SOLUTION TO FINAL EXAM

MAT 1322D, Fall 2011
Total = 55

1. (5 marks) (a) (3 marks) Find the value of the improper integral $\int_1^\infty x^2 e^{-x^3}\,dx$.

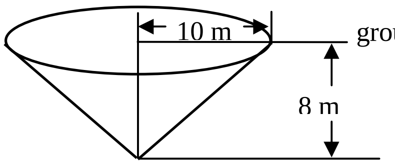
(b) (2 marks) Use comparison test to show that the improper integral $\int_0^1 \frac{1}{(1+\sin^2 x)x^{4/3}}\,dx$ is divergent.

Solution. (a) Let $u = -x^3$.

$$\int_1^\infty x^2 e^{-x^3}\,dx = \lim_{b\to\infty}\int_1^b x^2 e^{-x^3}\,dx = -\frac{1}{3}\lim_{b\to\infty}\int_{-1}^{-b^3} e^u\,du = -\frac{1}{3}\lim_{b\to\infty}\left[e^u\right]_{u=-1}^{-b^3} = \frac{1}{3e}.$$

(b) Since $1 + \sin^2 x \le 2$, $\frac{1}{(1+\sin^2 x)x^{4/3}} \ge \frac{1}{2x^{4/3}}$. Since $\int_0^1 \frac{1}{2x^{4/3}}\,dx = \frac{1}{2}\int_0^1 \frac{1}{x^{4/3}}\,dx$ diverges, $\int_0^1 \frac{1}{(1+\sin^2 x)x^{4/3}}\,dx$ diverges.

2. (5 marks) A conical tank has depth 8 meters and the radius of the top is 10 meters. It is filled with water of density $\rho = 1000$ kg / m^3. Set up and evaluate an integral to compute the work, in Joules, needed to pump the water to a point 2 meters above the ground level. (Use $g = 9.8$ m/sec^2, $\pi = 3.14$).



Solution. Let h be the distance from a layer of water to the bottom of the cone. Then the radius of this layer $r = (10 / 8)h = 5h / 4$. The area is $\pi r^2 = (5^2 / 4^2)\pi h^2$, and the volume of this layer with thickness dh is $\pi r^2 dh = (5^2 / 4^2)\pi h^2 dh$. The weight of this layer of water is $(5^2 / 4^2)g\rho\pi h^2 dh$. This layer of water is to be lifted for a height $10 - h$ meters. The work to be done to lift this layer of water to the ground level is $dW = (5^2 / 4^2)g\rho\pi h^2(10 - h)dh$. The total work is

$$W = \frac{5^2}{4^2} g\rho\pi\int_0^{10}(10h^2 - h^3)\,dh = \frac{25}{16} g\rho\pi\left[\frac{10}{3}h^3 - \frac{h^4}{4}\right]_{h=0}^{10} \approx 3.3\times 10^7 \text{ Joules.}$$

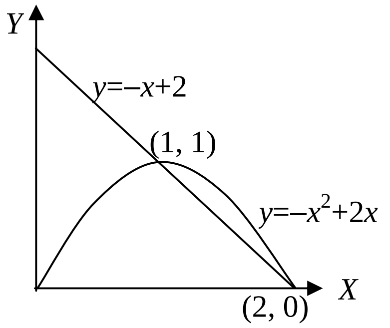
3. (5 marks) Consider the region R bounded by the graph of the function $y = -x^2 + 2x$ and the graph of the function $y = -x + 2$.

(a) (1 mark) Sketch this region.

(b) (2 marks) Find the area of this region.

(c) (2 marks) Find the volume of the solid obtained by revolving this region about the x-axis.

Solution. (a)



(b) Let $-x^2 + 2x = -x + 2$. $x^2 - 3x + 2 = 0$. $x = 1$, and $x = 2$. The area of R is

$$A = \int_1^2 [-x^2 + 2x - (-x+2)]dx = \int_1^2 (-x^2 + 3x - 2)dx = \left[-\frac{x^3}{3} + \frac{3x^2}{2} - 2x\right]_{x=1}^2 = \frac{1}{6}.$$

(c) $R_{\text{outer}} = -x^2 + 2x$, $R_{\text{inner}} = -x + 2$. The volume of the solid is

$$V = \pi\int_1^2 \left(\left(-x^2+2x\right)^2 - \left(-x+2\right)^2\right)dx = \pi\int_1^2 (x^4 - 4x^3 + 3x^2 + 4x - 4)dx$$

$$= \pi\left[\frac{x^5}{5} - x^4 + x^3 + 2x^2 - 4x\right]_{x=1}^2 = \frac{1}{5}\pi.$$

4. (6 marks) Determine whether each of the following series is convergent or divergent. Justify your answer by appropriate test method:

(a) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2+n}\right)$; (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^3 4^n}$; (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n + n}$.

