SCHULICH
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Schulich School of Engineering
FINAL EXAMINATION

ENGG 317: Mechanics of Solids (Winter 2008)

April 25, 2008 | 8:00 – 11:00 AM

Instructors: Drs. Q. Sun, S. Boyd, L. Li, S. Federico

Instructions:

- **Do all your work in the exam booklet provided.**
- Write your name and student ID and lecture section number on the exam booklet.
- **Clearly indicate your answer by drawing a box around it.**
- This is a closed book and closed notes exam.
- Schulich School of Engineering calculators are required.
- Formula sheet is attached.
- Answer all parts of all FOUR questions. Each sub-question answer should be clearly indicated.
- Exam duration is 180 minutes.

Stress – Axial loading and bearing stress

$$\sigma=\frac{P}{A},\ \sigma=\lim_{\Delta A\to x}\frac{\Delta F}{\Delta A},\ \tau_{ave}=\frac{P}{A},\ \sigma_b=\frac{P}{A_{proj}}=\frac{P}{td},\ FS=\frac{P_{fail}}{P_{allow}},\ K=\frac{\sigma_{max}}{\sigma_{avg}}$$

Stress and strain – Axial loading

$$\varepsilon=\lim_{\Delta x\to x}\frac{\Delta\delta}{\Delta x}=\frac{d\delta}{dx},\ \sigma=E\varepsilon,\ \delta=\frac{PL}{AE},\ \delta_T=\alpha(\Delta T)L,\ \tau=G\gamma,\ \varepsilon_x=\frac{\sigma_x}{E}-\frac{\nu\sigma_y}{E}-\frac{\nu\sigma_z}{E}$$

Torsion

$$\gamma=\frac{\rho\phi}{L},\ \gamma=\frac{\rho}{c}\gamma_{max},\ \tau=\frac{\rho}{c}\tau_{max},\ \tau=\frac{T\rho}{J},\ \phi=\frac{TL}{JG},\ T=\frac{4}{3}T_Y\left(1-\frac{1}{4}\frac{\rho_Y^3}{c^3}\right),\ T=\int\rho(\tau dA),\ P=T\omega,\ \omega=2\pi f,$$

$$T_p=\frac{4}{3}T_Y,\ \frac{\rho_Y}{c}=\frac{\phi_Y}{\phi},\ plasticity\ T=\int_0^{\rho_Y}\rho\tau dA+\int_{\rho_Y}\rho\tau_Y dA$$

Bending and Transverse Loading: $\sigma=-\frac{My}{I},\ \varepsilon_x=\frac{y}{c}\varepsilon_{max},\ \frac{1}{\rho}=\frac{M}{EI},\ M=-\int(y\sigma_x dA),\ \frac{y_Y}{y}=\frac{\sigma_Y}{\sigma},\ \tan\phi=\frac{I_z}{I_y}\tan\theta,$

$k={M_P}/{M_Y}$, plasticity $M=\int_0^{y_Y}y\sigma_x dA+\int_{y_Y}y\sigma_Y dA,$ $q=\frac{VQ}{I},\ \tau_{ave}=\frac{VQ}{It},\ F_{nail}=q\cdot s$

Transformations of Stress and Strain and Thin-walled Pressure Vessel

$$\sigma_{x'}=\frac{\sigma_x+\sigma_y}{2}+\frac{\sigma_x-\sigma_y}{2}\cos 2\theta+\tau_{xy}\sin 2\theta,\ \sigma_{y'}=\frac{\sigma_x+\sigma_y}{2}-\frac{\sigma_x-\sigma_y}{2}\cos 2\theta-\tau_{xy}\sin 2\theta,$$

$$\tau_{x'y'}=-\frac{\sigma_x-\sigma_y}{2}\sin 2\theta+\tau_{xy}\cos 2\theta,\ \tan 2\theta_p=\frac{2\tau_{xy}}{\sigma_x-\sigma_y},\ \tan 2\theta_s=-\frac{\sigma_x-\sigma_y}{2\tau_{xy}},$$

$$\sigma_{min,max}=\frac{\sigma_x+\sigma_y}{2}\pm\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}=\sigma_{ave}\pm R,\qquad \sigma_{hoop}=\frac{pr}{t},\ \sigma_{long}=\frac{pr}{2t}$$

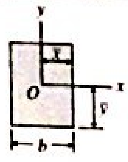
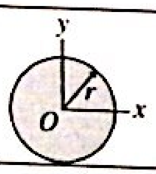
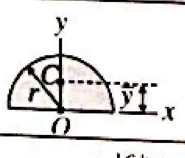
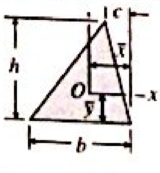
Design of Beams and Shafts for Strength

