Example of Proofs

Def An integer n is called odd if $n = 2k+1$ for some some integer k; even if $n = 2k$ for some

Direct Proof

Thm: Let n be an integer. If $\underbrace{n \text{ is odd}}_{p}$, then $\underbrace{n^2 \text{ is odd}}_{q}$

Proof Strategy

We are to prove $p \to q$ using a direct proof

Here; p: "n is odd"

q: "n^2 is odd"

So assume p (n is odd) and show q follows (n^2 is odd)

Proofs

Assume n is odd. Hence $n = 2k+1$ for some integer k

then $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$

$n^2 = 2m+1$ where $m = 2k^2 + 2k$

Since $n^2 = 2m+1$, where $m = 2k^2 + 2k$ is an integer (b/c k is a integer), we conclude that n^2 is odd. $\square$ Q.E.D.

Indirect Proofs (proofs by contraposition)

Thm: Let n be an integer. If $\underbrace{5n+4 \text{ is odd}}_{p}$, then $\underbrace{n \text{ is odd}}_{q}$

Proof Strategy

The statement is of the form $p \to q$, where

p "$5n+4$ is odd"

q "n is odd"

To prove $p \to q$ using an indirect proof, we prove the contrapositive $\neg q \to \neg p$ by a direct proof i.e assume $\neg q$, and show $\neg p$ follows.

Not $\neg p$: "$5n+4$ is even"
$\neg q$: "n is even"

Proof
Assume n is even, then $n = 2k$ for some integer k.
Now $5n+4 = 5(2k)+4 = 10k+4 = 2\underbrace{(5k+2)}_{m}$
$5n+4 = 2m$ for $m = 5k+2$

Since $5n+4 = 2m$ for $m = 5k+2$, which is integer b/c k is integer conclude $5n+4$ is even
We showed if n is even then $5n+4$, is even.
Hence if $5n+4$ is odd, then n is odd $\square$

Definition
A real number r is called rational if $r = p/q$ for some integers p and q with $q \neq 0$
A real number that is not rational is called irrational

Defintion
Let m, n be positive integers. If $n = km$ for some integer k, then we say: - n is a multiple of m
- m is a divisor of n
- m divides n (written m/n)

Thm

$\sqrt{2}$ is irrational

Proof Strategy

We are to prove where

p: "$\sqrt{2}$ is irrational"

For a proof by contradiction, assume $\neg p$ & show contradiction follows

Note $\neg p$ "$\sqrt{2}$ is rational"

Proof

Suppose $\sqrt{2}$ is rational then there exist integers p, q st

$\sqrt{2} = p/q$ and $q \neq 0$

Important

We may assume p/q is fully reduced (otherwise, we first reduce it) i.e p & q have no common divisors > 1.

then $\sqrt{2} = p/q$

$$2 = \frac{p^2}{q^2}$$

$$2q^2 = q^2.$$

So p^2 is even, hence p is even,

i.e $p = 2k$ where k is some integer

Then $2q^2 = (2k)^2$

$2q^2 = 4k^2$

$q^2 = 2k^2$

So q^2 is even, whence q is even

Hence $q = 2l$ for som integer l

We have $p = 2k$ & $q = 2l$

(k, l are integers), so

$5 | p$ & $2 | q$, contradicting the assumption that p & q have no common divisors > 1

Conclusion

Assuming $\sqrt{2}$ is a rational and have shown that doing so leads to a contradiction hence $\sqrt{2}$ is irrational

KNIGHTS & KNAVES QUESTION A, B, & C

A: "All of us are knaves"

B: "Exactly one of us is a knave"

Use proof by cases to show C is a knight

Proof Strategy

Define a "A is a knight"

b "B " "

c "C is a knight"

We are to prove $C \equiv T \rightarrow c$

Divide the problem into 4 cases

$p_1 = a \wedge b$ Observe that $p_1 \vee p_2 \vee p_3 \vee p_4 \equiv T$

$p_2 = a \wedge \neg b$ So $C \equiv T \rightarrow c \equiv (p_1 \vee p_2 \vee p_3 \vee p_4) \rightarrow c$

$p_3 = \neg a \wedge b$

$p_4 = \neg a \wedge \neg b$

Now $(p_1 \vee p_2 \vee p_3 \vee p_4) \rightarrow c \equiv \neg(p_1 \vee p_2 \vee p_3 \vee p_4) \vee c$

