Assignment 1

Due: Thursday, February 2, 2017 (in class)

1. (10 marks) In your own words, briefly describe (a) the "traditional" approach to macroeconomics that was dominant up to the 1970s, (b) the "Lucas critique" of this approach, and (c) how the modern "micro-founded" method of macroeconomic modeling attempts to address that critique.

   The traditional approach to macro involved ad hoc behavioural relations, such as "consumption" and "investment" functions, which directly specified how the corresponding variables were chosen by economic agents as a function of one or more economic aggregates.

   Lucas argued that these functions were not "invariant" to policy changes. In particular, these functions were estimated from historical data (e.g., the historical relationship between income and consumption determined the estimated consumption function). In the face of a policy change, though, the relationships between different macro aggregates may very well change, so that the historical relationship is not a good indicator of what will happen; that is, the behavioural relationships are not "invariant" to policy changes. As a result, one cannot sensibly do policy analysis.

   The modern micro-founded approach attempts to solve this problem by building up a macro model from fundamentals that *are* invariant to policy changes. For example, we can specify that households maximize utility subject to a budget constraint, and can then easily deduce what happes if the budget constraint changes due to, say, a change in tax policy.

2. (30 marks total) In this exercise, we will derive the famous "envelope theorem". Suppose you wish to (unconditionally) maximize some objective function $f(x, y; \alpha)$, where x and y are two variables you can choose, while α is some variable that is given exogenously. Note that, even though we don't get to choose α, it may still affect the optimal choice of x. An example of a variable like this would be the wage in the household problem we discussed in class: w affects the household's choice of c and l, but the household doesn't get to *choose* w.

   (a) (4 marks) Find the FOC's for the optimal choices of x and y , denoted respectively by x^* and y^*.

   The conditions are $f_x(x^*, y^*; \alpha) = 0$ and $f_y(x^*, y^*; \alpha) = 0$.

   (b) (5 marks) Consider the maximized value of the objective function, which is given by a function $Q(\alpha) \equiv f(x^*, y^*; \alpha)$. Note that x^* and y^*here implicitly depend on α: if α changes, the optimal values of x and y will generally change in response. Totally differentiate $f(x^*, y^*; \alpha)$ with respect to α to get an expression for $Q'(\alpha)$ that depends on $dx^*/d\alpha$ and $dy^*/d\alpha$.

Totally differentiating with respect to α, we get

$$Q'(\alpha) = f_x(x^*, y^*; \alpha)\frac{dx^*}{d\alpha} + f_y(x^*, y^*; \alpha)\frac{dy^*}{d\alpha} + f_\alpha(x^*, y^*; \alpha) \quad (1)$$

(c) (6 marks) Substitute the FOC's you got in part (a) into the expression from part (b). The resulting expression is effectively a statement of the envelope theorem. Interpret that theorem in your own words.

Since $f_x(x^*, y^*; \alpha) = 0$ and $f_y(x^*, y^*; \alpha) = 0$ from part (a), we have that $Q'(\alpha) = f_\alpha(x^*, y^*; \alpha)$. That is, a small change in α affects the maximized value of the objective function in three ways: (i) it affects the choice of x, which affects f; (ii) it affects the choice of y, which affects f; and (iii) it directly affects f. These effects are captured respectively by the three terms on the RHS of (1). The envelope theorem states effectively that we can ignore (i) and (ii), so that only (iii) affects $Q(\alpha)$.

(d) (15 marks) Suppose instead we have a constrained maximization problem. That is, suppose we want to maximize $f(x, y; \alpha)$ subject to the equality constraint $g(x, y; \alpha) = 0$. Redo (a)-(c) under this assumption in order to obtain the envelope theorem for constrained maximization. (HINT: Remember that the maximization of f subject to a constraint is mathematically equivalent to the *unconditional* maximization of the Lagrangian).

