Department of Mathematics and Statistics

Course	**Number**	**Section(s)**
Mathematics	202	All

Examination	**Date**	**Pages**
Final	December 2012	2

Instructors	**Course Examiner**
All	A. Kokotov

Special Instructions:

- Only approved calculators are allowed
- Justify and explain all your answers

1. [8] All the roots of the polynomial

$$x^4 - 16x^3 + 86x^2 - 176x + 105$$

are positive integers. Factorize this polynomial completely.

2. [8] Prove that all the real zeros of the polynomial

$$72x^4 + 42x^3 - 37x^2 - 13x + 6$$

lie between -2 and 1.

3. [7] Prove that the polynomial

$$x^5 + x^3 + x^2 - 1$$

has exactly one real positive zero.

4. [7] Find the sum

$$3 + 10 + 17 + 24 + \ldots + 7003 + 7010$$

5. [7] Using mathematical induction, prove that $3^n < n!$ if $n \geq 7$.

6. [8] Find all the horizontal and vertical asymptotes of

$$f(x) = \frac{x^2}{(x+2)(3-x)}$$

and sketch its graph.

7. [8] Express

$$z = \frac{(\sqrt{3} - i)^3}{(1-i)^6}(1 + \sqrt{3}\,i)^5$$

in the trigonometric form $z = \rho(\cos\phi + i\sin\phi)$.

8. [8] A polynomial

$$p(z) = 2z^5 + az^4 + bz^3 + cz^2 + dz + e$$

with real coefficients a, b, c, d, e has $2i$, $-i$ and 1 as roots. Find c.

9. [8] Find all the roots of the polynomial $z^5 - i = 0$

10. [8] Express the number 0.2012201220122012.... in the form $\frac{a}{b}$, where a and b are integers.

11. [8] Using binomial theorem, find the coefficient of the x^8-term of the polynomial

$$(x+1)^{11} - (x^2+1)^6$$

12. [7] How many hands of 5 cards with 2 red cards (i. e. hearts or diamonds) and 3 spades can be dealt from a 52-card bridge deck?

13. [8] How many five digit telephone numbers with all different digits?

MATH 202 Dec 2012 Final Exam

1. Step 1 Rational zeros that are positive have numerators as Factors of 105 : 1, 3, 5, 7, 15, 35, 105

den. " "

$\Rightarrow$ Fraction are with den = 1

Step 2

8 MARKS

$$\begin{array}{r|rrrrr} 1 & 1 & -16 & 86 & -176 & 105 \\ & & 1 & -15 & 71 & -105 \\ \hline & 1 & -15 & 71 & -105 & \boxed{0} \end{array}$$

$\Rightarrow x = 1$ is zero

$$\begin{array}{r|rrrr} 3 & 1 & -15 & 71 & -105 \\ & & 3 & -36 & 105 \\ \hline & 1 & -12 & 35 & \boxed{0} \end{array}$$

$\Rightarrow x = 3$ is a zero

$$\begin{array}{r|rrr} 5 & 1 & -12 & 35 \\ & & 5 & -35 \\ \hline & 1 & -7 & \boxed{0} \end{array}$$

$\Rightarrow x = 5$ is a zero

we now have $\dfrac{x^4-16x^3+86x^2-176x+105}{(x-1)(x-3)(x-5)} = (x-7)$

or $x^4-16x^3+86x^2-176x+105 = (x-1)(x-3)(x-5)(x-7)$

2.

$$\begin{array}{r|rrrrr} 1 & 72 & 42 & -37 & -13 & 6 \\ & & 72 & 114 & 77 & 64 \\ \hline & 72 & 114 & 77 & 64 & \boxed{70} \end{array}$$

$$\begin{array}{r|rrrrr} -2 & 72 & 42 & -37 & -13 & 6 \\ & & -114 & 114 & -154 & 334 \\ \hline & 72 & -72 & 77 & -167 & \boxed{+} \end{array}$$

8 MARKS

TEST Integers only (Integer bounds)

$x = 1$ is not a zero but it is an upper bound

$x = -2$ is not a zero but it is a lower bound

$\Rightarrow$ All zeros are between -2 and 1

---

3. $f(x) = \underbrace{x^5 + x^3}_{\text{No SC}} \underbrace{+ \, x^2}_{\text{No SC}} \underbrace{- \, 1}_{\text{1 SC.}}$

7 MARKS

$f(x)$ has one sign change

$\Rightarrow f(x)$ has one positive zero.