Solution. (a) Use the limit comparison test.

Since $\lim_{n\to\infty}\dfrac{\sin\left(\dfrac{1}{n^2+n}\right)}{\dfrac{1}{n^2}}=\lim_{n\to\infty}\left(\dfrac{\sin\left(\dfrac{1}{n^2+n}\right)}{\dfrac{1}{n^2+n}}\dfrac{\dfrac{1}{n^2+1}}{\dfrac{1}{n^2}}\right)=\lim_{n\to\infty}\left(\dfrac{\sin\left(\dfrac{1}{n^2+n}\right)}{\dfrac{1}{n^2+n}}\right)\lim_{n\to\infty}\left(\dfrac{\dfrac{1}{n^2+1}}{\dfrac{1}{n^2}}\right)=1$, and the series $\sum_{n=1}^{\infty}\dfrac{1}{n^2}$ converges, this series converges.

(b) Use the ratio test, $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left(\dfrac{(n+1)!}{(n+1)^3 4^{n+1}}\dfrac{n^3 4^n}{n!}\right)=\lim_{n\to\infty}\left(\dfrac{n^3}{4(n+1)^2}\right)=\infty$. This series diverges.

(c) Since $\dfrac{1}{2^n+n}<\dfrac{1}{2^n}$, and the geometric series $\sum_{n=1}^{\infty}\dfrac{1}{2^n}$ converges, this series converges.

5. (5 marks) Suppose the density of a rod is given by the $\delta(x)=\dfrac{1}{x^2+1}$, $0\le x\le 1$. Find the x-coordinate of the center of mass of the rod.

Solution. The total mass $m=\int_0^1\dfrac{1}{x^2+1}dx=$ arctan 1 − arctan 0 = $\pi/4$.

The moment $M=\int_0^1\dfrac{x}{x^2+1}dx=\dfrac{1}{2}(\ln 2-\ln 1)=\dfrac{\ln 2}{2}$.

The center of mass is at $\bar{x}=\dfrac{\ln 2/2}{\pi/4}=\dfrac{2\ln 2}{\pi}\approx 0.441$.

6. (6 marks) Two cars are emitting exhaust gas containing carbon monoxide (CO) in a closed garage with a total volume 800 m^3. One car is omitting 0.03 m^3 exhaust gas per minute with CO of concentration 5%, and the other car is omitting 0.05 m^3 exhaust gas per minute with CO of concentration 4%. Assume the exhaust gas is well mixed with the air in the garage, and a total of 0.08 m^3 of well mixed air is removed every minute from the garage by the ventilation system. At time t = 0, there is no CO in the garage.

(a) (2 marks) Let $V(t)$ be the volume of CO in the garage at time t. Find a differential equation satisfied by this function $V(t)$, and the initial condition.

(b) (1 mark) Find the equilibrium solution of this equation.

(c) (1 mark) Without solving this equation, sketch the graph of the solution.

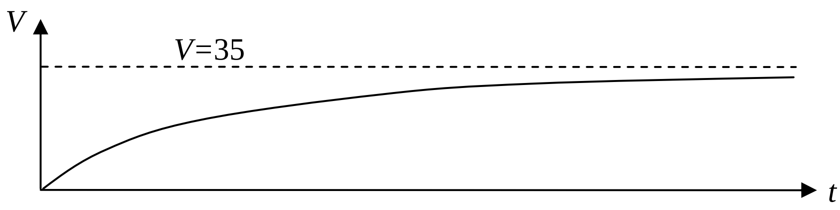
(d) (2 marks) Solve the initial-value problem in part (a).

Solution. (a) $r_{\text{in}} = 0.03 \times 0.05 + 0.05 \times 0.04 = 0.0035$ m^3 / min. $r_{\text{out}} = (0.08 / 800)V(t) = 0.0001V(t)$.

The equation is $Q'(t) = 0.0035 - 0.0001V(t)$.

(b) The equilibrium solution is obtained by $0.0035 - 0.0001V = 0$, or
$Q = 0.0035 / 0.0001 = 35$ m^3.

(c) The graph of the solution to this problem looks like the following:



(d) $\displaystyle\int \frac{dV}{0.0035 - 0.0001V} = \int dt$. $\displaystyle -\frac{1}{0.0001}\ln|0.0035 - 0.0001V| = t + C$.
$0.0035 - 0.0001V = Ke^{-0.0001t}$. When $t = 0$, $V = 0$. Hence, $K = 0.0035$.
$0.0001V = 0.0035\,(1 - e^{-0.0001t})$. $V = 35(1 - e^{-0.0001t})$.

