Math3705A: Supplementary materials 1

1. Notation for integration by parts

Method I.

The principal formula:

$$\int_a^b u(x)v'(x)dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)dx \tag{1}$$

We can directly follow it:

$$\int_0^\pi \underbrace{x^2}_{u(x)} \underbrace{\sin x}_{v'(x)} dx = \Big[v(x) = -\cos x, u'(x) = 2x\Big] =$$
$$= -x^2\cos x\Big|_0^\pi - 2\int_0^\pi (- \underbrace{\cos x}_{\text{new } v'(x)}) \underbrace{x}_{\text{new } u(x)} dx = \Big[v(x) = \sin x, u'(x) = 1\Big] =$$
$$= -\pi^2(-1) + 2x\sin x\Big|_0^\pi - 2\int_0^\pi \sin x dx = \pi^2 + 2\cos x\Big|_0^\pi = \pi^2 - 4.$$

Method II. (usually I use this one)

We can rewrite the formula (1) for integration by parts in the form

$$\int_a^b u dv = uv\Big|_a^b - \int_a^b v du \tag{2}$$

implying that $u = u(x)$ and $v = v(x)$: $du = u'(x)dx$, $dv = v'(x)dx$.

For example: $d\sin x = \cos x dx$, $d\cos x = -\sin x dx$

Then the same example

$$\int_0^\pi x^2 \sin x dx = -\int_0^\pi \underbrace{x^2}_{u} d\underbrace{\cos x}_{v} = -\underbrace{x^2}_{u}\underbrace{\cos x}_{v}\Big|_0^\pi + \int_0^\pi \underbrace{\cos x}_{v} d\underbrace{x^2}_{u} = \pi^2 +$$
$$2\int_0^\pi x\cos x dx = \pi^2 + 2\int_0^\pi \underbrace{x}_{\text{new } u} d\underbrace{\sin x}_{\text{new } v} = \pi^2 + 2\underbrace{x}_{\text{new } u}\underbrace{\sin x}_{\text{new } v}\Big|_0^\pi - 2\int_0^\pi \underbrace{\sin x}_{\text{new } v} d\underbrace{x}_{\text{new } u} =$$
$$\pi^2 + 2\cos x\Big|_0^\pi = \pi^2 - 4.$$

Applying the method II we don't need to specify $u'(x)$ and $v(x)$ at each step.

2. Partial Fractions

To determine the inverse Laplace transform $\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\}$, where $P(s)$ and $Q(s)$ are polynomials such that degree of polynomial $P(s) <$ degree of $Q(s)$, we need to resolve $\frac{P(s)}{Q(s)}$ into partial fractions:

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s-r_1} + \frac{A_2}{(s-r_1)^2} + \frac{B_1}{s-r_2} + \cdots + \frac{C_1 s + D_1}{as^2+bs+c} + \ldots,$$

where $r_i(i = 1, 2 \ldots)$ denote the simple roots of $Q(s)$ while as^2+bs+c are irreducible quadratic polynomials ($b^2 - 4ac < 0$). The reason is: we know how to determine the inverse LP of all these partial fractions.

So, all we need is to find the coefficients of decomposition $(A, B, C...)$.

I. The Cover-up Method (or Palm Closing Method)

Let's consider an example:

$$\frac{s^2+s+1}{(s-1)(s+2)(s-3)} = \frac{A}{s-1} + \frac{B}{s+2} + \frac{C}{s-3}$$

We notice that

$$\begin{aligned}
A &= \lim_{s\to 1}(s-1)\frac{s^2+s+1}{(s-1)(s+2)(s-3)} = \lim_{s\to 1}\frac{s^2+s+1}{(s+2)(s-3)} = \frac{1+1+1}{3\cdot(-2)} = -\frac{1}{2} \\
B &= \lim_{s\to -2}(s+2)\frac{s^2+s+1}{(s-1)(s+2)(s-3)} = \lim_{s\to -2}\frac{s^2+s+1}{(s-1)(s-3)} = \frac{4-2+1}{(-3)\cdot(-5)} = \frac{1}{5} \\
C &= \lim_{s\to 3}(s-3)\frac{s^2+s+1}{(s-1)(s+2)(s-3)} = \lim_{s\to 3}\frac{s^2+s+1}{(s-1)(s+2)} = \frac{9+3+1}{2\cdot 5} = \frac{13}{10}
\end{aligned}$$

Notice, to compute A we seem to "cover-up" $(s-1)$ in $\frac{s^2+s+1}{(s-1)(s+2)(s-3)}$ and then substitute $s = 1$. We did the same to compute B and C.

