



Version A
Answers

Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

DISCRETE MATHEMATICS FOR COMPUTING

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MAT1348A — Test 3 — Tuesday, February 28, 2017

INSTRUCTIONS

- Clearly write your name and student number on this test, and **sign it** below to confirm that you will read and follow these **instructions**:
- This is a 75-minute **closed-book** test. **No notes.** **No calculators.**
- **Put away everything** except a few pens or pencils, an eraser, and your student id. card.
- The exam consists of 9 questions on 10 pages (including this cover page). Total = 37 points.
- Questions 1–3 are **short-answer**. Write the final answer in the appropriate answer box. You do not need to show any other work.
- Questions 4–6 are **true-or-false**. In each question, you must select the correct response. You do not need to justify your answers.
- Questions 7–9 are **long-answer**. To receive full marks, your solution/proof must be complete, correct, and show all relevant details.
- Read all questions carefully and be sure to follow the instructions for the individual problems. You may ask for clarification.
- For rough work or additional work space, you may use the backs of pages. The last page (Page 10) contains a **Table of Set Identities**. You may detach Page 10. **Do not use any of your own scrap paper.**
- You must use **proper mathematical notation and terminology**.

Cellular phones, unauthorized electronic devices, or course notes are not allowed during this test. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession such as in your pockets. If caught with such a device or document, academic fraud allegations may be filed which may result in you obtaining zero for this test.

† **By signing below, you acknowledge that you have read, understand, and will comply with the above instructions.**

FAMILY NAME: ANSWERS	STUDENT NUMBER:
FIRST NAME:	†SIGNATURE:

DGD (circle): HGN LPR MRT **Prof** (If not Scott) _____

Please rewrite your name and student number here: (do not write in the table of points below)

FAMILY NAME:	FIRST NAME:
STUDENT NUMBER:	

Table of Points for marking purposes.

Question	Q1	Q2	Q3	MC	Q7	Q8	Q9	Total
Maximum points	4 pts	6 pts	4 pts	9 pts	5 pts	4 pts	5 pts	37 points
Marks obtained								

SHORT-ANSWER QUESTIONS.

Write your final answer in the answer box. No justification is needed.

(4 pts) **Q1.** Let A be the following set: $A = \{1, \{2\}, \{1, \{2\}\}\}$

Determine each of the following **cardinalities**:

$$|A| = 3$$

$$|\mathcal{P}(A)| = 2^3 = 8$$

$$|A \times A| = 3 \times 3 = 9.$$

$$|A \cap \mathcal{P}(A)| = 1 \text{ since } \{1, \{2\}\} \in A \cap \mathcal{P}(A), \text{ i.e. } \{1, \{2\}\} \subseteq A !$$

(6 pts) Q2. Complete each of the following definitions. Be precise.

The **power set** of a set S is

$$P(S) = \{X \mid X \subseteq S\}$$

The **Cartesian product** of sets A and B is

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

A function $f : X \rightarrow Y$ is called **injective** if

$$\forall a \in X \quad \forall a' \in X \quad [f(a) = f(a') \rightarrow a = a'] \quad \equiv$$

$$\forall a \in X \quad \forall a' \in X \quad [a \neq a' \rightarrow f(a) \neq f(a')]$$

A function $f : A \rightarrow B$ is called **surjective** if

$$\forall y \in B \quad \exists x \in A \quad [f(x) = y] \quad ,$$

i.e. for all $y \in B$ there exists $x \in A$ such that $f(x) = y$.

$$\text{i.e. } \text{Range}(f) = B$$

(4 pts) **Q3.** Finish the following sentences, by choosing one answer from the list of possible answers below. As usual, $\mathcal{P}(X)$ is the powerset of X .

• $A \not\subseteq B$ is equivalent to

c

$$A \not\subseteq B \equiv A \notin \mathcal{P}(B)$$

• $B \notin \mathcal{P}(A)$ is equivalent to

d

$$B \notin \mathcal{P}(A) \equiv B \not\subseteq A$$

• $A \not\subseteq \mathcal{P}(B)$ is equivalent to

a

$$A \not\subseteq \mathcal{P}(B) \equiv \exists a (a \in A \wedge a \not\subseteq B)$$

• $\mathcal{P}(B) \not\subseteq A$ is equivalent to

f

$$\mathcal{P}(B) \not\subseteq A \equiv \exists S (S \subseteq B \wedge S \notin A)$$

Choose your answer from among these possible answers to the 4 questions above

- (a) There exists some element of A that is not a subset of B .
- (b) There exists some subset of A which is not an element of B .
- (c) $A \not\subseteq \mathcal{P}(B)$.
- (d) There exists some element of B that is not an element of A .
- (e) There exists some element of B that is not a subset of A .
- (f) There exists some subset of B that is not an element of A .

