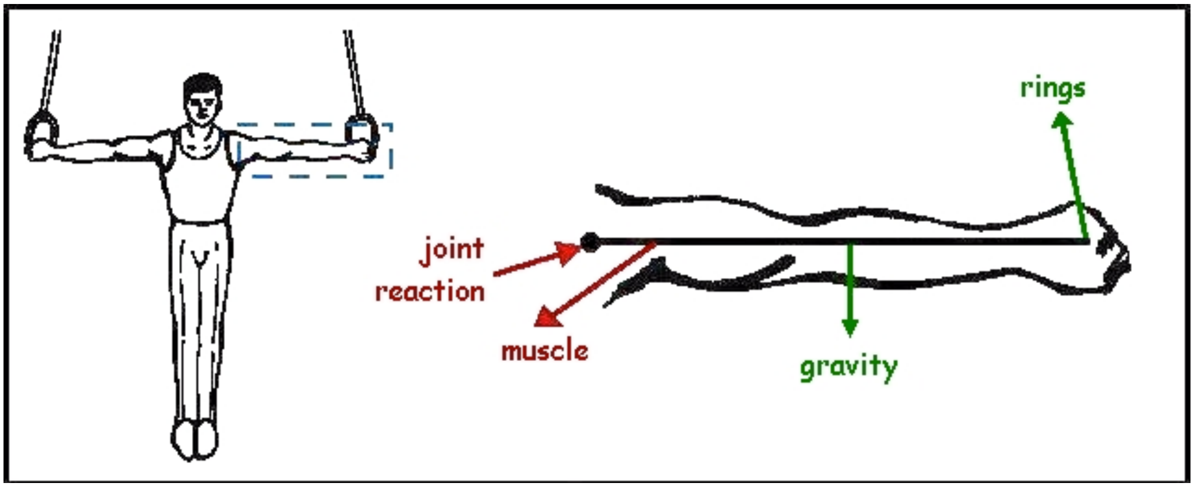
Static Equilibrium of Rigid Bodies (2-D)

The motion (or lack of motion) of any body is caused by a system of forces. In the case of a human limb segment, such as the left arm of the gymnast pictured below, this system of forces includes both internal and external forces.



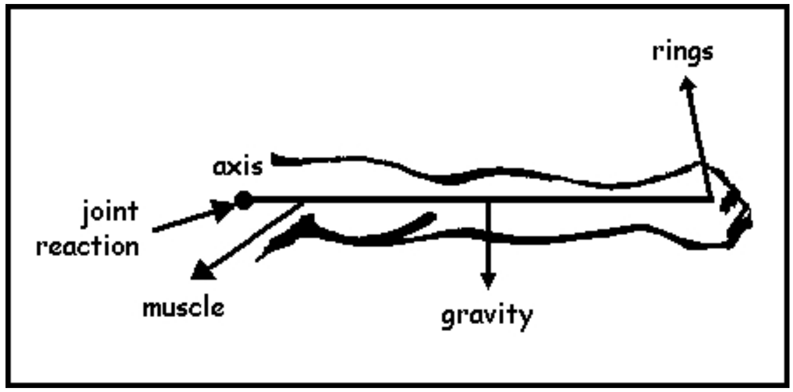
The external forces would be the force of the ring on the hand and the gravity force.

The internal forces would be the shoulder muscles and the joint reaction forces.

External forces can be measured with appropriate transducers but the internal forces are much more difficult to measure directly. Biomechanists have a method to calculate these very important forces and that is the topic of this lecture.

The motion (or lack of motion) of any body is caused by a system of forces.

A rigid body is in static equilibrium when the sum of all the forces and moments acting on it are equivalent to zero. Which means that:



$$\sum F_x = 0$$
$$\sum F_y = 0$$
$$\sum M = 0$$

In order to make the calculations in the above three equations, we must first resolve each of the forces into X and Y components and calculate the moment of each force about the axis. A Free Body Diagram is also a very important step in these calculations which will be explained next.

Free Body Diagrams:

A free body diagram (FBD) is an isolated drawing of an object that is assumed to be rigid. All models are simplifications of the actual situation but good models retain all of the characteristics that are relevant to the solution. Therefore, there is some subjectivity (art) in drawing an FBD but there are also some rules that cannot be ignored (science).

