COMP 2804 — Assignment 1

Due: Wednesday February 1, before 4:30pm, in the course drop box in Herzberg 3115.

Assignment Policy: Late assignments will not be accepted. Students are encouraged to collaborate on assignments, but at the level of discussion only. When writing the solutions, they should do so in their own words. Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.

Important note: When writing your solutions, you must follow the guidelines below.

- The answers should be concise, clear and neat.
- When presenting proofs, every step should be justified.
- Assignments should be stapled or placed in an unsealed envelope.

Substantial departures from the above guidelines will not be graded.

Question 1: On the first page of your assignment, write your name and student number.

Question 2: A password is a string of ten characters, where each character is a lowercase letter, a digit, or one of the eight special characters !, @, #, $, %, &, (, and).

A password is called *awesome*, if it contains at least one digit or at least one special character. Determine the number of awesome passwords.

Question 3: Determine the number of functions

$$f : \{1, 2, 3, 4\} \rightarrow \{a, b, c, \dots, z\},$$

such that $f(1) = f(2)$, or $f(3) = f(4)$, or $f(1) \neq f(3)$.

Question 4: Let $n \geq 1$ be an integer.

- Assume that n is odd. Determine the number of bitstrings of length n that contain more 0's than 1's. Justify your answer in plain English and at most three sentences.

  Hint: Symmetry.

- Assume that n is even.
  - Determine the number of bitstrings of length n in which the number of 0's is equal to the number of 1's.
  - Determine the number of bitstrings of length n that contain strictly more 0's than 1's.

– Argue that the binomial coefficient

$$\binom{n}{n/2}$$

is an even integer.

Question 5: Let m, n, k, and ℓ be integers such that $m \geq 1$, $n \geq 1$, and $1 \leq \ell \leq k \leq n$.

After a week of hard work, Elisa Kazan[1] goes to her neighborhood pub. This pub has m different types of beer and n different types of cider on tap. Elisa decides to order k pints: At most one pint of each type, and exactly ℓ pints of cider. Determine the number of ways in which Elisa can order these k pints. The order in which Elisa orders matters.

Question 6: Nick is not only your friendly TA, he also has a part-time job in a grocery store. This store sells n different types of India Pale Ale (IPA) and n different types of wheat beer, where $n \geq 2$ is an integer. Prove that

$$\binom{2n}{2} = 2\binom{n}{2} + n^2,$$

by counting, in two different ways, the number of ways to choose two different types of beer.

Question 7: A string consisting of characters is called *cool*, if exactly one character in the string is equal to x and each other character is a digit. Let $n \geq 1$ be an integer.

- Determine the number of cool strings of length n.
- Let k be an integer with $1 \leq k \leq n$. Determine the number of cool strings of length n that contain exactly $n - k$ many 0's.
- Use the above two results to prove that

$$\sum_{k=1}^{n} k\binom{n}{k}9^{k-1} = n \cdot 10^{n-1}.$$

Question 8: Determine the number of elements x in the set $\{1, 2, 3, \ldots, 99999\}$ for which the sum of the digits in the decimal representation of x is equal to 8. An example of such an element x is 3041.

You may use any result that was proven in class.

Question 9: Let $n \geq 2$ be an integer and let $G = (V, E)$ be a graph whose vertex set V has size n and whose edge set E is non-empty. The degree of any vertex u is defined to be the number of edges in E that contain u as a vertex.

[1]President of the Carleton Computer Science Society

- Prove that there exist at least two vertices in G that have the same degree.

  Hint: Consider the cases when G is connected and G is not connected separately. In each case, apply the Pigeonhole Principle. Alternatively, consider a vertex of maximum degree together with its adjacent vertices and, again, apply the Pigeonhole Principle.