TRUE-OR-FALSE QUESTIONS.

Circle the correct responses. No justification is needed.

(3 pts) **Q4.** Let A and B be **finite** sets.

- (a) If $f : A \rightarrow B$ is an injective (1-1) function, then $|A| \leq |B|$. T F
- (b) If $|A| \leq |B|$, then every function $f : A \rightarrow B$ is injective (1-1). T F
- (c) If $|A| \leq |B|$, then there exists an injective (1-1) function $f : A \rightarrow B$. T F
- (d) If $f : A \rightarrow B$ is a surjective (onto) function, then $|A| \leq |B|$. T F
- (e) If $|A| \leq |B|$, then there exists a surjective (onto) function $f : A \rightarrow B$. T F
- ↑ typo. We'll count either answer.*
- (f) If $|A| < |B|$, then no function $f : A \rightarrow B$ is surjective. T F
- (F if $|A| < |B|$, T if $|A| = |B|$)*
-

(3 pts) **Q5.** Let \mathcal{U} be a universal set \mathcal{U} with $|\mathcal{U}| \geq 2$.

Which of the following statements are true **for all subsets A and B of \mathcal{U}** ?

- (a) If $A \times B = \emptyset$, then $A = \emptyset$ or $B = \emptyset$. T F
- (b) $A \times B = B \times A$. T F
- (c) If $A \subseteq B$, then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. T F
- (d) $A \oplus B \subseteq A \cup B$. T F
- (e) $A \in \mathcal{P}(A)$. T F
- (f) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$. T F
-

(3 pts) Q6. Let f , g , and h be three functions defined as follows:

$$f: \mathbb{Z} \rightarrow (\mathbb{Z} \times \mathbb{Z})$$
$$f(x) = (x, 5 - x)$$

$$g: (\mathbb{Q}^+ \times \mathbb{Q}^+) \rightarrow \mathbb{Q}^+$$
$$g(r, s) = rs$$

$$h: \mathbb{Z} \rightarrow \mathbb{N}$$
$$h(n) = 2n^2 + 1$$

Note. \mathbb{Q}^+ denotes the set of positive rational numbers.

Which of the following statements are true regarding the above functions? Circle the correct response.

(a) f is injective (1-1).

T

F

(b) f is surjective (onto).

T

F

If $f(x) = (u, v) = (x, 5-x)$
Then $u+v=5$. $\therefore (0,0) \notin \text{range}(f)$

(c) g is injective (1-1).

T

F

$$g(1,2) = 1 \cdot 2 = 2 \cdot 1 = g(2,1)$$

(d) g is surjective (onto).

T

F

Given $q_0 \in \mathbb{Q}^+$, let $(r,s) = (1, q_0)$.
Then $g(r,s) = g(1, q_0) = q_0$ ✓

(e) h is injective (1-1).

T

F

$$h(1) = 2 \cdot 1^2 + 1 = h(-1)$$

(f) h is surjective (onto).

T

F

$h(n) \geq 1, \forall n$. $\therefore 0 \notin \text{range}(h)$

LONG-ANSWER QUESTIONS.

Detailed solutions are required.

- (5 pts) **Q7.** Let X , Y , and Z be subsets of the universal set \mathcal{U} . Prove the identity below using only the **set identities listed on Page 10**. Justify each step by giving the name or number of the corresponding identity.

*Do not skip steps or combine several distinct identities into a single step. You may apply the **same** identity multiple times in one single step, but never use more than one distinct Law per step. Do not omit necessary parentheses.*

Note. Your solution should only need 6 to 12 steps, assuming each step makes use of exactly one Law.

$$\overline{(X - Z) \cup (\bar{Y} - Z)} = Z \cup (\bar{X} \cap Y)$$

$$\begin{aligned} \overline{(X - Z) \cup (\bar{Y} - Z)} &= \overline{X - Z} \cap \overline{\bar{Y} - Z} && 14 \text{ (De Morgan's)} \\ &= \overline{X \cap \bar{Z}} \cap \overline{\bar{Y} \cap \bar{Z}} && 18 \text{ (Difference Law)} \\ &= (\bar{X} \cup \bar{\bar{Z}}) \cap (\bar{\bar{Y}} \cup \bar{\bar{Z}}) && 15 \text{ De Morgan's} \\ &= (\bar{X} \cup Z) \cap (Y \cup Z) && 7 \text{ Double Complement} \\ &= (Z \cup \bar{X}) \cap (Z \cup Y) && 8 \text{ Commutative Law} \\ &= Z \cup (\bar{X} \cap Y) && 13 \text{ Distributive Law} \end{aligned}$$

(4 pts) Q8. Let $f : (\mathbb{R}^+ \times \mathbb{R}^+) \rightarrow (\mathbb{R}^+ \times \mathbb{R}^+)$ be the function defined by $f(r, s) = (2r, rs)$.