Write the Lagrangian as $\mathcal{L}(x, y, \lambda; \alpha) \equiv f(x, y; \alpha) - \lambda g(x, y; \alpha)$. The FOC's are $\mathcal{L}_x(x^*, y^*, \lambda^*; \alpha) = 0$, $\mathcal{L}_y(x^*, y^*, \lambda^*; \alpha) = 0$ and $\mathcal{L}_\lambda(x^*, y^*, \lambda^*; \alpha) = 0$. Let $Q(\alpha) \equiv \mathcal{L}(x^*, y^*, \lambda^*; \alpha)$ be the maximized value of the Lagrangian. Totally differentiating with respect to α, we get

$$Q'(\alpha) = \mathcal{L}_x(x^*, y^*, \lambda^*; \alpha)\frac{dx^*}{d\alpha} + \mathcal{L}_y(x^*, y^*, \lambda^*; \alpha)\frac{dy^*}{d\alpha} + \mathcal{L}_y(x^*, y^*, \lambda^*; \alpha)\frac{d\lambda^*}{d\alpha} + \mathcal{L}_\alpha(x^*, y^*, \lambda^*; \alpha)$$

Substituting in the FOC's then yields simply

$$Q'(\alpha) = \mathcal{L}_\alpha(x^*, y^*, \lambda^*; \alpha) = f_\alpha(x^*, y^*; \alpha) - \lambda^* g_\alpha(x^*, y^*; \alpha)$$

which is the envelope theorem for the constrained maximization problem.

3. (60 marks total) In the model discussed in Lecture Note 1, we assumed that the government raised revenues through a lump-sum tax on households. In this question, we will explore what happens if the government can also use another type of tax. In particular, suppose that, in addition to the nominal lump-sum tax T (i.e., τ in real terms), the government also levies a proportional labour income tax on the household. That is, suppose the household must pay a fraction τ_w (with $0 \leq \tau_w \leq 1$) of its wage earnings to the government.

NOTE: Assume throughout this question that none of the NNC's ever bind.

(a) (5 marks) Write down the nominal (i.e., expressed in units of money) and real (i.e., expressed in units of goods) household budget constraints. As we did in class, use the time constraint $N^s + l = 1$ to substitute out any value of the labour supply N^s that might appear in your expression. As we also did in class, write these budget constraints as equalities (rather than inequalities).

The household's labour earnings are $WN^s = W(1-l)$, of which $\tau W(1-l)$ is paid in the form of taxes, leaving households with net labour earnings of $(1-\tau_w)W(1-l)$. Thus, the nominal budget constraint is $Pc = (1-\tau_w)W(1-l) + \Pi - T$, where Π is nominal firm profits and T is the nominal lump-sum tax. Dividing both sides by P gives the real budget constraint, $c = (1-\tau_w)w(1-l) + \pi - \tau$, where π and τ are real profits and lump-sum taxes, respectively.

(b) (6 marks) Set up the Lagrangian for the household's problem and use it to obtain the household FOC governing its consumption-leisure choice (i.e., analogous to equation (9) on p.11 of Lecture Note 1). Give an economic interpretation of the condition you found.

The Lagrangian is $\mathcal{L}(c,l,\lambda) = U(c,l) + \lambda[(1-\tau_w)w(1-l) + \pi - \tau - c]$. This yields the FOC's

$$U_c(c^*, l^*) = \lambda^*$$

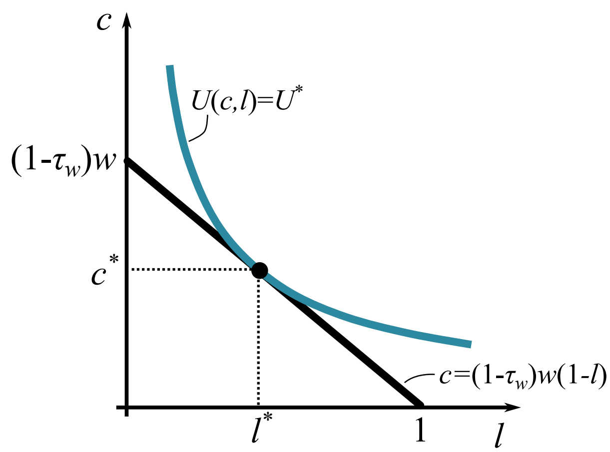
$$U_l(c^*, l^*) = (1-\tau_w)w\lambda^*$$

Combining these to eliminate λ, we get the desired condition,

$$(1-\tau_w)w = \frac{U_l(c^*, l^*)}{U_c(c^*, l^*)}$$

The RHS of this expression is the MRS of leisure for consumption, which tells us how much extra consumption the household would need to be given in order to compensate them for giving up a unit of leisure (i.e., for working one more unit of time). The LHS of the expression, meanwhile, is the actual amount of extra consumption the household gets for doing so: by working an additional unit, it gets a wage w, but it has to pay a fraction τ_w of it to the government, leaving it with $(1-\tau_w)w$. The FOC says these two things must be equal at an optimum.