ART (subjective part)

The rigid body must have a shape that is sufficient to include the points of application of all of the forces acting on the body. In many cases, a straight line is sufficient but in other cases, a more complicated shape is required.

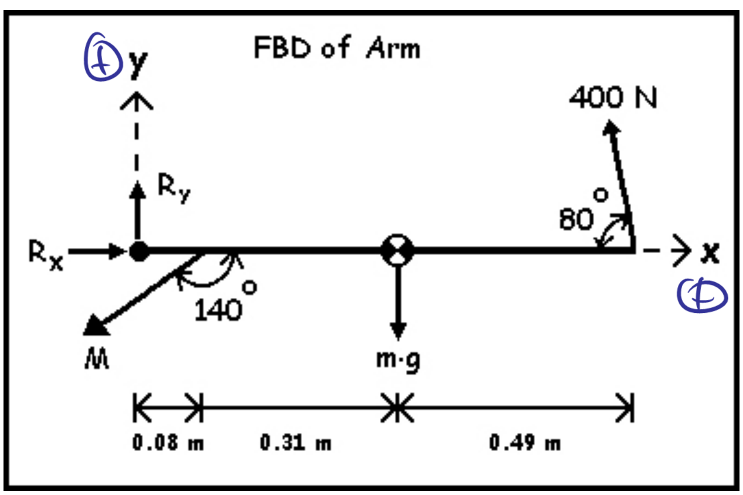
The orientation of the body must be similar to the actual orientation.

SCIENCE (every FBD must include the following)

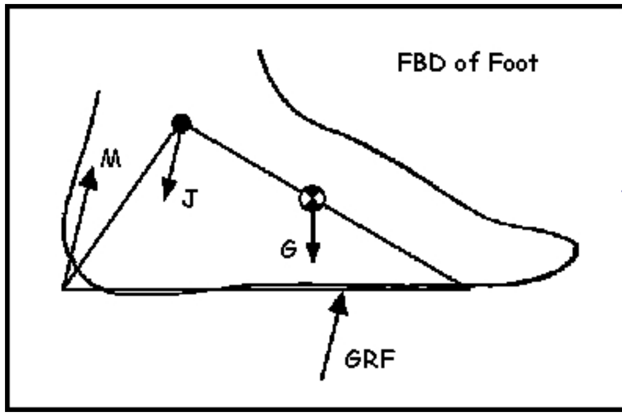
- rigid body de-coupled from all other objects (free)
- axis system
- gravitational force
- forces at each coupling (i.e. joints)
- all other forces and moments (internal and external)
- magnitudes or labels for all forces and moments
- dimensions
- accelerations (assumed to be zero in static equilibrium)

Examples of FBDs:

Often, the rigid body can be drawn as a straight line. This is the case when all of the forces can be drawn such that the point of application of each force intersects with a straight line. The right arm of the gymnast can be modeled this way. Notice that the dimensions and external forces are known. Since the magnitudes of the internal forces are not known, letters are used instead of numbers.



Other times, the rigid body is best represented by a simple shape. This is the case when all points of application of the forces cannot be shown to intersect with a straight line. The FBD of the foot is a good example of this because the points of application of the ground reaction force (GRF), joint force (J) and gravity force (G), etc. are best connected with a triangle.

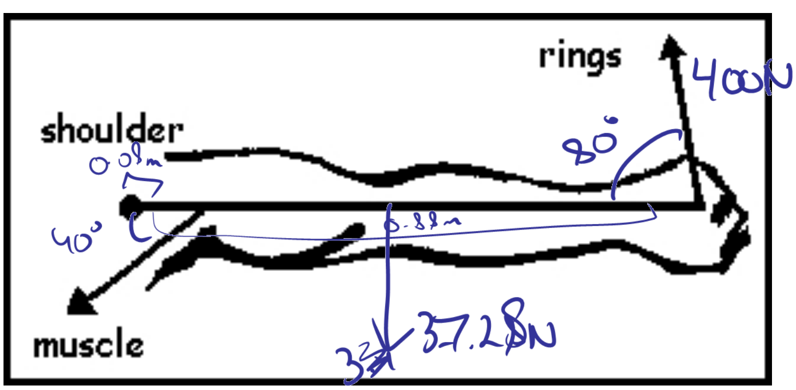


Triangle used for ease of alignment of forces

Sample Problem:

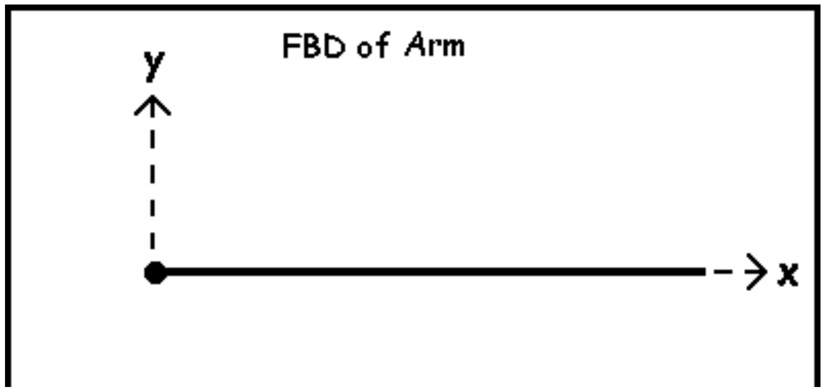
Using the diagram of the right arm of the gymnast, find the magnitude of the muscle force (M) and the joint reaction forces at the shoulder.

<u>Given</u>
arm is held horizontal
force of rings = 400 N located 0.88 m from shoulder
force of rings acts at 80 degrees above negative X axis
arm mass = 3.4 kg and located 0.39 m from the shoulder
muscle force acts 0.08 m from the shoulder
muscle force acts at 40 degrees below negative X axis



Solution

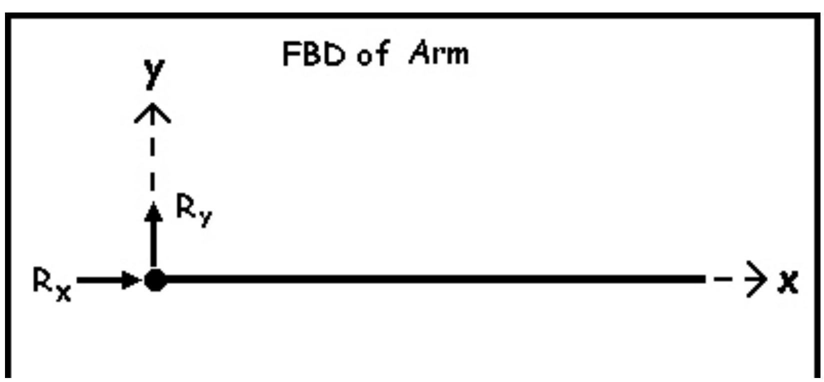
The first step of every static equilibrium solution is to draw the FBD.



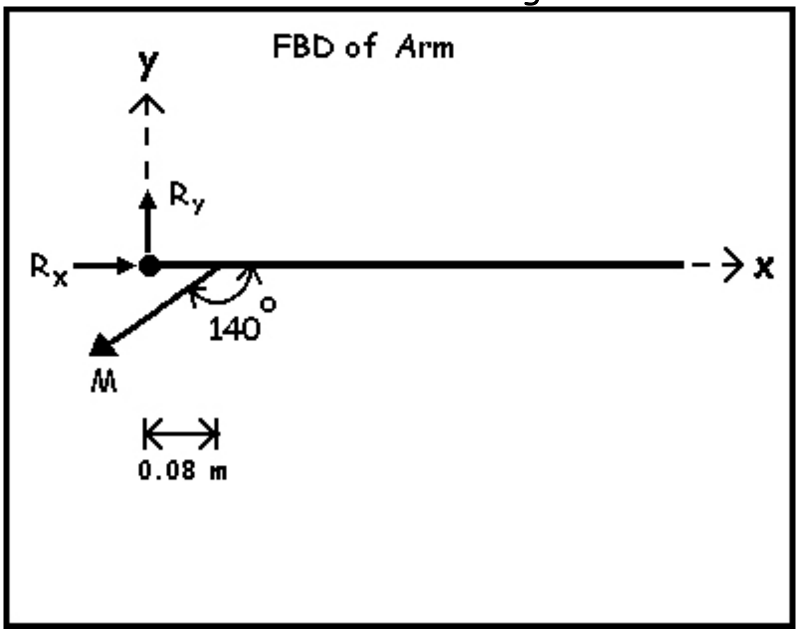
Here, I have represented the arm with a straight line. It has a horizontal orientation according to the given information. The shoulder axis is shown and positive X and Y axes are shown.