$\frac{dV}{dx}=-w,\ \frac{dM}{dx}=V,\ V_D-V_C=-\{$area under load curve C to D$\}$, $M_D-M_C=-\{$area under shear curve C to D$\}$

First Moment of Area, Second Moment of Area

$Q_x=\int_A ydA=A\bar{y},\ I_x=\int_A y^2 dA$, thin-walled tube $J_o=2\pi r^3 t=2I_{xx}$

Properties of Plane Areas

	$\bar{x}=\frac{b}{2}$, $\bar{y}=\frac{b}{2}$	$A=bh$	$I_{xx}=\frac{bh^3}{12},I_{yy}=\frac{hb^3}{12}$	$J_O=\frac{bh}{12}(h^2+b^2)$
		$A=\pi r^2$	$I_{xx}=I_{yy}=\frac{\pi r^4}{4}=\frac{\pi d^4}{64}$	$J_O=\frac{\pi r^4}{2}=\frac{\pi d^4}{32}$
	$\bar{y}=\frac{4r}{3\pi}$	$A=\frac{\pi r^2}{2}$	$I_{xx}=I_{yy}=\frac{\pi r^4}{8}$	$J_O=\frac{\pi r^4}{4}$
	$\bar{x}=\frac{b+c}{3}$, $\bar{y}=\frac{h}{3}$	$A=\frac{bh}{2}$	$I_{xx}=\frac{bh^3}{36}$, $I_{yy}=\frac{bh}{36}(b^2-bc+c^2)$	$J_O=\frac{bh}{36}(h^2+b^2-bc+c^2)$

Question 1

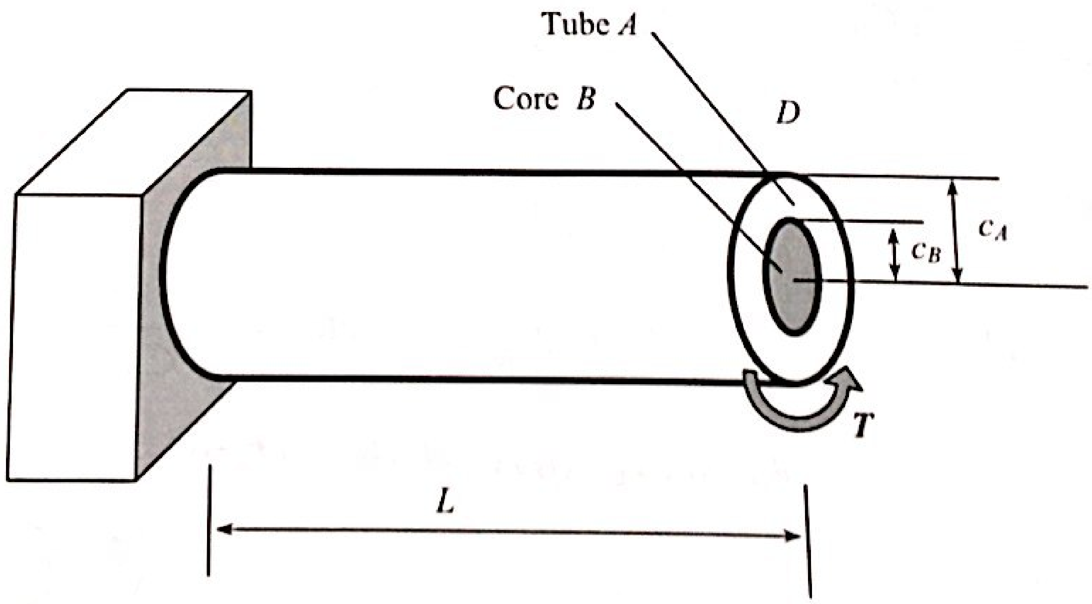
A composite shaft of length L consists of a tube A and a solid core B perfectly bonded together at their interface. Tube A has an outer radius c_A and an inner radius c_B. Radius c_B is also the radius of the solid core B. A torque T is applied at end D of the composite shaft. Both materials are assumed to be elastoplastic, and all material constants are given in the table below.

1.1 [10 Marks] Determine the maximum shear stress caused by an applied torque T if both materials are within their elastic limits.

1.2 [6 Marks] If the applied torque has caused one material to be completely plastic and the other exactly at its elastic limit, find the angle of twist of end D with respect to the fixed end. Draw the shear stress distribution along a radial line from the center of the cross section, indicating the magnitudes where necessary.

1.3 [4 Marks] What is the magnitude of the applied torque T to cause the stress distribution as described in 1.2? Express your findings in terms of c_A, c_B, and the material constants from the table below.