---

4. TEST for Arith $10-3=7$

$17-10=7$

$\vdots$

$\Rightarrow$ Arithmetic, $d = 7$

$a = 3$

7 MARKS

$S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{1002} = \frac{1002}{2}[2(3) + (1002-1)7]$

$S_{1002} = 3513513$

where $n = ?$

$a_n = a + (n-1)d$

$7010 = 3 + (n-1)7$

$7010 = 3 + 7n - 7$

$7010 = 7n - 4$

$7010 + 4 = 7n$

$n = 1002$

Math 202 Dec 2012

5. $P(n): 3^n < n! \quad \forall n \geq 7$

Step 1 $\quad P(7):$ $\quad LHS = 3^7 = 2187$, $\quad RHS = 7! = 5040$ $\Rightarrow LHS < RHS \Rightarrow P(7) = T$

Step 2 $\quad$ Assume $P(k)$ for any one $k \geq 7$: $\quad 3^k < n!$

Step 3 $\quad$ Prove $P(k) \to P(k+1)$; $\quad 3^k < k! \Rightarrow 3^{k+1} < (k+1)!$

Step 4 $\quad 3^k < k!$

$$\Rightarrow 3^k (k+1) < k!\,(k+1) = (k+1)!$$

Since $k \geq 7 \Rightarrow 3 < (k+1)$

7 marks

$$\Rightarrow 3^k \ast 3^k < 3^k (k+1) < (k+1)!$$

$$3^{k+1} < (k+1)!$$

$$\Rightarrow P(k+1)$$

Since $P(7) = T$, $P(k) \Rightarrow P(k+1)$ for $n \geq 7$

$\Rightarrow P(n)$ is true for all n.

---

6. $f(x) = \dfrac{x^2}{(x+2)(3-x)}$ $\qquad$ V.A: $(x+2)(3-x) = 0$

$x = -2 \mid x = 3$

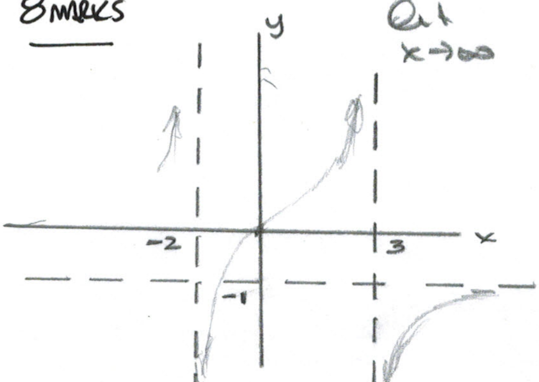
Since DEN $\neq 0$ for either one $\Rightarrow$ we have 2 V.A. $\begin{cases} x = -2 \\ x = 3 \end{cases}$

HA: $\lim\limits_{x \to \infty} \dfrac{x^2}{(x+2)(3-x)} = \dfrac{\infty}{\infty}$

8 marks

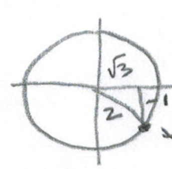
$$\lim_{x \to \infty} \frac{\frac{x^2}{x^2}}{\left(\frac{x}{x} + \frac{2}{x}\right)\left(\frac{3}{x} - \frac{x}{x}\right)} = \lim_{x \to \infty} \frac{1}{\left(1 + \frac{2}{x}\right)\left(\frac{3}{x} - 1\right)} = \frac{1}{(1)(-1)} = -1$$

$\Rightarrow$ HA: $y = -1$



---

7. $\dfrac{(\sqrt{3} - i)^3}{(1 - i)^6} (1 + \sqrt{3} i)^5$



$\theta = -30°$

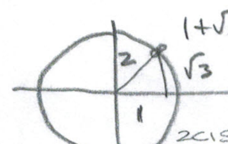
$r = \sqrt{\sqrt{3}^2 + (-1)^2}$

$r = 2$

$2 \operatorname{cis}(-30)$

$2 \operatorname{cis}(330°)$

$2 \operatorname{cis} \frac{11\pi}{6}$



$\theta = 60°$

$r = \sqrt{1^2 + \sqrt{3}^2}$

$r = 2$

$2 \operatorname{cis} 60°$

$2 \operatorname{cis} \frac{\pi}{3}$

$\theta = -45°$

$r = \sqrt{1^2 + (-1)^2}$

$r = \sqrt{2}$

$\sqrt{2} \operatorname{cis}(-45°)$

$\sqrt{2} \operatorname{cis} 315°$

$\sqrt{2} \operatorname{cis} \frac{7\pi}{4}$

$$= \frac{(2 \operatorname{cis} -30°)^3 (2 \operatorname{cis} 60°)^5}{[\sqrt{2} \operatorname{cis}(-45°)]^6}$$

