$

$\vec{r}'(x) = \vec{i} - \vec{j}$

$\vec{F}(\vec{r}(x)) \cdot \vec{r}'(x) = x^2(2-x) - (2-x)$
 $= 2x^2 - x^3 - 2 + x$

$\int_C \vec{F} \cdot d\vec{r} = \int_0^2 (2x^2 - x^3 - 2 + x) dx = \left[\frac{2x^3}{3} - \frac{x^4}{4} - 2x + \frac{x^2}{2} \right]_0^2$
 $= \frac{16}{3} - \frac{16}{4} - 4 + 2 = \frac{16}{3} - 6 = \frac{16-18}{3} = -\frac{2}{3}$

7. The square surface $x = 0, 0 \leq y \leq 1, 0 \leq z \leq 1$ is oriented with unit normal \vec{i} . Consider the vector field $\vec{F}(x, y, z) = x^2\vec{i} + yz\vec{j} + x\vec{k}$. Then the value of the flux integral $\iint_S \vec{F} \cdot d\vec{S}$ is

A. 0

B. 1

C. -1

D. \vec{j}

E. π

F. None of the above

$$\vec{F}(0, y, z) = yz\vec{j} \perp \vec{x}$$

$$\text{So } \iint_S \vec{F} \cdot d\vec{S} = 0$$

8. Let f, u and v be differentiable functions such that

$$u(0, 0) = 4$$

$$v(0, 0) = 1$$

$$u_x(0, 0) = -3$$

$$u_y(0, 0) = 4$$

$$v_x(0, 0) = 2$$

$$v_y(0, 0) = 3$$

$$u_y(4, 1) = -2$$

$$v_y(4, 1) = -5$$

$$\vec{\nabla}f(4, 1) = 2\vec{i} + 3\vec{j}$$

$$\vec{\nabla}f(0, 0) = \vec{0}$$

If $z(x, y) = f(u(x, y), v(x, y))$, then what is the value of $\frac{\partial z}{\partial y}(0, 0)$? $= \frac{\partial f}{\partial u}(u(0,0), v(0,0)) \frac{\partial u}{\partial y}(0,0) + \frac{\partial f}{\partial v}(u(0,0), v(0,0)) \frac{\partial v}{\partial y}(0,0)$

A. 0

B. 13

C. -17

D. 17

E. -19

F. -21

$$= \frac{\partial f}{\partial u}(4, 1)(4) + \frac{\partial f}{\partial v}(4, 1)(3)$$

$$= 2 \cdot (4) + 3 \cdot (3) = 8 + 9 = 17$$

9. Consider the two-dimensional region D drawn below, whose boundary is the oriented curve C , also drawn. Let $\vec{F}(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j}$ be a vector field with continuous partial derivatives. Then which of the following equations corresponds to Green's theorem?

A. $\int_C Q dx + P dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$

B. $\int_C P dx + Q dy = \iint_D \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) dx dy$

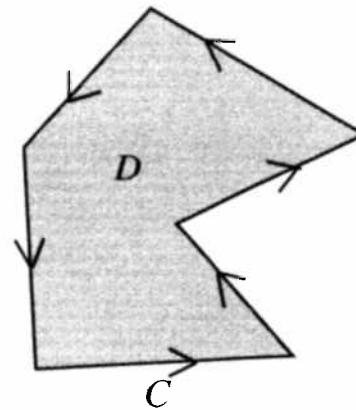
C. $\int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$

D. $\int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x} \right) dx dy$

E. $\int_C P dx + Q dy = \iint_D \left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} \right) dx dy$

F. $\int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y} \right) dx dy$

See notes



10. What is the volume of the region bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 2$, $z = 0$ and the graph of the function $z = f(x, y) = 3x^2 + 6y$

A. 6

B. 8

C. 10

D. 12

E. 14

F. 16

$$\int_0^1 \int_0^2 (3x^2 + 6y) dy dx =$$

$$\int_0^1 3x^2 y + 3y^2 \Big|_0^2 dx = \int_0^1 (6x^2 + 12) dx$$

$$= 6x \Big|_0^1 + 12x \Big|_0^1 = 2 + 12 = 14$$

11. Using any valid technique, find the global extrema of the function

$$f(x, y) = x^2 + y^2 - 2y$$

on the closed and bounded domain

$$D = \{(x, y) \mid \underbrace{x^2 + y^2}_{g(x, y)} \leq 4\}$$

$f_x = 2x$, $f_y = 2y - 2$
 \Rightarrow Critical point is $(0, 1)$ which belongs to D , so keep it

