3.6 Cauchy - Euler DE.

Problem 5. Solve the following Cauchy-Euler DE: $x^2y'' + 4xy' + 2y = 4\ln(x)$, $0 < x < \infty$.

Solution. The equation is Cauchy - Euler. First we solve the homogeneous DE:

$$x^2y'' + 4xy' + 2y = 0 \quad , \quad y = x^m$$

$$\Rightarrow x^2\, m(m-1)\cdot x^{m-2} + 4x\, m x^{m-1} + 2x^m = 0$$

$$\Rightarrow \cancel{x^m}\left(m(m-1) + 4m + 2\right) = 0 \quad , \quad x > 0$$

$$\Rightarrow \quad m^2 + 3m + 2 = 0 \quad , \quad m_{1,2} = \frac{-3 \pm \sqrt{9-8}}{2}$$

$m_1 = -2$, $m_2 = -1$

$$y_c = c_1 x^{-2} + c_2 x^{-1}$$

general solution of the hom. d.e.

Second. $y_p = u_1 x^{-2} + u_2 x^{-1}$

Standard form:

$$y'' + \frac{4}{x} y' + \frac{2}{x^2} y = \frac{4\ln(x)}{x^2}$$

$$y_1 = x^{-2} \quad , \quad y_2 = x^{-1}$$

$$u_1' = \frac{\begin{vmatrix} 0 & x^{-1} \\ \frac{4\ln(x)}{x^2} & -x^{-2} \end{vmatrix}}{\begin{vmatrix} x^{-2} & x^{-1} \\ -2x^{-3} & -x^{-2} \end{vmatrix}} = \frac{-4\ln(x)\,x^{-3}}{x^{-4}} = -4x\ln(x)$$

$$u_1 = -4\int x\ln(x)\,dx = -4\int \ln(x)\,d\frac{x^2}{2} = -4\ln(x)\cdot\frac{x^2}{2}$$
$$+4\int \frac{x^2}{2}\,d\ln(x) = -2x^2\ln(x) + 2\int x^2\frac{1}{x}\,dx$$
$$= -2x^2\ln(x) + x^2$$

$$u_2' = \frac{\begin{vmatrix} x^{-2} & 0 \\ -2x^{-3} & 4x^{-2}\ln(x) \end{vmatrix}}{x^{-4}} = \frac{4x^{-4}\ln(x)}{x^{-4}} = 4\ln(x)$$

$$u_2 = \int 4\ln(x)\,dx = 4x\ln(x) - 4\int x\,d\ln(x)$$
$$= 4x\ln(x) - 4x$$

$$y_p = u_1y_1 + u_2y_2 = (-2x^2\ln(x) + x^2)\,x^{-2}$$
$$+(4x\ln(x) - 4(x))\,x^{-1} = -2\ln(x) + 1 + 4\ln(x) - 4$$
$$= 2\ln(x) - 3$$

$$y = y_p + y_c = \underbrace{2\ln(x) - 3}_{y_p} + \underbrace{C_1x^{-2} + C_2x^{-1}}_{y_c}$$

!!!|| Nothe that here the method of undetermined coeffitients is not applicable (the DE is not with constant coeffitients!).

Example 2

Problem. Find the general solution by VP:

$$2x^2y'' + 5xy' + y = x^2 - x; \quad x > 0.$$

Solution. Cauchy-Euler d.e.

(A) $2x^2y'' + 5xy' + y = 0$, $y = x^m$

$$2\cancel{x^2}\, m(m-1)\cancel{x^{m-2}} + 5\cancel{x}\, m\, \cancel{x^{m-1}} + \cancel{x^m} = 0$$

$$2m^2 + 3m + 1 = 0; \quad m_{1,2} = \frac{-3 \pm \sqrt{9-8}}{4}$$

$$m_{1,2} = \frac{-3 \pm 1}{4}, \quad m_1 = -1, \quad m_2 = -\frac{1}{2}$$

$$y_c = c_1 x^{-1} + c_2 x^{-1/2}$$

$$y_c = c_1 y_1 + c_2 y_2; \quad y_1 = x^{-1}, \ y_2 = x^{-1/2}$$

(B) Variation of parameters to find y_p:

Standard form: $y'' + \frac{5}{2x} y' + \frac{1}{2x^2} y = \frac{x^2 - x}{2x^2} = \frac{1}{2} - \frac{1}{2x}$

$$y'' + \frac{5}{2x} y' + \frac{1}{2x^2} y = \frac{1}{2} - \frac{1}{2}x^{-1}$$

$$W(y_1, y_2) = \begin{vmatrix} x^{-1} & x^{-1/2} \\ -x^{-2} & -\frac{1}{2}x^{-3/2} \end{vmatrix} = -\frac{1}{2}x^{-5/2} + x^{-5/2} = \frac{1}{2}x^{-5/2}$$

$$u_1' = \frac{\begin{vmatrix} 0 & x^{-1/2} \\ \frac{1}{2} - \frac{1}{2}x^{-1} & -\frac{1}{2}x^{-1/3} \end{vmatrix}}{W(y_1, y_2)} = -\frac{\left(\frac{1}{2} - \frac{1}{2}x^{-1}\right)x^{-1/2}}{\frac{1}{2}x^{-5/2}} = (x^{-1} - 1)x^2$$

$$u_1' = x - x^2 \Rightarrow u_1 = \int (x - x^2)\,dx = \frac{x^2}{2} - \frac{x^3}{3}$$

$$u_2' = \frac{\begin{vmatrix} x^{-1} & 0 \\ -x^{-2} & \frac{1}{2} - \frac{1}{2}x^{-1} \end{vmatrix}}{W(y_1, y_2)} = \frac{x^{-1}\left(\frac{1}{2} - \frac{1}{2}x^{-1}\right)}{\frac{1}{2}x^{-5/2}}$$

$$= x^{3/2}(1 - x^{-1}) = x^{3/2} - x^{1/2}$$

$$u_2 = \int \left(x^{3/2} - x^{1/2}\right) dx = \frac{2}{5}x^{5/2} - \frac{2}{3}x^{3/2}$$

$$y_p = u_1 y_1 + u_2 y_2 = \left(\frac{x^2}{2} - \frac{x^3}{3}\right) \cdot x^{-1}$$

$$+ \left(\frac{2}{5}x^{5/2} - \frac{2}{3}x^{3/2}\right) x^{-1/2} = \frac{x}{2} - \frac{x^2}{3} + \frac{2}{5}x^2 - \frac{2}{3}x$$

$$y_p = + \frac{x^2}{15} - \frac{x}{6}$$

<u>The general solution:</u>

$$y = y_c + y_p = c_1 y_1 + c_2 y_2 + y_p$$

$$= \underbrace{c_1 x^{-1} + c_2 x^{-1/2}}_{y_c} + \underbrace{\frac{x^2}{15} - \frac{x}{6}}_{y_p}$$