(c) (6 marks) For the special case of $\pi = \tau$, draw a graph showing the determination of the household's optimal choice of consumption and leisure. Be sure to label carefully all components of the graph, including the axes, any curves or lines, and any intercept values.



At the optimum, the household's indifference curve (blue curve) is just tangent to the budget constraint (black line).

(d) (5 marks) Write down the government's (real) balanced-budget constraint that equates its real tax revenues with its real spending.

The government's real spending continues to be given by g. It now gets tax revenues from two sources. As before, it gets lump-sum revenues τ. It now also receives labour income tax revenues equal to $\tau_w w(1-l)$. Thus, the balanced-budget constraint is now $g = \tau + \tau_w w(1-l)$.

(e) (38 marks total) Suppose the government has decided that it will definitely spend $g = \bar{g}$, but it is interested in determining what the best way is to raise the necessary tax revenues: should it use only the lump-sum tax, only the labour income tax, or some combination of the two? We will work toward answering this question in steps. **NOTE: In what follows, assume that the wage w is fixed (i.e., it doesn't change at all in response to changes in tax policy).**

i. (3 marks) With g fixed at $\bar{g}$, for any given value of τ_w, there is an associated value of τ that makes the government's budget constraint hold (that is, given τ_w, the government gets a certain amount of revenue from labour income taxes; it must then raise the remainder from lump-sum taxes, τ, to cover its total expenditure $\bar{g}$). Write down an expression for this value as a function of τ_w, i.e., write it as $\tau = \phi(\tau_w)$, where ϕ is some function. (NOTE: The household's optimal choice of leisure, l^*, should appear in this expression. Note that l^* depends on τ_w; that is, l^* is implicitly a function of τ_w.)

Using the balanced-budget constraint from (d), the level of lump-sum taxes needed to balance the budget is $\tau = \phi(\tau_w) \equiv \bar{g} - \tau_w w(1-l^*)$.

ii. (4 marks) Differentiate the expression you found in (i) with respect to τ_w to obtain $\phi'(\tau_w)$. (NOTE: Since the household's optimal choice of leisure depends on τ_w,

$dl^*/d\tau_w$ will appear in this expression.)

We get

$$\phi'(\tau_w) = -w(1-l^*) + \tau_w w \frac{dl^*}{d\tau_w}$$

iii. (6 marks) Substitute the expression you found in (i) into the household's budget constraint to eliminate τ. Suppose τ_w changes (but τ also changes by the right amount to keep the government budget in balance, i.e., we continue to have $\tau = \phi(\tau_w)$). Can the household continue to afford the same bundle of c and l it was consuming previously? Is it possible that it could afford more of both c and l? What do your answers to these questions say about the *income effect* of such a change in tax policy?

Substituting $\tau = \bar{g} - \tau_w w(1-l^*)$ into the household budget constraint $c = (1-\tau_w)w(1-l) + \pi - \tau$, we get $c = w(1-l) + \pi - \bar{g}$. This expression does not depend on τ_w (or τ), and is thus independent of the precise way in which the government chooses to levy its taxes. In particular, if τ_w changes, the existing combination of c and l will continue to satisfy the budget constraint (with equality). Thus, it can afford the same bundle, and it's not possible for it to be able to afford more of both goods. In other words, there is no income effect coming from this change.

iv. (5 marks) Given your answer to (iii), and the fact that an increase in τ_w effectively lowers the relative cost of leisure (by lowering the effective wage the household receives), argue that l^* will always increase in response to an increase in τ_w (assuming again that the lump-sum tax continues to be given by $\tau = \phi(\tau_w)$, so that it changes as well).