Boundary analysis: $\vec{\nabla} f = \lambda \vec{\nabla} g$
 $g = 4$

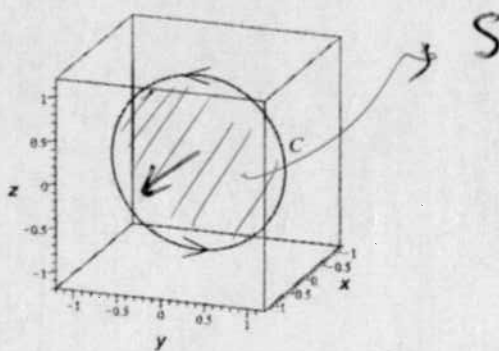
$$\begin{array}{l} 2x = 2\lambda x \longrightarrow 2x(1-\lambda) = 0 \Rightarrow x=0 \text{ or } \lambda=1 \\ 2y = 2\lambda y - 2\lambda \\ x^2 + y^2 = 4 \end{array} \left. \begin{array}{l} y^2 = 4 \\ y = \pm 2 \\ \lambda = 2 \text{ or } -\frac{2}{3} \\ (0, 2), (0, -2) \end{array} \right| \begin{array}{l} 2y = 2y - 2 \\ \times \\ \text{impossible} \end{array}$$

Check points

(x, y)	$f(x, y) = x^2 + y^2 - 2y$
$(0, 1)$	-1 \leftarrow global min
$(0, 2)$	0
$(0, -2)$	8 \leftarrow global max

12. Consider the vector field $\vec{F}(x, y, z) = ze^{x+y^2} \vec{i} + (2yze^{x+y^2} - z^3) \vec{j} + (e^{x+y^2} + y^3) \vec{k}$.

- (a) Compute the divergence and the curl of \vec{F} , i.e. compute $\vec{\nabla} \cdot \vec{F}$ and $\vec{\nabla} \times \vec{F}$.
- (b) Using Stokes' theorem, compute the line integral $\int_C \vec{F} \cdot d\vec{r}$, where C is the circle $y^2 + z^2 = 1$ in the yz -plane, oriented as illustrated in the figure.



$$\begin{aligned} \text{a) } \vec{\nabla} \cdot \vec{F} &= \left(\frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k} \right) \cdot (ze^{x+y^2} \vec{i} + (2yze^{x+y^2} - z^3) \vec{j} + (e^{x+y^2} + y^3) \vec{k}) \\ &= ze^{x+y^2} + 2ze^{x+y^2} + 4y^2ze^{x+y^2} = z(3+4y^2)e^{x+y^2} \end{aligned}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ze^{x+y^2} & 2yze^{x+y^2} - z^3 & e^{x+y^2} + y^3 \end{vmatrix}$$

$$\begin{aligned} &= \vec{i} (2ye^{x+y^2} + 3y^2 - 2ye^{x+y^2} + 3z^2) - \vec{j} (e^{x+y^2} - e^{x+y^2}) \\ &\quad + \vec{k} (2yze^{x+y^2} - 2yze^{x+y^2}) = \vec{i} (3y^2 + 3z^2) \end{aligned}$$

(Extra page)

b) C is the boundary of the oriented surface S
(see diagram). By Stokes' theorem

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} \quad \text{Parametrize } S:$$

$$\vec{r}(a, \theta) = a \cos \theta \vec{j} + a \sin \theta \vec{k}$$

$$0 \leq a \leq 1$$

$$0 \leq \theta \leq 2\pi$$

$$\vec{r}_a = \cos \theta \vec{j} + \sin \theta \vec{k}, \quad \vec{r}_\theta = -a \sin \theta \vec{j} + a \cos \theta \vec{k}$$

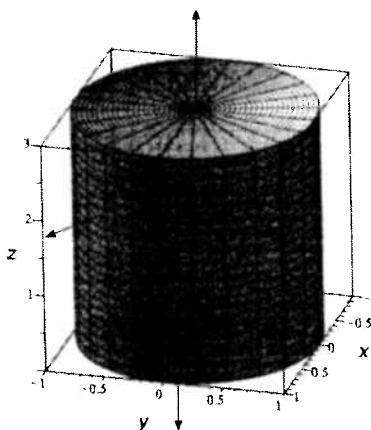
$$\vec{r}_a \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & \cos \theta & \sin \theta \\ 0 & -a \sin \theta & a \cos \theta \end{vmatrix} = \vec{i} a$$

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_0^{2\pi} \int_0^1 \vec{i} (3a^2) \cdot \vec{i} a \, da \, d\theta = \int_0^{2\pi} \int_0^1 3a^3 \, da \, d\theta$$

$$= \int_0^{2\pi} \left. \frac{3a^4}{4} \right|_0^1 d\theta = \int_0^{2\pi} \frac{3}{4} d\theta = \frac{3}{4} \cdot 2\pi = \frac{3\pi}{2}$$

13. Consider the vector field $\vec{F}(x, y, z) = (e^y + xz)\vec{i} + (y^2 + z^3)\vec{j} + \sin(xy)\vec{k}$.

- (a) Compute the divergence and the curl of \vec{F} , i.e. compute $\vec{\nabla} \cdot \vec{F}$ and $\vec{\nabla} \times \vec{F}$.
- (b) Let E denote the three-dimensional solid region bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 3$, and let S denote the surface which is the boundary of E (i.e. S is the "skin" of E), oriented outwards. Compute the flux integral $\iint_S \vec{F} \cdot d\vec{S}$ using Gauss' Divergence theorem.