	Tube A	Core B
Shear Modulus of Rigidity	$3G$	G
Yield Strength	τ_{YA}	τ_{YB}
Yield Strain	γ_{YA}	γ_{YB}



ENGG317 2008 Final Exam Question # 1

1.1. Equilibrium $T_A + T_B = T$ —— ①

Compatibility $\phi_A = \phi_B$ —— ②

Linear elastic materials, from ②

$$\frac{T_A L}{G_A J_A} = \frac{T_B L}{G_B J_B}$$

knowing $G_A = 3G$, $J_A = \frac{\pi}{2}(C_A^4 - C_B^4)$

$G_B = G$, $J_B = \frac{\pi}{2} C_B^4$

$$T_A = 3\frac{J_A}{J_B} T_B \quad \text{—— ③}$$

③ → ①

$$3\frac{J_A}{J_B} T_B + T_B = T$$

$$\therefore \quad T_B = \frac{J_B}{3J_A + J_B} T = \frac{C_B^4}{3(C_A^4 - C_B^4) + C_B^4} T$$

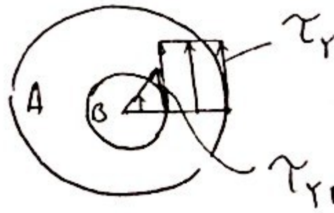
$$T_A = \frac{3J_A}{3J_A + J_B} T = \frac{3(C_A^4 - C_B^4)}{3C_A^4 - 2C_B^4} T$$

$$\tau_{A,max} = \frac{T_A C_A}{J_A} = \frac{3C_A}{3J_A + J_B} T$$

$$\tau_{B,max} = \frac{T_B C_B}{J_B} = \frac{C_B}{3J_A + J_B} T$$

1.2. (I) If A is plastic: $C_B \phi = \gamma_{YB} L$; $\phi = \frac{\gamma_{YB} L}{C_B}$

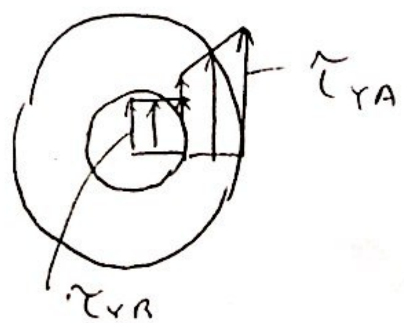
the shear stress distribution



OR (II) If B is plastic: $C_A \phi = \gamma_{YA} L$,

$$\phi = \frac{\gamma_{YA} L}{C_A}$$

Shear stress



1.3. For (I)

$$T = \int_A \tau \rho \, dA + \int_B \tau \rho \, dA = \int_{c_B}^{c_A} \tau_{YA} \rho (2\pi\rho) \, d\rho + \int_0^{c_B} \frac{\tau_{YB}}{c_B} \rho \cdot \rho (2\pi\rho)$$

$$= 2\pi \tau_{YA} \frac{1}{3} (c_A^3 - c_B^3) + \frac{2\pi}{c_B} \tau_{YB} \left(\frac{c_B^4}{4} \right)$$

$$= \frac{2\pi \tau_{YA}}{3} (c_A^3 - c_B^3) + \frac{\pi \tau_{YB}}{2} c_B^3$$

For (II)

$$T = \int_A \tau \rho \, dA + \int_B \tau \rho \, dA$$

$$= \int_{c_B}^{c_A} \frac{\tau_{YA}}{c_A} \rho \cdot \rho (2\pi\rho) d\rho + \int_0^{c_B} \tau_{YB} \rho (2\pi\rho) d\rho$$

$$= \frac{\pi \tau_{YA}}{2 c_A} (c_A^4 - c_B^4) + \frac{2\pi \tau_{YB}}{3} c_B^3$$

2.1 or $I_z = \frac{1}{12}\times 75\times 75^3 - 2\times\frac{1}{12}\times 30\times 45^3$