Next, I must draw the reaction forces. Since the arm is now free from the rest of the body, I represent the forces that the body exerts on the arm with a horizontal and a vertical force acting at the shoulder. I use the Rx and Ry labels because I do not know the magnitudes of these forces.

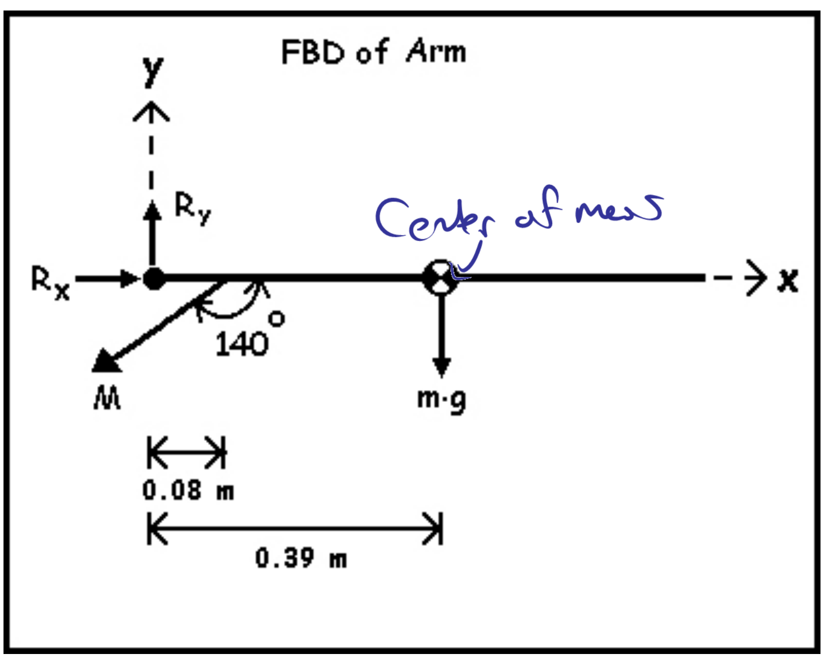
Joint rxn forces
Newton's 3rd Law



Next, I must draw the muscle force. The muscle exerts a force located 0.08 m from the shoulder axis. I used a 140 degree label which is the same as 40 degrees below the negative X axis. I used the M label because I do not know the magnitude of this force.

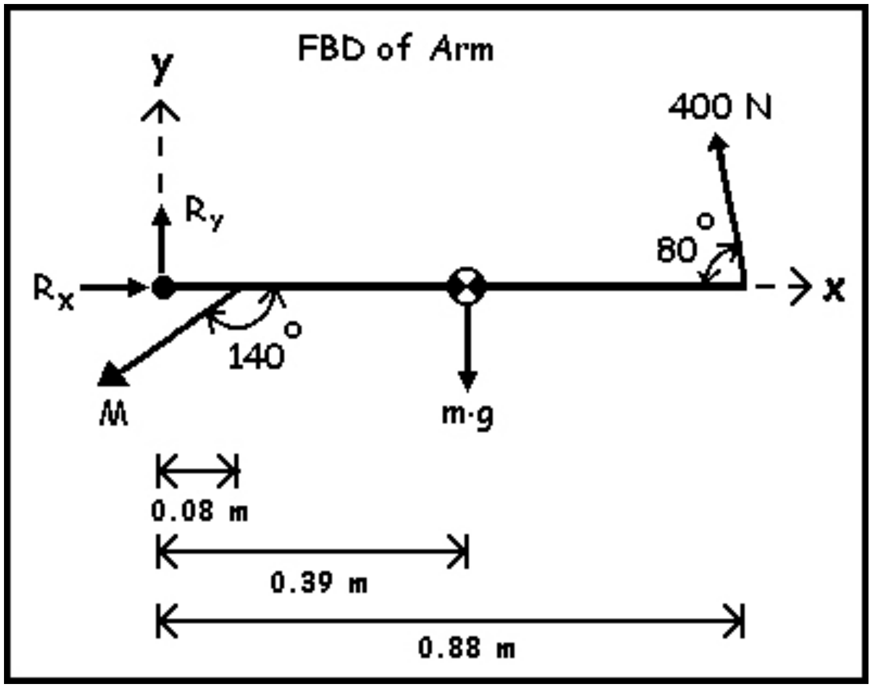


Next, I must draw the gravitational force. Gravity exerts a force located 0.39 m from the shoulder axis. It acts straight down with a magnitude equal to the product of mass (3.4 kg) and g (9.81 m/s/s).