7. (3 marks) Consider the initial-value problem: $y' = (2t - y)^2$, $y(0) = 2$. Use Euler's method with step-size $h = 0.1$ to find an approximation of $y(0.3)$. Use at least three digits after decimal point in your calculation.

Solution. $y_{i+1} = y_i + 0.1(2t_i - y_i)^2$. $y(0) = y_0 = 2.000$. $t_0 = 0$.

$y(0.1) \approx y_1 = 2 + 0.1 \times (2 \times 0 - 2)^2 = 2.400$, $t_1 = 0.1$.

$y(0.2) \approx y_2 = 2.4 + 0.1 \times (2 \times 0.1 - 2.4)^2 = 2.884$. $t_2 = 0.2$.

$y(0.3) \approx y_3 = 2.884 + 0.1 \times (2 \times 0.2 - 2.884)^2 \approx 3.501$.

8. (5 marks) Find the radius of convergence and the interval of convergence of the power series

$$\sum_{n=1}^{\infty} (-1)^n \frac{(x-1)^n}{2^n\sqrt{n}}.$$

Solution. The center of the series is 1. The radius of convergence is $\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}\sqrt{n+1}}{2^n\sqrt{n}}\right| = 2$. This series is absolutely convergent in the interval (1 – 2, 1 + 2) = (–1, 3).

When $x = -1$, the series is $\sum_{n=1}^{\infty}(-1)^n\frac{(-2)^n}{2^n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$. This is a p-series with $p = 1/2$. By the integral test, this series is divergent.

When $x = 3$, the series is $\sum_{n=1}^{\infty}(-1)^n\frac{2^n}{2^n\sqrt{n}} = \sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$. By the alternating series test, it is convergent. The interval of convergence of this series is (–1, 3].

9. (4 marks) Consider function $f(x) = \frac{x^3}{(1-x)^2}$.

(a) (3 marks) Find the Maclaurin series of this function.

(b) (1 mark) Find the value of $f^{(5)}(0)$, i.e., the fifth derivative of this function at $x = 0$.

Solution. (a) Take the derivative of the identity $\frac{1}{1-x} = \sum_{n=0}^{\infty}x^n$ on both sides, we have

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty}nx^{n-1}.$$

Hence, $\frac{x^3}{(1-x)^2} = \sum_{n=1}^{\infty}nx^{n+2} = x^3 + 2x^4 + 3x^5 + \dots$.

(b) The coefficient of x^n is $n - 2 = \frac{f^{(n)}(0)}{n!}$. Hence, $f^{(5)}(0) = 3 \times 5! = 3 \times 5! = 360$.

10. (7 marks) Consider the 2-variable function $z = f(x, y) = 2xy - x^2 + 2y$.

(a) (1 mark) Find the partial derivatives z_x and z_y.

(b) (1 mark) Find the gradient vector of this function at the point (2, 1).

(c) (3 marks) Find the directional derivative of z at point (2, 1) in the direction of the vector $\mathbf{v} = \sqrt{3}\,\mathbf{i} + \mathbf{j}$.

(d) (2 marks) Find the equation of the plane tangent to the graph of this function at a point where $x = 2$, and $y = 1$.

Solution. (a) $z_x = 2y - 2x$, $z_y = 2x + 2$.

(b) The gradient vector at the point (2, 1) is (–2, 6).

(c) The unit vector in the direction **v** is **u** = $\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$. The directional derivative is

$$D_{\mathbf{v}}(z) = -2\times\frac{\sqrt{3}}{2} + 6\times\frac{1}{2} = 3 - \sqrt{3}.$$

(d) When $x = 2$, and $y = 1$, $z = 2$. The equation of the tangent plane is $z = -2(x - 2) + 6(y - 1) + 2$, or $z = -2x + 6y$.

11. (4 marks) Consider 2-variable function $z = xe^{xy} + 2y^2$.

(a) (2 mark) Find the partial derivatives z_x and z_y.

(b) (2 marks) Suppose x and y are functions of independent variables s and t as $x = s + t$ and $y = st^2$. Use the chain rule to find the partial derivatives $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ at the point $s = t = 1$.

Solution. (a) $z_x = e^{xy} + xye^{xy}$, $z_y = x^2e^{xy} + 4y$.

(b) When $s = t = 1$, $x = 2$, $y = 1$. $z_x = 3e^2$, $z_y = 4(e^2 + 1)$. $x_s = x_t = 1$, $y_s = t^2 = 1$, $y_t = 2st = 2$. Therefore, by the chain rule,

$z_s = z_x x_s + z_y y_s = 3e^2 + 4(e^2 + 1) = 7e^2 + 4$,
$z_t = 3e^2 + 8(e^2 + 1) = 11e^2 + 8$.