$= 2181093.75\ mm^4$

$\approx 2.181\times 10^{-6}\ m^4$

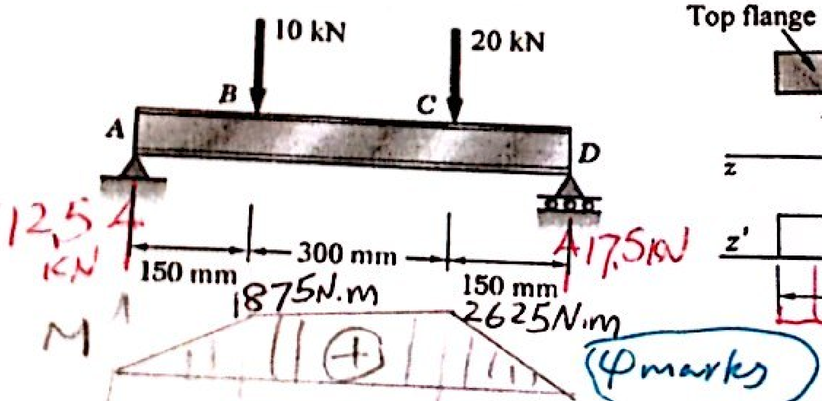
Question 2

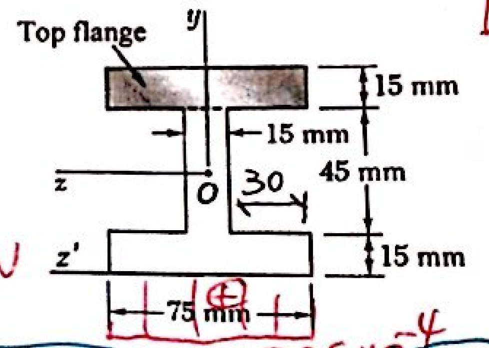
A simply supported beam is subjected to two concentrated loads at positions *B* and *C*. Its cross-section is shown below right. Neglect the effect of stress concentrations.

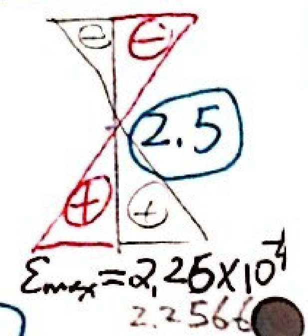
2.1 [12 Marks] Determine the maximum normal stress due to bending, if the entire beam is made of steel with a Young's modulus of 200 GPa. (Note: your bending-moment diagram must be shown).

2.2 [4 Marks] Show normal strain distributions, respectively, along the *y* and the *z'* axes for the cross-section at *C* knowing that the material is the same as above. Indicate the magnitude of maximum normal strain.

2.3 [4 Marks] The top flange is now replaced by an alloy with a Young's modulus of 120 GPa and is perfectly bonded to the steel. Locate the neutral axis of the cross-section with respect to the *z'* axis.







2.1 $R_A = \frac{1}{600}(20\times 150 + 10\times 450) = 12.5\ kN$ (2marks)

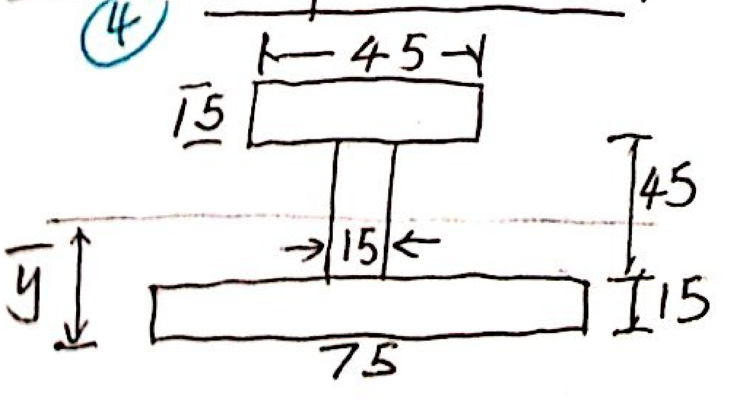
$R_B = \frac{1}{600}(10\times 150 + 20\times 450) = 17.5\ kN$

$M_{max} = R_D \times 150 = 2625\times 10^3\ N.mm$

$I_z = \frac{1}{12}\times 15\times 45^3 + 2\times(\frac{1}{12}\times 75\times 15^3 + 75\times 15\times 30^2) = 2181094\ mm^4$ 3.75 (4)

$\sigma_{max} = \frac{M_{max}}{S} = \frac{M_{max}}{I_z/37.5} = \frac{2625000\ N.mm}{58162.5\ mm^3} = 45.132\ MPa$ (2)

2.3 (4) transformed section



$\bar{y} = \frac{75\times 15\times 7.5 + 15\times 45\times 37.5 + 45\times 15\times 67.5}{75\times 15 + 15\times 45 + 45\times 15}$

