University of Ottawa
School of Electrical Engineering and Computer Science

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Assignment #1 Solution

Question 50: An ancient Sicilian legend says that the barber in a remote town who can be reached only by traveling a dangerous mountain road shaves those people, and only those people, who do not shave themselves. Can there be such a barber?

Answer: No. This is a classical paradox. (We will use the male pronoun in what follows, assuming that we are talking about males shaving their beards here, and assuming that all men have facial hair. If we restrict ourselves to beards and allow female barbers, then the barber could be female with no contradiction.) If such a barber existed, who would shave the barber? If the barber shaved himself, then he would be violating the rule that he shaves only those people who do not shave themselves. On the other hand, if he does not shave himself, then the rule says that he must shave himself. Neither is possible, so there can be no such barber.

Question 18: When planning a party you want to know whom to invite. Among the people you would like to invite are three touchy friends. You know that if Jasmine attends, she will become unhappy if Samir is there, Samir will attend only if Kanti will be there, and Kanti will not attend unless Jasmine also does. Which combinations of these three friends can you invite so as not to make someone unhappy?

Answer: We will translate these conditions into statements in symbolic logic, using j , s , and k for the propositions that Jasmine, Samir, and Kanti attend, respectively. The first statement is $j \rightarrow \neg s$. The second statement is $s \rightarrow k$. The last statement is $\neg k \vee j$, because "unless" means "or." (We could also translate this as $k \rightarrow j$. From the comments following Definition 5 in the text, we know that $p \rightarrow q$ is equivalent to "q unless $\neg p$. In this case p is $\neg j$ and q is $\neg k$.) First, suppose that s is true. Then the second statement tells us that k is also true, and then the last statement forces j to be true. But now the first statement forces s to be false. So we conclude that s must be false; Samir cannot attend. On the other hand, if s is false, then the first two statements are automatically true, not matter what the truth values of k and j are. If we look at the last statement, we see that it will be true as long as it is not the case that k is true and j is false. So the only combinations of friends that make everybody happy are Jasmine and Kanti, or Jasmine alone (or no one!).

Question 12: Show that each conditional statement in Exercise 10 is a tautology without using truth tables.

Answer: We argue directly by showing that if the hypothesis is true, then so is the conclusion. An alternative approach, which we show only for part (a), is to use the equivalences listed in the section and work symbolically.
a) Assume the hypothesis is true. Then p is false. Since $p \vee q$ is true, we conclude that q must be true. Here is a more "algebraic" solution:

$[\neg p \wedge (p \vee q)] \rightarrow q \equiv \neg[\neg p \wedge (p \vee q)] \vee q \equiv \neg\neg p \vee \neg(p \vee q)] \vee q \equiv p \vee \neg(p \vee q) \vee q \equiv (p \vee q) \vee \neg(p \vee q) \equiv \mathbf{T}$

The reasons for these logical equivalences are, respectively, Table 7, line 1; De Morgan's law; double negation; commutative and associative laws; negation law.

b) We want to show that if the entire hypothesis is true, then the conclusion $p \rightarrow r$ is true. To do this, we need only show that if p is true, then r is true. Suppose p is true. Then by the first part of the hypothesis, we conclude that q is true. It now follows from the second part of the hypothesis that r is true, as desired.

c) Assume the hypothesis is true. Then p is true, and since the second part of the hypothesis is true, we conclude that q is also true, as desired.

d) Assume the hypothesis is true. Since the first part of the hypothesis is true, we know that either p or q

is true. If p is true, then the second part of the hypothesis tells us that r is true; similarly, if q is true, then the third part of the hypothesis tells us that r is true. Thus in either case we conclude that r is true.

Question 24: Show that $(p \rightarrow q) \vee (p \rightarrow r)$ and $p \rightarrow (q \vee r)$ are logically equivalent.

Answer: We determine exactly which rows of the truth table will have T as their entries. Now $(p \rightarrow q) \vee (p \rightarrow r)$ will be true when either of the conditional statements is true. The conditional statement will be true if p is false, or if q in one case or r in the other case is true, i.e., when $q \vee r$ is true, which is precisely when $p \rightarrow (q \vee r)$ is true. Since the two propositions are true in exactly the same situations, they are logically equivalent.

Question 10: Let C(x) be the statement "x has a cat," let D(x) be the statement "x has a dog," and let F(x) be the statement "x has a ferret." Express each of these statements in terms of C(x), D(x), F(x), quantifiers, and logical connectives. Let the domain consist of all students in your class.

a) A student in your class has a cat, a dog, and a ferret.

b) All students in your class have a cat, a dog, or a ferret.

c) Some student in your class has a cat and a ferret, but not a dog.

d) No student in your class has a cat, a dog, and a ferret.

e) For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has this animal as a pet.