Last, I must draw the force of the rings. The 400 N force is located 0.88 m from the shoulder axis. It acts at angle of 80 degrees above the negative X axis.

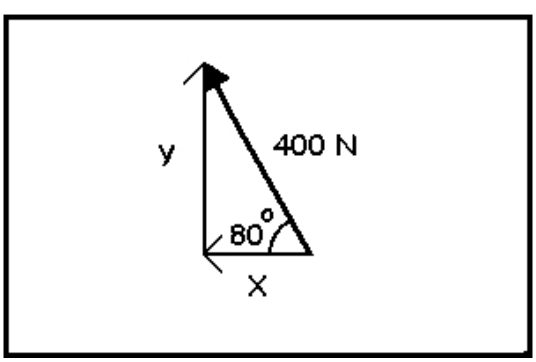
This a complete FBD with all forces and dimensions clearly shown.

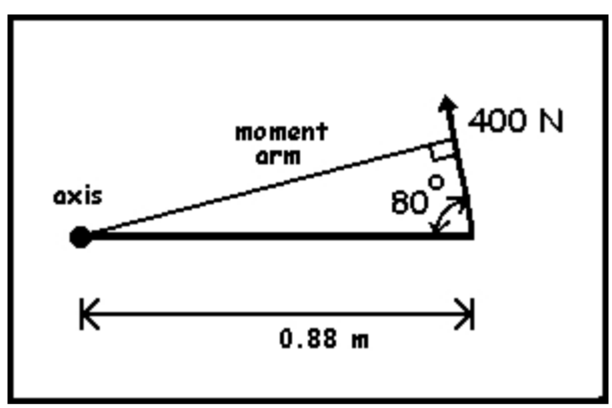


The second step of every static equilibrium problem is to fill in the force table. The force table lists the X and Y components of each force in the FBD and also the moment of force about the axis of each of those forces.

Let's start with the force of the rings.

Resolving the force of the rings into X and Y components.



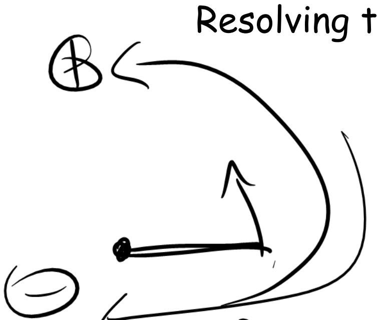


For ease of organization

FORCE TABLE			
FORCE	X component	Y component	Moment
rings	-400 cos 80°	400 sin 80°	400 (0.88 sin 80°)
gravity	0	-37.3 N	
muscle			
Rx			
Ry			

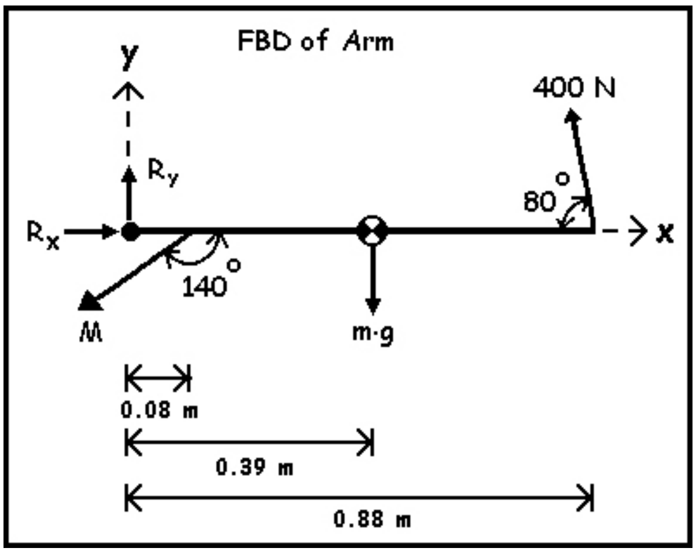
$M = F \times \underbrace{MA}_{\text{moment arm}}$