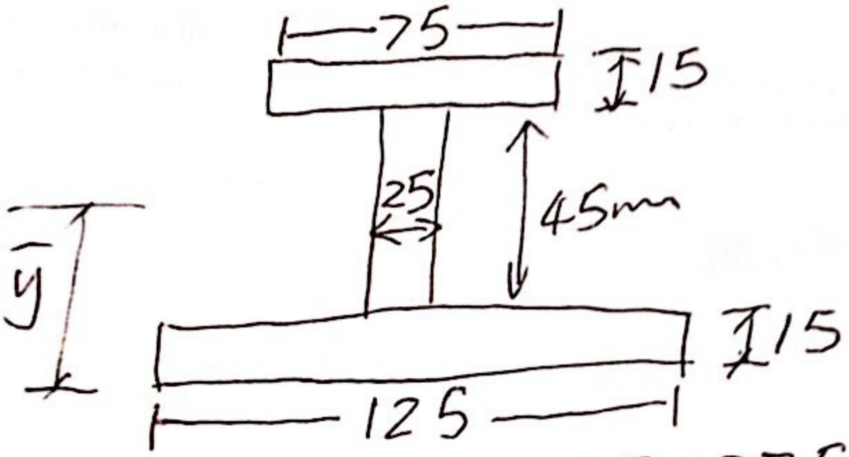
$= 32.045\ mm$

use $n = \frac{120}{200} = 0.6$

Other method for $\boxed{2.3}$

Reference material Alloy

$$n = \frac{200}{120} = 1.6667$$



$$\bar{y} = \frac{125\times15\times7.5 + 25\times45\times37.5 + 75\times15\times67.5}{125\times15 + 25\times45 + 75\times15}$$

$= 32.045^{4545}$ mm

ENGG 317 cont'd

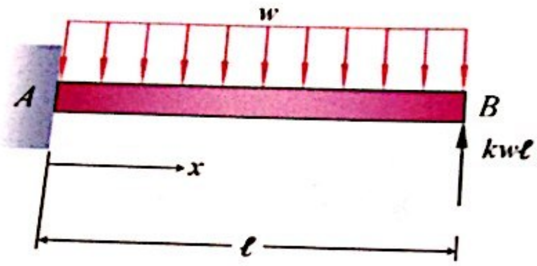
Question 3

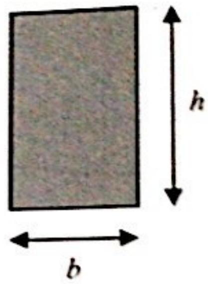
A cantilever beam of length ℓ and bending stiffness EI, is loaded by a uniformly distributed load w (downward) and a concentrated load $kw\ell$ (upward) applied at B (k is a non-dimensional parameter).

a) [8 marks] Find the value of k for which the vertical deflection of end B is zero.

b) [6 marks] For $k = 0$, draw the shear force V and bending moment M diagrams and determine the maximum values of V and M.

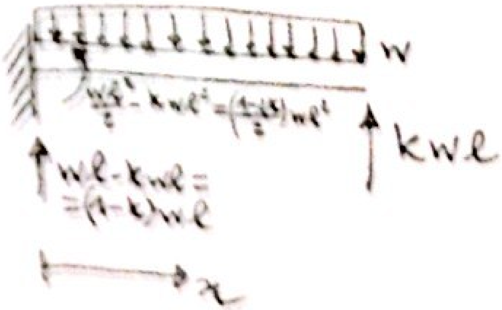
c) [6 marks] For $k = 0$, let τ_{all} and σ_{all} be the allowable shear and normal stresses respectively, and $\sigma_{all} = 20\tau_{all}$. Considering the beam has a rectangular cross-section (shown below right) with height h, find the minimum width b for which both normal and shear stresses are within the allowable stresses. Assume such that $\ell/h >> 10$.





ENGG 317 2008 Final exam - Question 3 Solution

a)



$$\curvearrowright M: -\left(\frac{1-2k}{2}\right)w\ell^2 + (1-k)w\ell x - \frac{wx^2}{2}$$

$$\frac{d^2y}{dx^2}(x) = \frac{1}{EI}M(x)$$

$$\theta(x) = \frac{1}{EI}\left(-\left(\frac{1-2k}{2}\right)w\ell^2 x + \frac{1-k}{2}w\ell x^2 - \frac{wx^3}{6}\right) + C_1$$

$$\theta(0) = 0 \Rightarrow C_1 = 0$$

$$y(x) = \frac{1}{EI}\left(-\left(\frac{1-2k}{4}\right)w\ell^2 x^2 + \frac{1-k}{6}w\ell x^3 - \frac{wx^4}{24}\right) + C_2$$

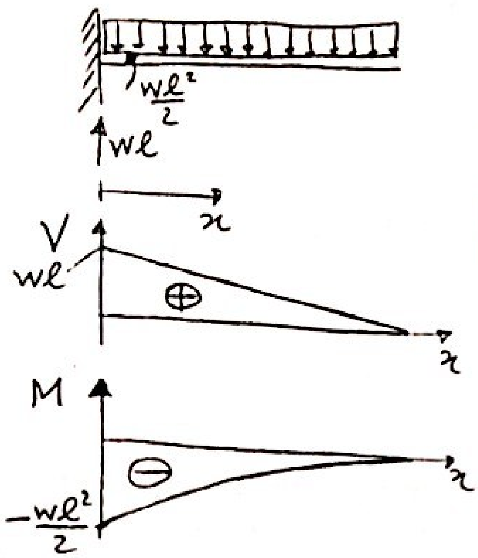
$$y(0) = 0 \Rightarrow C_2 = 0$$

$$y(\ell) = \frac{w\ell^4}{EI}\left(-\frac{1-2k}{4} + \frac{1-k}{6} - \frac{1}{24}\right) = 0 \Rightarrow$$

