

Solutions to 2012 ENGR 233/4 Final

#1 (a) $P_1(2,1,0)$ $P_2(3,4,0)$ $P_3(1,1,1)$

$$\vec{P_1P_2} = (1, 3, 0) \quad \vec{P_3P_1} = (1, 0, -1)$$

$$\vec{N} = \vec{P_1P_2} \times \vec{P_3P_1} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 0 \\ 1 & 0 & -1 \end{vmatrix} = -3\hat{i} + \hat{j} - 3\hat{k}$$

$$-3x + y - 3z = -3(2) + 1(1) + 3(0)$$

$$-3x + y - 3z = -6 + 1 = -5$$

$$\boxed{3x - y + 3z = 5}$$

(b) $\vec{r}(t) = (3, 4, 0) + t(-3, 1, -3)$

$$\begin{cases} x = 3 - 3t \\ y = 4 + t \\ z = 0 - 3t \end{cases}$$

#2. (a) $W = \sqrt{x^3 + y} + e^{xz}$, $x = 2t$, $y = t^2$, $z = t^{-1}$

$$\frac{dW}{dt} = \frac{\partial W}{\partial x} \frac{dx}{dt} + \frac{\partial W}{\partial y} \frac{dy}{dt} + \frac{\partial W}{\partial z} \frac{dz}{dt} = \frac{\partial W}{\partial x}(2) + \frac{\partial W}{\partial y}(2t) + \frac{\partial W}{\partial z}(-t^{-2})$$

$$\frac{\partial W}{\partial x} = \frac{1}{2}(x^3 + y)^{-1/2} (3x^2) + ze^{xz} \quad \text{at } t=1: x=2 \quad y=1 \quad z=1$$

$$\frac{\partial W}{\partial y} = \frac{1}{2}(x^3 + y)^{-1/2} \quad \frac{\partial W}{\partial y}(2, 1, 1) = \frac{1}{2}(8+1)^{-1/2} (3.4) + e^2$$

$$\frac{\partial W}{\partial z} = xe^{xz} \quad \frac{\partial W}{\partial z}(2, 1, 1) = 2e^2 = \frac{1}{2} \cdot \frac{1}{2} \cdot 2 \cdot 4 + e^2 = (2 + e^2)$$

$$\frac{dW}{dt} = (2 + e^2)2 + \frac{1}{6}(2 \cdot 1) + 2e^2(-1^{-2}) = 4 + 2e^2 + \frac{1}{3} - 2e^2 = \boxed{\frac{13}{3}}$$

(2)

#2(b) direction of fastest cooling: $-\vec{\nabla}T(2,3)$ (or unit vector)

$$\vec{\nabla}T = \frac{\partial T}{\partial x} \hat{i} + \frac{\partial T}{\partial y} \hat{j} = (10x + 4e^{xy}) \hat{i} + xe^{xy} \hat{j}$$

$$-\vec{\nabla}T(2,3) = -(20 + 3e^6) \hat{i} - 2e^6 \hat{j}$$

$$\hat{u} = \frac{-\vec{\nabla}T(2,3)}{\|\vec{\nabla}T\|} = \frac{-(20 + 3e^6) \hat{i} - 2e^6 \hat{j}}{\sqrt{(20 + 3e^6)^2 + (2e^6)^2}}$$

$$\text{directional derivative } D_{\hat{u}}T = \vec{\nabla}T \cdot \hat{u} = \frac{-\|\vec{\nabla}T\|^2}{\|\vec{\nabla}T\|}$$

$$= -\|\vec{\nabla}T\| = -\sqrt{400 + 120e^6 + 9e^{12} + 4e^{12}}$$

$$= -\sqrt{400 + 120e^6 + 13e^{12}}$$

#3. $g(x, y, z) = x^2 + y^2 + z = 10 = \text{constant}$ $\vec{\nabla}g = 2x\hat{i} + 2y\hat{j} + \hat{k}$ is $\perp$ to S

$$(2x, 2y, 1) = c(1, \frac{3}{2}, \frac{1}{2}) \Rightarrow c = 2$$

$$2x = 2$$

$$x = 1$$

$$2y = 3$$

$$y = \frac{3}{2}$$

$$z = 10 - x^2 - y^2 = 10 - 1 - \frac{9}{4}$$

$$= 9 - \frac{9}{4} = 9 \cdot \frac{3}{4} = \frac{27}{4}$$

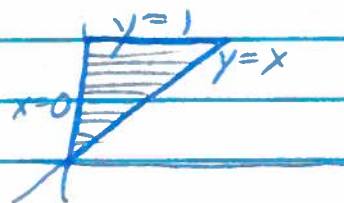
$$(1, \frac{3}{2}, \frac{27}{4})$$

$$\#4. I_y = \iint_R x^2 \rho(x, y) dx dy$$

$$= \int_0^1 \int_0^y x^2 \sqrt{1+y^4} dx dy$$

$$= \int_0^1 \frac{y^3}{3} \sqrt{1+y^4} dy = \frac{1}{3} \cdot \frac{2}{3} (1+y^4)^{3/2} \cdot \frac{1}{4} \Big|_0^1 = \frac{1}{18} (5^{3/2} - 1^{3/2})$$

$$= \frac{\sqrt{125} - 1}{18}$$



③

#5 (a) $\text{grad } f = \vec{\nabla} f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}$

$$= 2x e^{x^2} \cos z \hat{i} + z^4 \cos y \hat{j} + (e^{x^2} \sin z + 4z^3 \sin y) \hat{k}$$