$$a) \vec{\nabla} \cdot \vec{F} = \left(\frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k} \right) \cdot \left((e^y + xz)\vec{i} + (y^2 + z^3)\vec{j} + \sin(xy)\vec{k} \right)$$

$$= z + 2y$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^y + xz & y^2 + z^3 & \sin(xy) \end{vmatrix} =$$

$$\vec{i} (x \cos(xy) - 3z^2) - \vec{j} (y \cos(xy) - x) + \vec{k} (-e^y)$$

(Extra page)

b) By Gauss' divergence theorem

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E (\nabla \cdot \vec{F}) dV$$

The region E is
described as follows
in cylindrical
coordinates

$$0 \leq r \leq 1$$

$$0 \leq z \leq 3$$

$$0 \leq \theta \leq 2\pi$$

$$\iiint_E \nabla \cdot \vec{F} dV = \int_0^{2\pi} \int_0^1 \int_0^3 (z + 2r \sin^4 \theta) r dr dz d\theta$$

$$= \dots = \frac{9\pi}{2}$$

14. Show that the vector field $\vec{F}(x, y) = (2xy + 2)\vec{i} + (x^2 + 1)\vec{j}$ is conservative. Then, find a scalar function $f(x, y)$ which is such that $\vec{F}(x, y) = \nabla f(x, y)$, i.e. find a potential for \vec{F} . Finally, compute the line integral

$$\int_C \vec{F} \cdot d\vec{r}$$

where C is some curve which starts at the point $(1, 2)$ and ends at the point $(3, 1)$.

$$\frac{\partial Q}{\partial x} = 2x = \frac{\partial P}{\partial y} = 2x \Rightarrow \text{Conservative}$$

$$\frac{\partial f}{\partial x} = 2xy + 2 \Rightarrow f(x, y) = x^2y + 2x + h(y)$$

$$\frac{\partial f}{\partial y} = x^2 + h'(y) = Q(x, y) = x^2 + 1$$

$$\Rightarrow h'(y) = 1 \Rightarrow h(y) = y + C$$

$$\Rightarrow \boxed{f(x, y) = x^2y + 2x + y + C} \quad \text{all potentials}$$

$$\int_C \vec{F} \cdot d\vec{r} = f(3, 1) - f(1, 2)$$

$$= (9 \cdot 1 + 2 \cdot 3 + 1) - (2 + 2 + 2)$$

$$= 16 - 6 = \boxed{10}$$

15. A spherical shell is defined by the inequalities

$$16 \leq x^2 + y^2 + z^2 \leq 25.$$

This shell is made out of a material with mass density

$$\delta(x, y, z) = z^2.$$

Find the total mass of this shell.

In spherical coordinates, the region is described as

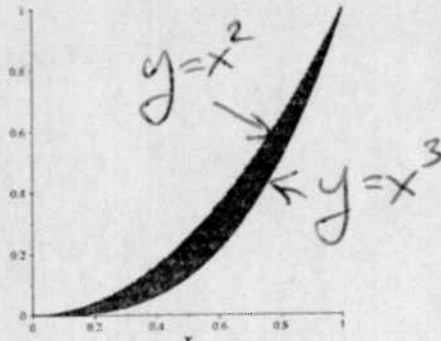
$$\begin{aligned} 4 &\leq \rho \leq 5 \\ 0 &\leq \theta \leq 2\pi \\ 0 &\leq \phi \leq \pi \end{aligned}$$

$$\text{Mass} = \int_0^{\pi} \int_0^{2\pi} \int_4^5 \rho^2 \cos^2 \phi \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$$

$$\begin{aligned} &= \int_0^{\pi} \int_0^{2\pi} \int_4^5 \rho^4 \cos^2 \phi \sin \phi \, d\rho \, d\theta \, d\phi = \dots \\ &= \frac{8404}{15} \pi \end{aligned}$$

16. Consider the region R illustrated below between the graphs of the functions $y = x^2$ and $y = x^3$ for x between 0 and 1. Let $f(x, y) = x + 3y$. Evaluate the double integral

$$\iint_R f(x, y) dA.$$



$$\begin{aligned} \int_0^1 \int_{x^3}^{x^2} (x+3y) dy dx &= \int_0^1 \left(xy + \frac{3y^2}{2} \right) \Big|_{x^3}^{x^2} dx = \int_0^1 \left(x^3 + \frac{3x^4}{2} - \left(x^4 - \frac{3x^6}{2} \right) \right) dx \\ &= \int_0^1 \left(x^3 + \frac{1}{2}x^4 - \frac{3}{2}x^6 \right) dx = \frac{x^4}{4} + \frac{x^5}{10} - \frac{3}{2} \frac{x^7}{7} \Big|_0^1 \\ &= \frac{1}{4} + \frac{1}{10} - \frac{3}{14} \\ &= \frac{19}{140} \end{aligned}$$

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