$$\Rightarrow \frac{-6+12k+4-4k-1}{24} = 0 \Rightarrow$$

$$\Rightarrow 8k - 3 = 0 \Rightarrow k = \boxed{\frac{3}{8} = k_0}$$

b) $k=0$

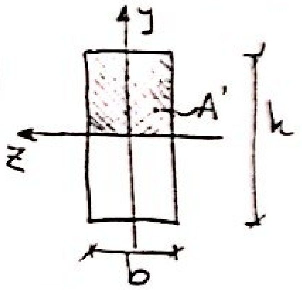


$\curvearrowleft M: \quad M(x) = -\frac{w\ell^2}{2} + w\ell x - \frac{wx^2}{2}$

$\uparrow V: \quad V(x) = w\ell - wx$

$$\boxed{\begin{aligned} M_{max} &= M(0) = -\frac{w\ell^2}{2} \\ V_{max} &= V(0) = w\ell \end{aligned}}$$

c) for a rectangular cross-section:



$I = \frac{1}{12} bh^3$

$Q^{A'} = b\frac{h}{2}\frac{h}{4} = \frac{bh^2}{8}$

The maximum shear stress due to V is at $y=0$ (on the z axis):

$$\tau_{max} = \frac{VQ^{A'}}{tI} = \frac{w\ell \; \cancel{b}h^{\cancel{2}}/8}{\cancel{b}\cdot\frac{1}{12}bh^{\cancel{3}}} = \frac{3}{2}\frac{w\ell}{bh}$$

the maximum normal stress due to M is at $y = \frac{h}{2}$

$$\sigma_{max} = \frac{|M|}{I}\frac{h}{2} = \frac{w\ell^2/2}{\frac{1}{12}bh^{\cancel{3}2}}\frac{\cancel{h}}{2} = 3\frac{w\ell^2}{bh^2}$$

If both σ_{max} and τ_{max} have to be below the respective allowable values σ_{all} and τ_{all}, then:

$$\sigma_{all} = 20\,\tau_{all} \Rightarrow \begin{cases} \sigma_{max} = 3\frac{w\ell^2}{bh^2} < \sigma_{all} \\ \tau_{max} = \frac{3}{2}\frac{w\ell}{bh} < \frac{1}{20}\sigma_{all} \end{cases}$$

$\sigma: \quad b_{min} = 3\,\frac{w}{\sigma_{all}}\frac{\ell^2}{h^2}$

$\tau: \quad b_{min} = 30\,\frac{w}{\sigma_{all}}\frac{\ell}{h}$

Since $\frac{\ell}{h} > 10 \rightarrow \boxed{b_{min} = \frac{3w}{\sigma_{all}}\frac{\ell^2}{h^2}}$ (from σ)

Question 4

A rigid bar *AB* is welded to a solid cylindrical member *BC* of radius 15 mm. A load *P* is applied at *A* with $\theta = 90°$ as shown.

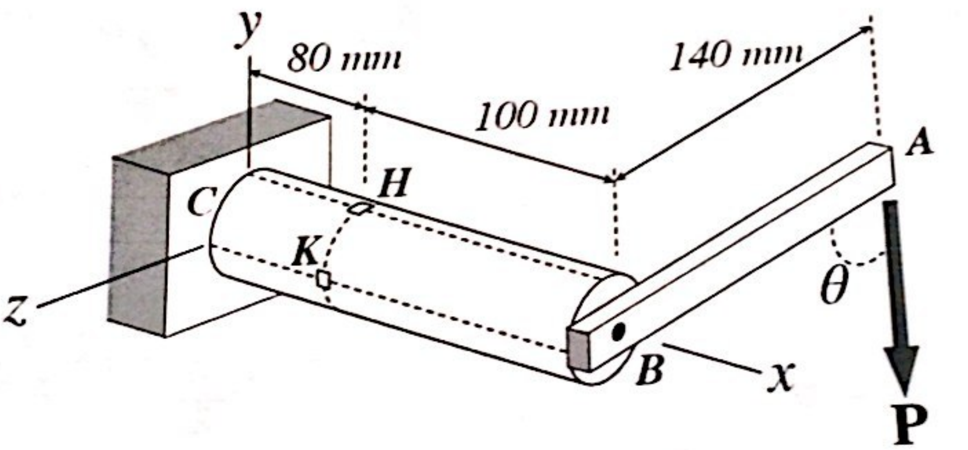
4.1 [10 Marks] Determine the state of plane stress at point *K* when $P = 800$ N (illustrate your results on a representative element *K* as shown below bottom right).