(b) $\text{div}(\text{grad } f) = \vec{\nabla} \cdot \vec{\nabla} f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

$$= \frac{\partial}{\partial x} (2x e^{x^2} \cos z) + \frac{\partial}{\partial y} (z^4 \cos y) + \frac{\partial}{\partial z} (-e^{x^2} \sin z + 4z^3 \sin y)$$

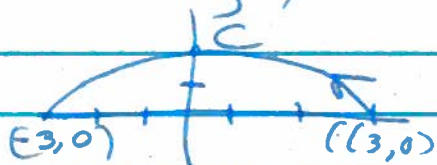
$$= 2e^{x^2} \cos z + 4x^2 e^{x^2} \cos z - z^4 \sin y - e^{x^2} \cos z + 12z^2 \sin y$$

(c) $\text{div } f$ not possible since f is scalar, not vector

(d) $\text{curl}(\text{grad } f) = \vec{\nabla} \times \vec{\nabla} f = 0$ by identity proved in class

(e) $\text{grad}(\text{grad } f)$
not possible since $\text{grad } f$ is a vector and we cannot apply grad to a vector field, only to scalar functions

#6 (a) $\int_C -y dx + x dy$ $P = -y$ $Q = x$ $\frac{\partial Q}{\partial x} = 1 \neq -1 = \frac{\partial P}{\partial y}$



$$x = 3 \cos t$$

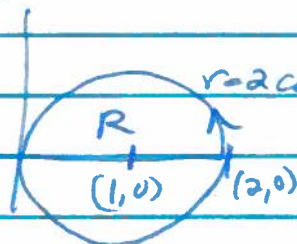
$$y = 2 \sin t$$

$$0 \leq t \leq \pi$$

$$\int_0^\pi -2 \sin t (-3 \sin t dt) + 3 \cos t (2 \cos t dt)$$

$$= \int_0^\pi 6 (\sin^2 t + \cos^2 t) dt = 6t \Big|_0^\pi = \boxed{6\pi}$$

(b)



Green's Thm

$$\oint_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\oint_C -y dx + x dy = \iint_R 1 - (-1) dA$$

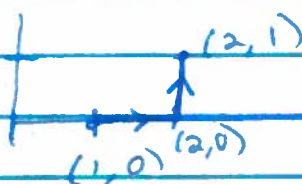
$$= 2 \iint_R 1 dA = \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} r dr d\theta$$

$$= 2(\text{area of } R) = 2(\pi \cdot 1^2) = \boxed{2\pi}$$

#7. $P = 1 + e^{-y}$, $Q = -(xe^{-y} + 4y)$

$$\frac{\partial Q}{\partial x} = -e^{-y} = \frac{\partial P}{\partial y} \quad \text{and } P, Q \text{ smooth in the whole plane} \Rightarrow \text{indep. of path}$$

$$\int_{(1,0)}^{(2,1)} P dx + Q dy = \int_1^2 (1 + e^{-y}) dx + \int_0^1 -(xe^{-y} + 4y) dy$$



$$= 2 \int_1^2 dx - \int_0^1 (2e^{-y} + 4y) dy$$

$$= 2x \Big|_1^2 - (-2e^{-y} + 2y^2) \Big|_0^1 = 2 - (-2e^{-1} + 2)$$

$$= \boxed{2 + 2e^{-1}}$$

(using potential: $\phi(x,y) = \int (1 + e^{-y}) dx = x + xe^{-y} + g(y)$)

$$\frac{\partial \phi}{\partial y} = -xe^{-y} + g'(y) = Q = -xe^{-y} - 4y \Rightarrow g'(y) = -4y$$