But what if we have something like that

$$\frac{s^2-s+1}{(s-1)^2(s+2)^2} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+2} + \frac{D}{(s+2)^2} \qquad (3)$$

According to the Cover-up method:

$$\begin{aligned} B &= \lim_{s\to 1} \frac{s^2-s+1}{(s+2)^2} = \frac{1-1+1}{3^2} = \frac{1}{9} \\ D &= \lim_{s\to -2} \frac{s^2-s+1}{(s-1)^2} = \frac{4+2+1}{(-3)^2} = \frac{7}{9} \end{aligned}$$

How to determine A and C? We can use

II. "Plug-in" Method

Pick up and substitute in (3) any s (obviously except a root of the denominator!):

$\underline{s=0}$: $\quad \frac{1}{4} = -A + \frac{1}{9} + \frac{C}{2} + \frac{7}{9\cdot 4} \Rightarrow -A + \frac{C}{2} = -\frac{1}{18}$,

$\underline{s=-1}$: $\quad \frac{1+1+1}{4} = -\frac{A}{2} + \frac{1}{36} + C + \frac{7}{9} \Rightarrow -\frac{A}{2} + C = -\frac{1}{18}$

from the linear system for A and C we have: $A = \frac{1}{27}$, $C = -\frac{1}{27}$.

Or use

III. Method of Comparing Coefficients

Reduce to (3) common denominator $(s-1)^2(s+2)^2$, then we can set the right and left numerators equal each other:

$s^2 - s + 1 = A(s-1)(s+2)^2 + \frac{1}{9}(s+2)^2 + C(s-1)^2(s+2) + \frac{7}{9}(s-1)^2$.

Setting equal the sum of coefficients at the same powers of s, we obtain a linear system for A and C:

at s^3 : $\quad A + C = 0$

at s^2 : $\quad 3A + \frac{1}{9} + \frac{7}{9} = 1 \Rightarrow A = \frac{1}{27}$, and hence $C = -\frac{1}{27}$.

Which method is the best one?

- First try the Cover-up Method if possible. This one is faster, although is not universal: it works only for the highest powers of the linear factors: $s - r_i$.
- Both the "Plug-in" and the Comparing Coefficients methods are universal (but may turn out tangled at implementation).

Exercises from class

Exercise 1.

We know that $\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}$. Compute $\mathcal{L}\{\cos(at)\}$ applying the derivative theorem for the Laplace transform.

Solution:

According the derivative theorem (Theorem 1.3 in the textbook)

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) \tag{4}$$

and we pick up $f(t) = \sin(at)$, so $f'(t) = (\sin(at))' = a\cos(at)$. Then
$a\mathcal{L}\{\cos(at)\} = s\mathcal{L}\{\sin(at)\} - \sin 0 = \frac{as}{s^2+a^2}$.
It follows $\mathcal{L}\{\cos(at)\} = \frac{1}{a}\frac{as}{s^2+a^2} = \frac{s}{s^2+a^2}$.

Exercise 2.

Let $f(t) = |t-3|$. Evaluate separately the left and the right parts in (4) for function $f(t)$ and its piecewise continuous derivative $f'(t)$. Ensure that the equality holds.

Solution:

Here $f(t)$ is continuous (and definitely can be bounded by any exponential function e^{at}, $a > 0$):

$$f(t) = \begin{cases} 3-t, & 0 \le t < 3 \\ t-3, & t \ge 3 \end{cases}$$

while $f'(t)$ is piecewise continuous:

$$f'(t) = \begin{cases} -1, & 0 \le t < 3 \\ 1, & t \ge 3 \end{cases}$$

Let's compute $\mathcal{L}\{f'(t)\}$ using the definition of the LT:

$$\int_0^3 (-1)e^{-st}dt + \int_3^\infty 1e^{-st}dt = \frac{1}{s}e^{-st}\Big|_0^3 - \frac{1}{s}e^{-st}\Big|_3^\infty = \frac{1}{s}e^{-3s} - \frac{1}{s} - \\ -\frac{1}{s}\lim_{T\to\infty} e^{-sT} + \frac{1}{s}e^{-3s} = \frac{2}{s}e^{-3s} - \frac{1}{s}. \tag{5}$$

On the other side, computation of $\mathcal{L}\{f(t)\}$ according the definition gives us

$$\begin{aligned}\int_0^3 (3-t)e^{-st}dt + \int_3^\infty (t-3)e^{-st}dt = -\frac{1}{s}\int_0^3 (3-t)de^{-st} - \frac{1}{s}\int_3^\infty (t-3)de^{-st} = \\ = -\frac{1}{s}(3-t)e^{-st}\Big|_0^3 - \frac{1}{s}\int_0^3 e^{-st}dt - \frac{1}{s}(t-3)e^{-st}\Big|_3^\infty + \frac{1}{s}\int_3^\infty e^{-st}dt = \\ = \frac{3}{s} + \frac{1}{s^2}e^{-st}\Big|_0^3 - \frac{1}{s^2}e^{-st}\Big|_3^\infty = \frac{3}{s} + \frac{e^{-3s}}{s^2} - \frac{1}{s^2} + \frac{e^{-3s}}{s^2} = \\ = \frac{1}{s}\Big(3 + \frac{2}{s}e^{-3s} - \frac{1}{s}\Big),\end{aligned} \tag{6}$$

hence $s\mathcal{L}\{f(t)\} = 3 + \dfrac{2e^{-3s}}{s} - \dfrac{1}{s}$.
Notice that $f(0) = 3$, and we observe:
$s\mathcal{L}\{f(t)\} - 3 = \dfrac{2e^{-3s}}{s} - \dfrac{1}{s} = \mathcal{L}\{f'(t)\}$ according to (5).