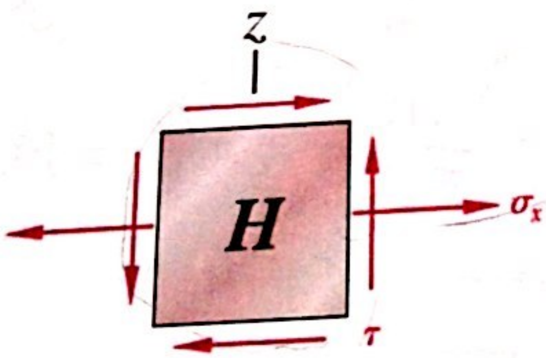
4.2 [8 Marks] A new load *P* results in a state of plane stress at *H* with $\sigma_x = 42$ MPa and $\tau = 60$ MPa in the directions shown on element *H* (bottom left). Determine the principal stresses and their orientation.

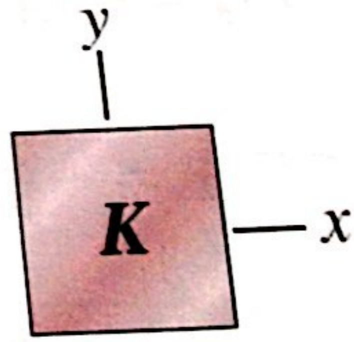
4.3 [2 Marks] If an *additional* centric tensile load was applied at *B* on the cylindrical member *BC*, would the maximum in-plane shear stress, τ_{max} (choose only one):

a) increase
b) decrease
c) stay the same

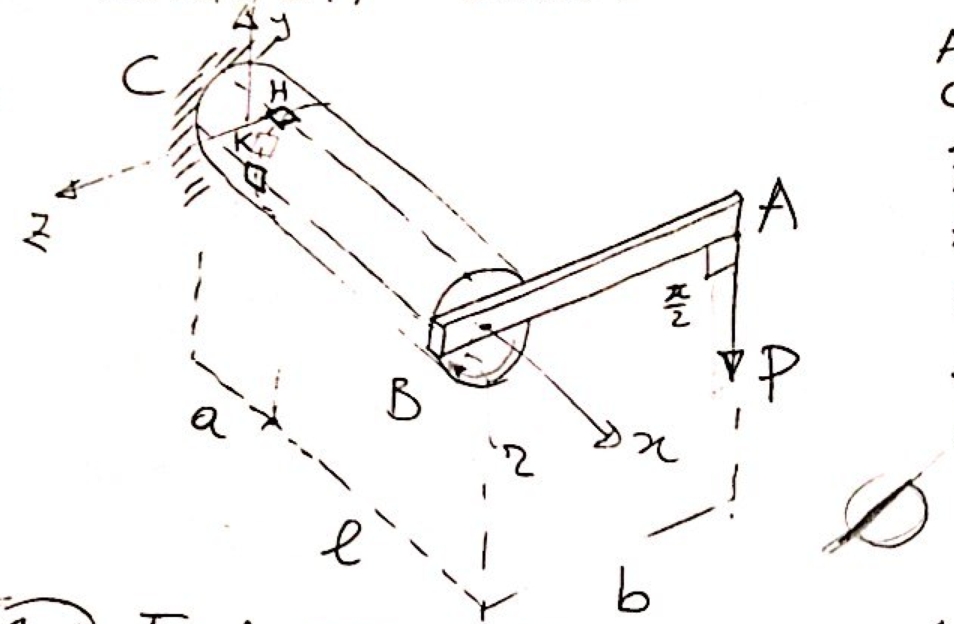
Show your work in your exam booklet.







ENGG 317 2008 Final Q4



AB: rigid
CB: Solid cylinder radius = 15 mm

$P = 800\,N$

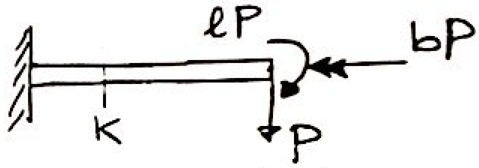
$r = 15\,mm$

$b = 140\,mm$

$l = 100\,mm$

$a = 80\,mm$

4.1 Find state of stress at K



$N = 0 \Rightarrow \sigma_x^{(N)} = 0$

$V_y = -P \Rightarrow \tau_{xz}^{(V_y)} = -\frac{PQ'}{t\,I_z} = -\frac{P\,\frac{\pi r^2}{2}\cdot\frac{4r}{3\pi}}{2r\cdot\frac{\pi r^4}{4}} = -\frac{4}{3}\frac{P}{\pi r^2}$