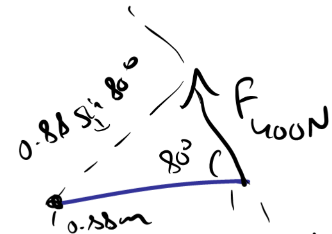
Resolving the force of gravity into X and Y components:



Curl your fingers in rotation of force

·thumb = rotational axis



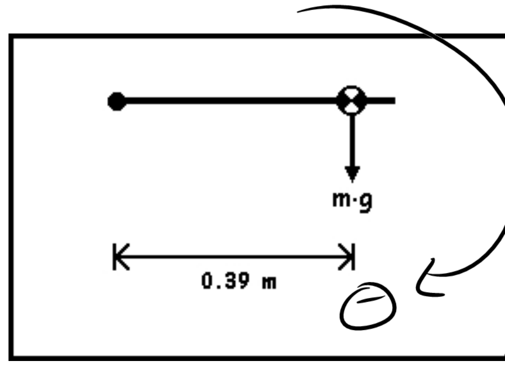


Moment arm = perpendicular distance from axis of rotation to force

The magnitude of the gravitational force is always equal to the mass of the object multiplied by the gravitational constant of 9.81. This force always acts directly downward (negative Y component) and has a zero X component.

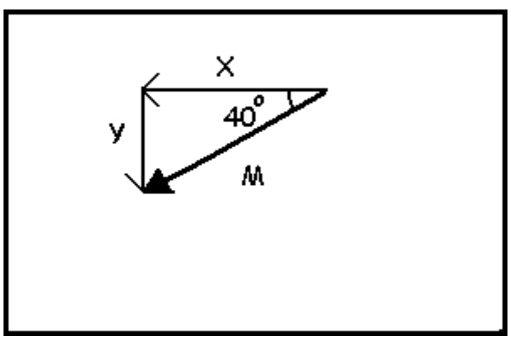
FORCE TABLE			
FORCE	X component	Y component	Moment
rings	-400 cos 80°	400 sin 80°	400 (0.88 sin 80°)
gravity	0	-3.4(9.81)	
muscle			
Rx			
Ry			

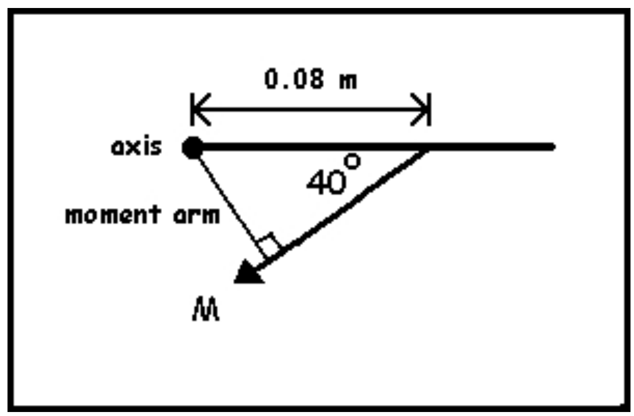
The moment of gravity about the shoulder axis.



FORCE TABLE			
FORCE	X component	Y component	Moment
rings	$-400 \cos 80^{o}$	$400 \sin 80^{o}$	$400\ (0.88 \sin 80^{o})$
gravity	0	$-3.4(9.81)$	$-3.4(9.81)(0.39)$
muscle			
Rx			
Ry			

Resolving the muscle force into X and Y components.