Answer:

a) We assume that this means that one student has all three animals: $\forall x\,(C(x) \wedge D(x) \wedge F(x))$.

b) $\forall x(C(x) \vee D(x) \vee F(x))$

c) $\exists x(C(x) \wedge F(x) \wedge \neg D(x))$

d) This is the negation of part (a): $\neg\exists x(C(x) \wedge D(x) \wedge F(x))$.

e) Here the owners of these pets can be different: $(\exists x C(x)) \wedge (\exists x D(x)) \wedge (\exists x F(x))$. There is no harm in using the same dummy variable, but this could also be written, for example, as : $(\exists x C(x)) \wedge (\exists y D(y)) \wedge (\exists z F(z))$

Question 42: Express each of these system specifications using predicates, quantifiers, and logical connectives.

a) Every user has access to an electronic mailbox.

b) The system mailbox can be accessed by everyone in the group if the file system is locked.
c) The firewall is in a diagnostic state only if the proxy server is in a diagnostic state.
d) At least one router is functioning normally if the throughput is between 100 kbps and 500 kbps and the proxy server is not in diagnostic mode.

Answer: There are many ways to write these, depending on what we use for predicates.
a) Let $A(x)$ be "User x has access to an electronic mailbox." Then we have $\forall x A(x)$.
b) Let $A(x, y)$ be "Group member x can access resource y," and let $S(x, y)$ be "System x is in state y."
Then we have $S(\text{file system, locked}) \rightarrow \forall x A(x, \text{system mailbox})$.
c) Let $S(x, y)$ be "System x is in state y." Recalling that "only if" indicates a necessary condition, we have $S(\text{firewall, diagnostic}) \rightarrow S(\text{proxy server, diagnostic})$.
d) Let $T(x)$ be "The throughput is at least x kbps," where the domain of discourse is positive numbers, let $M(x, y)$ be "Resource x is in mode y," and let $S(x, y)$ be "Router x is in state y." Then we have

$(T(100) \wedge \neg T(500) \wedge \neg M(\text{proxy server, diagnostic})) \rightarrow \exists x S(x, \text{normal})$.

Question 46: Establish these logical equivalences, where x does not occur as a free variable in A. Assume that the domain is nonempty.

a) $(\forall x P(x)) \vee A \equiv \forall x(P(x) \vee A)$

b) $(\exists x P(x)) \vee A \equiv \exists x(P(x) \vee A)$

Answer: a) There are two cases. If A is true, then $\forall x(P(x) \vee A)$ is true, and since $P(x) \vee A$ is true for all x,
$\forall x (P(x) \vee A)$ is also true. Thus both sides of the logical equivalence are true (hence equivalent). Now suppose that A is false. If $P(x)$ is true for all x, then the left-hand side is true. Furthermore, the right-hand side is also true (since $P(x) \vee A$ is true for all x). On the other hand, if $P(x)$ is false for some x, then both sides are false. Therefore again the two sides are logically equivalent.

b) There are two cases. If A is true, then $\exists x (P(x) \vee A)$ is true, and since $P(x) \vee A$ is true for some (really all) x, $\exists x(P(x) \vee A)$ is also true. Thus both sides of the logical equivalence are true (hence equivalent). Now suppose that A is false. If $P(x)$ is true for at least one x, then the left-hand side is true. Furthermore, the right-hand side is also true (since $P(x) \vee A$ is true for that x). On the other hand, if $P(x)$ is false for all x, then both sides are false. Therefore again the two sides are logically equivalent.

Question 20: Express each of these statements using predicates, quantifiers, logical connectives, and mathematical operators where the domain consists of all integers.
a) The product of two negative integers is positive.
b) The average of two positive integers is positive.

Answer:

a) $\forall x \forall y((x < 0) \wedge (y < 0) \rightarrow (xy > 0))$

b) $\forall x \forall y ((x > 0) \wedge (y > 0) \rightarrow ((x + y)/2 > 0))$

Question 32: Express the negations of each of these statements so that all negation symbols immediately precede predicates.

b) $\exists x \exists y P(x, y) \land \forall x \forall y Q(x, y)$

Answer:

b)

$$\begin{aligned}\neg(\exists x \exists y\, P(x,y) \land \forall x \forall y\, Q(x,y)) &\equiv \neg\exists x \exists y\, P(x,y) \lor \neg\forall x \forall y\, Q(x,y)\\ &\equiv \forall x \neg\exists y\, P(x,y) \lor \exists x \neg\forall y\, Q(x,y)\\ &\equiv \forall x \forall y\, \neg P(x,y) \lor \exists x \exists y\, \neg Q(x,y)\end{aligned}$$

Question 46: Determine the truth value of the statement $\exists x \forall y(x \leq y2)$ if the domain for the variables consists of

a) The positive real numbers.

b) The integers.

c) The nonzero real numbers.

Answer:

This statement says that there is a number that is less than or equal to all squares.

a) This is false, since no matter how small a positive number x we might choose, if we let $y=\sqrt{x/2}$, then $x=2y^2$, and it will not be true that $x <= y^2$.

b) This is true, since we can take $x = -1$, for example.

c) This is true, since we can take $x = -1$, for example.