Note. \mathbb{R}^+ denotes the set of positive real numbers.

Q8a. Prove that f is injective (one-to-one).

Suppose $f(r, s) = f(r', s')$. Then $(2r, rs) = (2r', r's')$.

$\therefore 2r = 2r'$, so $r = r'$. Also

$$rs = r's'.$$

$\therefore rs = r's'$ since $r' = r$.

$\therefore \frac{rs}{r} = \frac{r's'}{r}$ since $r > 0$!

$\therefore s = s'$ by cancelling the r .

Q8b. Prove that f is surjective (onto). (Be sure to check your answer is well-defined and solves the problem).

Let $(u, v) \in \mathbb{R}^+ \times \mathbb{R}^+$. We must show: $\exists (r, s) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $f(r, s) = (u, v)$. This means:

$\exists (r, s) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $(2r, rs) = (u, v)$.

So we must have: $2r = u \therefore r = u/2$

also, $rs = v \therefore s = v/r = 2(v/u)$. This is well-defined since $u \in \mathbb{R}^+ \therefore u \neq 0$.

Check: Given (u, v) as above, let $r = \frac{u}{2}$, $s = 2\left(\frac{v}{u}\right)$.

Then

$$f(r, s) = (2r, rs) = \left(2\left(\frac{u}{2}\right), \left(\frac{u}{2}\right) \cdot 2\left(\frac{v}{u}\right)\right) = (u, v) \checkmark$$

as required.

Q9. Let A and B be subsets of the universal set.
(5 pts) Prove each of the following two theorems, starting your proofs as indicated.

Q9a. Theorem 1. If $(A \cup B) \subseteq A$, then $B \subseteq (A \cap B)$.

Proof of Theorem 1. (complete this proof)

Assume $(A \cup B) \subseteq A$. Our goal is to prove that

$$B \subseteq (A \cap B)$$

To achieve our goal, we will take an arbitrary element $x \in B$, and we will prove

that

$$x \in A \cap B$$

Let $x \in B$ be an arbitrary element of B . Then...

(complete this proof)

$$x \in A \cup B, \text{ since } B \subseteq A \cup B.$$

$$\text{But we assumed } A \cup B \subseteq A \therefore x \in A \therefore (x \in A) \wedge (x \in B).$$

$$\therefore x \in A \cap B.$$

Q9b. Theorem 2. If $B \subseteq (A \cap B)$, then $(A \cup B) \subseteq A$.

Proof of Theorem 2. (complete this proof)

Assume $B \subseteq (A \cap B)$. Our goal is to prove that

$$A \cup B \subseteq A$$

To achieve our goal, we will take an arbitrary element $x \in (A \cup B)$, and we will prove

that

$$x \in A$$

Let $x \in (A \cup B)$ be an arbitrary element of $A \cup B$. Then...

(complete this proof)

$$\text{either } x \in A \text{ or } x \in B.$$

$$\text{Case 1. If } x \in A, \text{ we're done } \checkmark.$$

$$\text{Case 2. If } x \in B \text{ then since } B \subseteq A \cap B \text{ (which we assumed), then } x \in A \cap B \therefore x \in A \checkmark$$

$$\text{So in either case, we see } x \in A.$$

Q9c. What have you proved in Q9a. and Q9b. ? Give a single statement that combines Theorem 1 and Theorem 2 using appropriate connective words such as *and*, *or*, *if*, *only if*, *if and only if*, etc.

The overall Theorem proved in Q9:

$$(A \cup B) \subseteq A \text{ if and only if } B \subseteq (A \cap B)$$

Table of Set Identities

You may detach this page

1. 2.	$A \cup \emptyset = A$ $A \cap \mathcal{U} = A$	Identity Laws
3. 4.	$A \cup \mathcal{U} = \mathcal{U}$ $A \cap \emptyset = \emptyset$	Domination Laws
5. 6.	$A \cup A = A$ $A \cap A = A$	Idempotent Laws
7.	$\overline{(\overline{A})} = A$	(Double) Complementation Law
8. 9.	$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative Laws
10. 11.	$A \cup (B \cap C) = (A \cup B) \cap C$ $A \cap (B \cup C) = (A \cap B) \cup C$	Associative Laws
12. 13.	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive Laws
14. 15.	$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's Laws
16. 17.	$A \cup \overline{A} = \mathcal{U}$ $A \cap \overline{A} = \emptyset$	Complement Laws
18.	$A - B = A \cap \overline{B}$	Difference Law

(end of Test 3)