Since such an increase τ_w has no income effect, it will have a pure substitution effect. Since leisure has become relatively cheaper, this subsitution effect causes the household to want to increase its leisure, i.e., l^* must increase.

v. (6 marks) Let $Q(\tau_w, \tau)$ be the maximized value of the household's Lagrangian when the tax policy is given by τ_w and τ (note that we are not assuming $\tau = \phi(\tau_w)$ here). That is, Q is the value of the Lagrangian at the optimal choices of consumption, leisure, and the Lagrange multiplier. Find expressions for the partial derivatives $Q_{\tau_w}(\tau_w, \tau)$ and $Q_\tau(\tau_w, \tau)$. (HINT: Use the envelope theorem you found in question 2(d)).

Writing the Lagrangian as $\mathcal{L}(c, l, \lambda, \tau_w, \tau) \equiv U(c, l) + \lambda[(1-\tau_w)w(1-l) + \pi - \tau - c]$, the envelope theorem implies that $Q_{\tau_w} = \mathcal{L}_{\tau_w} = -\lambda^* w(1-l^*)$ and $Q_\tau = \mathcal{L}_\tau = -\lambda^*$.

vi. (8 marks) Let $R(\tau_w) = Q(\tau_w, \phi(\tau_w))$ denote the maximized value of the household's Lagrangian when the labour income tax is τ_w and the lump-sum tax is set to balance the government budget (i.e., when $\tau = \phi(\tau_w)$). Find an expression for $R'(\tau_w)$ as a function of $dl^*/d\tau_w$. (HINT: Differentiate $R(\tau_w) = Q(\tau_w, \phi(\tau_w))$ with respect to τ_w, then substitute in your answers from (v) and (ii)).

Differentiating $R(\tau_w) = Q(\tau_w, \phi(\tau_w))$, we get

$$R'(\tau_w) = Q_{\tau_w}(\tau_w, \phi(\tau_w)) + Q_\tau(\tau_w, \phi(\tau_w))\,\phi'(\tau_w)$$

Substituting in Q_{τ_w} and Q_τ from (v) and ϕ' from (ii), we get

$$\begin{aligned} R'(\tau_w) &= -\lambda^* w(1-l^*) - \lambda^*\left[-w(1-l^*) + \tau_w w \frac{dl^*}{d\tau_w}\right] \\ &= -\lambda^* \tau_w w \frac{dl^*}{d\tau_w} \end{aligned}$$

vii. (6 marks) Using your answer to (vi), argue that household welfare always decreases when τ_w increases, and therefore the government should optimally set $\tau_w = 0$ and $\tau = \bar{g}$. (HINT: You will need to use your reasoning from (iv).)

Note from part (b) that $\lambda^* = U_c > 0$. Further, from (iv), we know that $dl^*/d\tau_w > 0$, i.e., an increase in τ_w (with a corresponding decrease in τ to keep the government budget balanced) always leads to an increase in leisure. Thus, we see that $R'(\tau_w) < 0$, i.e., the maximized value of the Lagrangian falls when τ_w increases. Further,

$$\begin{aligned} R(\tau_w) &= Q(\tau_w, \phi(\tau_w)) \\ &= \mathcal{L}(c^*, l^*, \lambda^*, \tau_w, \phi(\tau_w)) \\ &= U(c^*, l^*) + \lambda^*\left[(1-\tau_w)w(1-l^*) + \pi - \phi(\tau_w) - c^*\right] \end{aligned}$$

Since the household budget constraint is necessarily satisfied with equality at an optimum, the term in square brackets on the last line is equal to zero, and thus $R(\tau_w) = U(c^*, l^*)$. That is, $R(\tau_w)$ gives the value of household welfare when the labour income tax is τ_w and the lump-sum tax is $\tau = \phi(\tau_w)$. Thus, a rise in τ_w leads to a fall in household welfare, and therefore if $\tau_w > 0$ the household can always be made better if the government lowers τ_w. As a result, the optimal choice of τ_w is $\tau_w = 0$, in which case $\tau = \phi(0) = \bar{g}$.