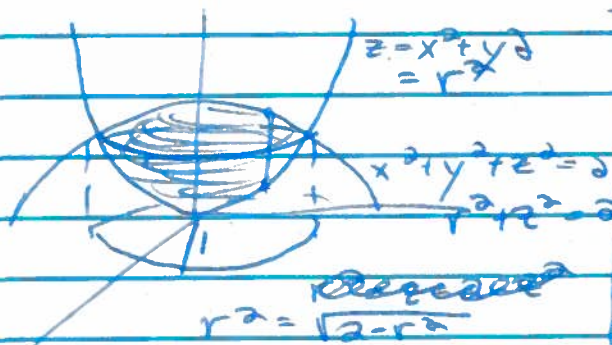
$$\Rightarrow g(y) = -2y^2 + C$$

$$\phi(x,y) = x + xe^{-y} - 2y^2 + C$$

$$\int_C P dx + Q dy = \phi(x,y) \Big|_{(1,0)}^{(2,1)} = \phi(2,1) - \phi(1,0) = 2 + 2e^{-1} - 2 - (1 + 1e^0 - 0)$$

$$= \boxed{2e^{-1} - 2} + (1 + 1e^0 - 0)$$

#8. Divergence Thm: $\oint \vec{F} \cdot \vec{n} ds = \iiint_D (\nabla \cdot \vec{F}) dv$



$$r^2 = \frac{-1 \pm \sqrt{1+8}}{2}$$

$$r^4 = 2 - r^2$$

$$r^4 + r^2 - 2 = 0$$

$$r^2 = \frac{-1 \pm \sqrt{1+8}}{2}$$

$$\boxed{r=1}$$

$$= \frac{-1 \pm 3}{2} = -2 \text{ or } 1$$

$$= \iiint_D \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \right) dv$$

$$= \iiint_D 3 dz dr d\theta$$

$$= 3 \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{2-r^2}} (2 - r^2 - r^2) r dr d\theta$$

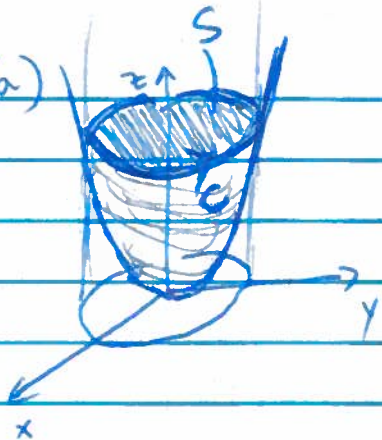
$$= 3 \int_0^{2\pi} \left[\frac{2}{3} (2 - r^2)^{3/2} - \frac{r^4}{4} \right]_0^1 d\theta$$

$$= 3 \left(-\frac{1}{3} - \frac{1}{4} + \frac{1}{3} 2^{3/2} \right) 2\pi = \left[6\pi \left(\frac{\sqrt{8}-1}{3} - \frac{1}{4} \right) \right]$$

Stokes Theorem: $\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n} dS$

(5)

#10 (a)



$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

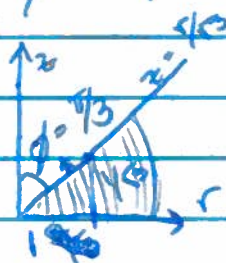
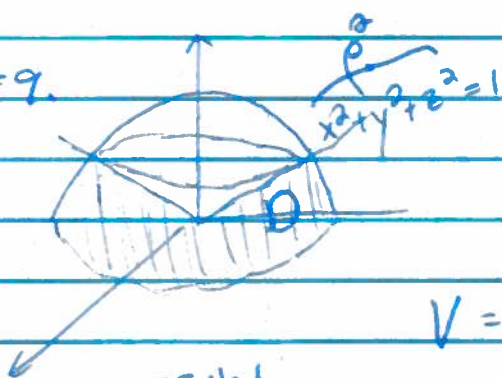
$$= \left(\frac{\partial(xy)}{\partial y} - \frac{\partial(xz)}{\partial z} \right) \hat{i} - \left(\frac{\partial(xy)}{\partial x} - \frac{\partial(yz)}{\partial z} \right) \hat{j} + \left(\frac{\partial(xz)}{\partial x} - \frac{\partial(yz)}{\partial y} \right) \hat{k}$$

$$= (x-x)\hat{i} - (y-y)\hat{j} + (z-z)\hat{k} = \vec{0}$$

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n} dS = \boxed{0}$$

(b) The answer is the same ⁽¹⁰⁾ by independence of path.

#9.



$$3z^2 = x^2 + y^2$$

$$3z^2 = r^2$$

$$z = \frac{r}{\sqrt{3}}$$

$$\tan \phi = \frac{r}{z} = \sqrt{3} \quad \frac{1}{\sqrt{3}} \quad \frac{\pi/3}{\sqrt{3/2}}$$

$$\phi = \pi/3$$

$$V = \iiint_D 1 dV$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \int_0^{\sqrt{3} \tan \phi} \rho^2 \sin \phi d\rho d\phi d\theta$$

$$= \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \frac{\rho^3}{3} \bigg|_0^{\sqrt{3} \tan \phi} \sin \phi d\phi d\theta = \frac{1}{3} \left(\frac{\rho^3}{3} \right) \bigg|_{\pi/3}^{\pi/2} d\theta = \frac{1}{3} \left(-0 + \frac{1}{2} \right) 2\pi = \boxed{\frac{\pi}{3}}$$