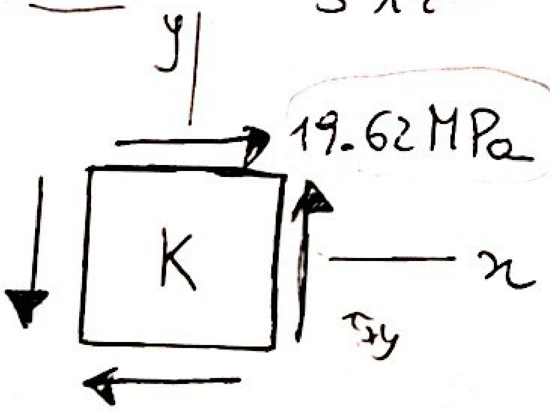
$V_z = 0 \Rightarrow \tau_{xy}^{(V_z)} = 0$

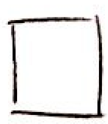
$T = -bP \Rightarrow \tau_{xz}^{(T)} = +\frac{|T|}{J}r = \frac{bP}{\pi\frac{r^4}{2}}r = 2\frac{bP}{\pi r^3}$

$M_y = 0 \Rightarrow \sigma_x^{(M_y)} = 0$

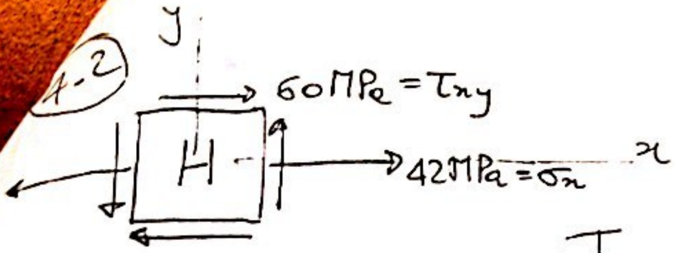
$M_z = -lP \Rightarrow \sigma_x^{(M_z)} = -\frac{-lP}{I_z}y = 0 \quad (y_K = 0!!)$

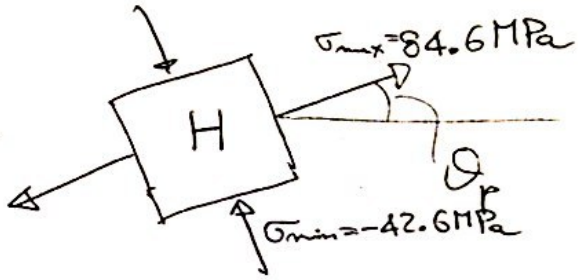
$$\tau_{xz} = -\frac{4}{3}\frac{P}{\pi r^2} + 2\frac{Pb}{\pi r^3} = \frac{P}{\pi r^2}\left(-\frac{4}{3} + \frac{2b}{r}\right) = 19.62\,MPa$$





4.2



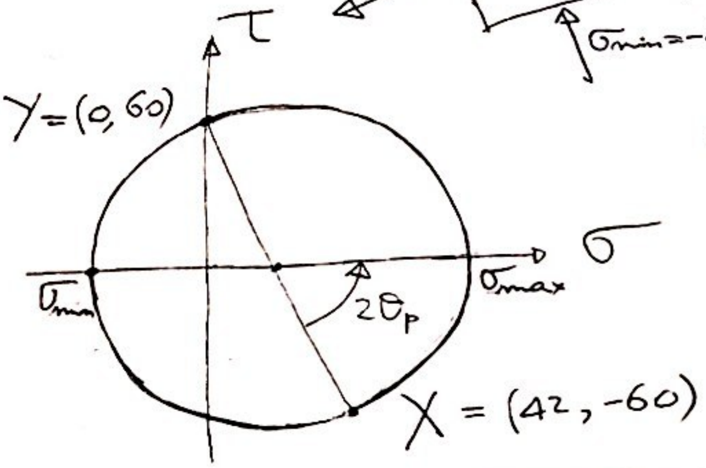


$\sigma_{ave} = \dfrac{\sigma_x + \sigma_y}{2} =$

$= \dfrac{\sigma_x}{2} = 21\,MPa$

$R = \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} =$

$= 63.6\,MPa$



$\sigma_{min} = \sigma_{ave} - R =$

$= -42.6\,MPa$

$\sigma_{max} = \sigma_{ave} + R =$

$= 84.6\,MPa$

$\theta_p = \dfrac{1}{2} \arctan \dfrac{2\tau_{xy}}{\sigma_x - \sigma_y} =$

$= 35.35°$

4.3 If an additional centric Tensile load is applied at B on BC, would the maximum in-plane shear stress at H

$\Rightarrow$ $\boxed{\text{a) increase}}$

b) decrease

c) stay the same

$\longrightarrow$ obvious!