8 marks

$$= \frac{\cancel{2^3} \operatorname{cis}(-90°)\, 2^5 \operatorname{cis} 300°}{\cancel{2^3} \operatorname{cis}(-270°)}$$

$$= 2^5 \operatorname{cis}(-90° + 300° + 270°)$$

$$= 2^5 \operatorname{cis} 480°$$

$$= 2^5 \operatorname{cis} 120°$$

$$\frac{\left(2 \operatorname{cis} \frac{11\pi}{6}\right)^3 \left(2 \operatorname{cis} \frac{\pi}{3}\right)^5}{\left(\sqrt{2} \operatorname{cis} \frac{7\pi}{4}\right)^6}$$

$$\frac{\cancel{2^3} \operatorname{cis} \frac{33\pi}{6}\, 2^5 \operatorname{cis} \frac{5\pi}{3}}{\cancel{2^3} \operatorname{cis} \frac{42\pi}{4}}$$

$$= 2^5 \operatorname{cis} -\frac{20\pi}{6}$$

$$= 2^5 \operatorname{cis} -\frac{10\pi}{3} = 2^5 \operatorname{cis}\left(-\frac{4\pi}{3}\right) = 2^5 \operatorname{cis} \frac{2\pi}{3}$$

8. $2i$ a zero $\Rightarrow (x-2i)$ is factor

$\Rightarrow -2i$ " " $\Rightarrow (x+2i)$ " "

$-i$ a zero $\Rightarrow (x+i)$ "

$\Rightarrow$ i " " $\Rightarrow (x-i)$ "

1 a zero $\Rightarrow (x-1)$ "

$$\Rightarrow p(x) = (x-2i)(x+2i)(x+i)(x-i)(x-1)$$
$$= (x^2+4)(x^2+1)(x-1)$$
$$= (x^4+x^2+4x^2+4)(x-1)$$
$$= (x-1)(x^4+5x^2+4)$$
$$= x^5+5x^3+4x-x^4-5x^2-4$$
$$p(x) = x^5 - x^4 + 5x^3 - 5x^2 + 4x - 4$$

8 marks

to get coeff of $x^5 = 2$ we need $f(x) = 2px$

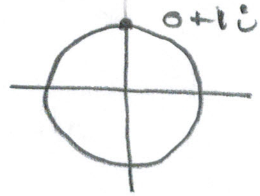
$$f(x) = 2x^5 - 2x^4 + 10x^3 - 10x^2 + 8x - 8$$

Since coefficient of x^2 is now $-10 \Rightarrow c = -10$

---

9. we need $z^5 = i$ or $z = (0+1i)^{\frac{1}{5}}$

Step 1 Change to polar



$0+1i$

$\theta = 90^\circ$

$r = \sqrt{0^2+1^2}$

$r = 1$

we want $(0+i)^{\frac{1}{5}}$

$= (1 \text{ cis } 90^\circ)^{\frac{1}{5}}$

8 marks

Step 2 there are 5 roots:

$1^{\frac{1}{5}} \text{cis } \frac{90^\circ}{5}$ $\quad 1 \text{ cis } \frac{90+360}{5}$ $\quad 1 \text{ cis } \frac{90+720}{5}$ $\quad 1 \text{ cis } \frac{90^\circ +}{5}$ $\quad 1 \text{ cis } \frac{90+}{5}$

$\text{cis } 18^\circ$ $\quad \text{cis } 90^\circ$ $\quad \text{cis } 162^\circ$ $\quad \text{cis } 234^\circ$ $\quad \text{cis } 306^\circ$

---

10. $.201220122012\ldots = .2012 + .00002012 + .0000\,0000\,2012 + \ldots$

$$= \frac{2012}{10000} + \frac{2012}{10000\,0000} + \ldots$$

$$= \frac{2012}{10000} + \frac{2012}{10000} * \frac{1}{10000} + \frac{2012}{10000} * \left(\frac{1}{10000}\right)^2 + \ldots$$

8 marks

$\Rightarrow$ Geom series $a = \frac{2012}{10000}$, $r = \frac{1}{10000}$

Since $-1 < r < 1 \Rightarrow \lim_{n\to\infty} S_n = \dfrac{\frac{2012}{10000}}{1 - \frac{1}{10000}}$

$$= \frac{2012}{10000} \cdot \frac{10000}{9999} = \frac{2012}{9999}$$

---

11. $(x+1)^{11} = \ldots + \frac{11!}{8!3!} x^8 (1)^3 + \ldots$

8 marks

$(x^2+1)^6 = \ldots + \frac{6!}{4!2!} (x^2)^4 (1)^2$

$$\Rightarrow \text{Coeff} \left[\frac{11!}{8!3!} - \frac{6!}{4!2!}\right] x^8 = (165-15)x^8 = 150x^8$$

---

12. Choose 2 Red cards: ${}_{26}C_2$, choose 3 spades ${}_{13}C_3 \Rightarrow$ total ${}_{26}C_2 * {}_{13}C_3$

7 marks

---

13. — — — — — ${}_{10}P_5$

8 marks