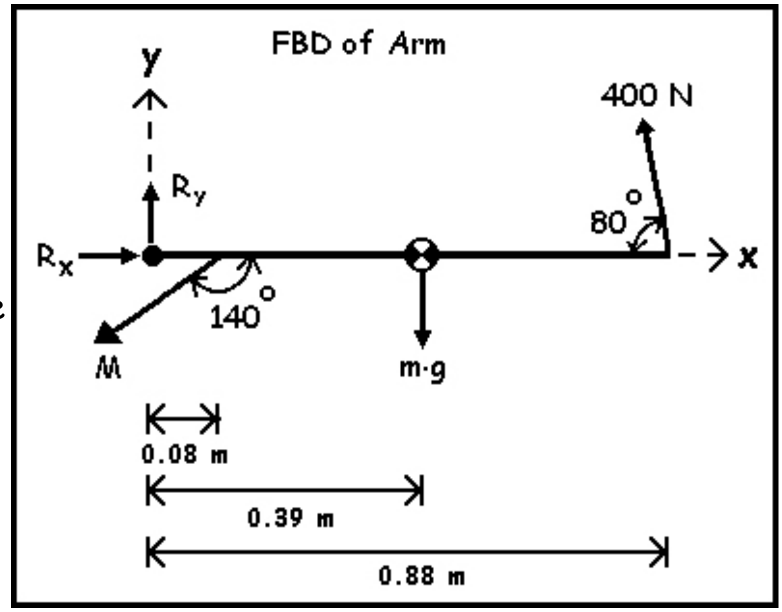


FORCE TABLE			
FORCE	X component	Y component	Moment
rings	$-400 \cos 80^{o}$	$400 \sin 80^{o}$	$400\ (0.88 \sin 80^{o})$
gravity	0	$-3.4(9.81)$	$-3.4(9.81)(0.39)$
muscle	$-M \cos 40^{o}$	$-M \sin 40^{o}$	$-M(0.08 \sin 40^{o})$
Rx	Rx	x	0
Ry	x	Ry	

because they act on the axis itself!

The joint reaction forces are positive and they each have a zero moment because they act at the axis.

The force Table is now complete and should be consistent with the FBD.



FORCE TABLE			
FORCE	X component	Y component	Moment
rings	$-400 \cos 80^\circ$	$400 \sin 80^\circ$	$400\,(0.88 \sin 80^\circ)$
gravity	0	$-3.4(9.81)$	$-3.4(9.81)(0.39)$
muscle	$-M \cos 40^\circ$	$-M \sin 40^\circ$	$-M(0.08 \sin 40^\circ)$
Rx	Rx	0	0
Ry	0	Ry	0

The last step is to use this table to solve the three equations of static equilibrium.

M, Rx, Ry unknown

$\Sigma F_x = 0$

$\Sigma F_y = 0$

$\Sigma M = 0$

$$\Sigma F_y = 400 \sin 80 - (3.4)(9.81) - M \sin 40 + R_y = 0$$

$$\Sigma M = (400)(0.88 \sin 80) - (3.4)(9.81)(0.39) - M(0.08 \sin 40) = 0$$

Isolate for M

Using the third equation, we can solve for the magnitude of the muscle force (M):

$\sum M = 0$ (sum of the moment column equals zero)

$\therefore$ $400\,(0.88 \sin 80^{o}) - 3.4(9.81)(0.39) - M(0.08 \sin 40^{o}) + 0 + 0 = 0$

Rearranging,

$$400\,(0.88 \sin 80^{o}) - 3.4(9.81)(0.39) = M(0.08 \sin 40^{o})$$

$$\frac{400\,(0.88 \sin 80^{o}) - 3.4(9.81)(0.39)}{(0.08 \sin 40^{o})} = M$$

$$6488 = M$$

Using the first equation, we can solve for Rx knowing M = 6488 N.

$Rx - M\cos 40 - 400\cos 80 = 0$

$\sum F_X = 0$ (sum of the X component column equals zero)

$\therefore$ $-400 \cos 80o + 0 - 6488 \cos 40o + Rx + 0 = 0$

Rearranging,

$$Rx = 400 \cos 80o + 6488 \cos 40o$$

$$Rx = 5040$$

Using the second equation, we can solve for Ry knowing M = 6488 N.

$\sum F_Y = 0$ (sum of the Y component column equals zero)

$\therefore$ $400 \sin 80^{o} - 3.4(9.81) - 6488 \sin 40^{o} + 0 + Ry = 0$

Rearranging,

$$Ry = -400 \sin 80^{o} + 3.4(9.81) + 6488 \sin 40^{o}$$

$$Ry = 3810$$

SUMMARY: It is practically impossible to directly measure internal forces but the ability to solve static equilibrium problems yields an estimate of these forces. In the example of the gymnast, we found that the muscle force required to hold that position was extremely large and this, in turn, caused very large forces (stress) within the shoulder joint. In industry, this method is used quite often to determine the safety of lifting tasks by estimating the muscle and joint stresses in the lumbar spine.

REVIEW Solving static equilibrium problems always uses the following steps:

1) Draw the Free Body Diagram of the rigid body showing all forces and dimensions.

2) Complete the Force Table by resolving all forces into X, Y components and moments.

3) Use the three equations of static equilibrium to solve for the unknowns.