$\equiv (\neg p_1 \wedge \neg p_2 \wedge \neg p_3 \wedge \neg p_4) \vee c$

$\equiv (\neg p_1 \vee c) \wedge (\neg p_2 \vee c) \wedge (\neg p_3 \vee c) \wedge (\neg p_4 \vee c)$

$\equiv (p_1 \rightarrow c) \wedge (p_2 \rightarrow c) \wedge (p_3 \rightarrow c) \wedge (p_4 \rightarrow c)$

Hence, to prove C, we need to prove each of $p_1 \rightarrow c$

$p_2 \rightarrow c$

$p_3 \rightarrow c$

$p_4 \rightarrow C$

Proof

Various proof of $p_1 \to c$

Case 1: To prove $p_1 \to c$

Assume $p_1 = T$ then $a = T$ ($\&$ b is T) & A is a knight then A is telling the truth, so A, B, C are all knaves - a contradiction

Hence $p_1 = F$, so $p_1 \to c$ is T

Case 2: To prove $p_2 \to c$

(identical to p_1)

Case 3: To prove $p_3 \to c$

Assume $P_3 = T$ then $a = F$ & $b = T$. so A is a knave, and B is a knight, and so exactly one of A, B, C is a knave. So it must be A so C is a knight, this proves $c = T$, and so $p_3 \to c$ is T

Case 4: To prove $p_4 \to c$

Assume $P_4 = T$, then $a = F$ & $b = F$, so A is a knave, and so not all of A, B, C are knaves, and B is a knave. Hence C is a knight.

Thus $c = T$ and $p_4 \to c$ is true as well.

All of our $P_n \to C$ are true, thus so is

$(p_1 \vee p_2 \vee p_3 \vee p_4) \to c \equiv c$ is T $\square$

Proof of Equivalence

Thm: Let m & n be positive integers. Then:

$$m = n \iff m \mid n \text{ and } n \mid m$$

Proof Strategy

We are to prove $p \leftrightarrow q$, where

p "$m = n$"

q "$m \mid n$ & $n \mid m$"

We need to prove both $p \to q$ & $q \to p$

Proof

To prove $p \to q$

Assume p "$m = n$"

Then $n = 1 \cdot m$ so $m \mid n$

and $n = 1 \cdot n$ so $n \mid m$

So q follows

To prove $q \Rightarrow p$

Assume q "$m \mid n$ and $n \mid m$"

Hence $n = km$ for some integer k and $m = ln$ for some integer l

Then

$$n = km = k(ln) = (kl)n$$

$$n - (kl)n = 0$$

$$n(1 - kl) = 0$$

Since $n > 0$, we must have $1 - kl = 0$, i.e. $kl = 1$. Since k, l are positive integers we have $k = l = 1$

Thus $n = 1 \cdot m = m$ so p follows

Scanned by CamScanner

$m \mid n$ m divides n

$\uparrow$ m is a divisor of n

"divides" $n = km$ for int. k

Thm

The equation $x^3 + x + 1 = 0$ has no rational roots.

Proof (By contradiction, then by cases)

Suppose $x^3 + x + 1 = 0$ has a rational root r. then there exist $r = a/b$, where a, b are integers $b \neq 0$, st $r^3 + r + 1 = 0$

We may assume a/b is reduced, i.e. a & b have no common divisors > 1.

Then: $(a/b)^3 + a/b + 1 = 0$

$a^3 + ab^2 + b^3 = 0$

Case 1 : a is even

then $\underbrace{a^3}_{\text{even}} + \underbrace{ab^2}_{\text{even}} + \underbrace{b^3}_{\text{even}} = 0$ so b^3 must be even, so b is even

So a & b are both even, contradicting the assumption that a/b is reduced

Case 2 : a is odd

then $\underbrace{a^3}_{\text{odd}} + \underbrace{a^2b}_{\text{odd}} + \underbrace{b^3}_{\text{odd}} = 0$

Case 2.1 b odd $\rightarrow$ $\rightarrow\leftarrow$ (contradiction)

Case 2.2 b even $\rightarrow$ $\rightarrow\leftarrow$ (contradiction)