COMP 335: Theoretical Computer Science
Fall 2013 Assignment 2

Solutions to Assignment 2

1. (15 marks) Give regular expressions for the following languages on $\Sigma = \{a, b\}$.

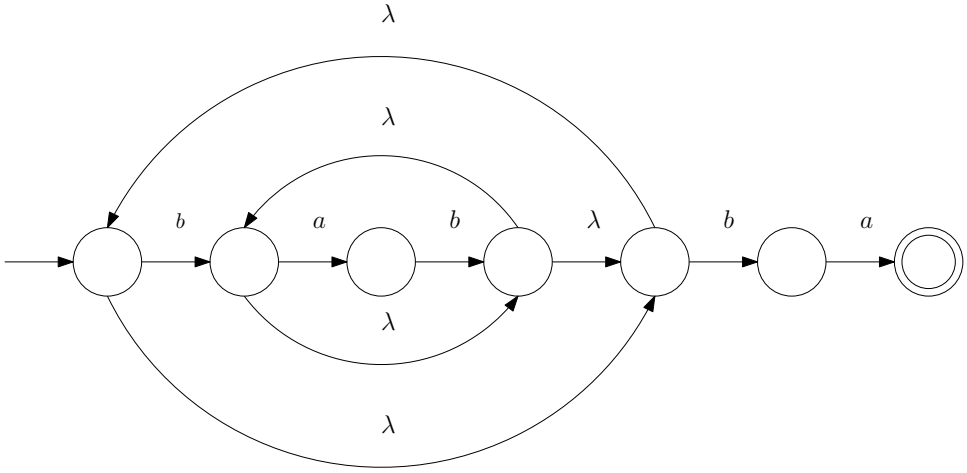
   (a) $\{w \in \{a,b\}^\star : w$ has neither the substring aa nor the substring $bb\}$.
   Soln. $a(ba)^\star(b+\lambda) + b(ab)^\star(a+\lambda)$

   (b) $\{w : (n_a(w) - n_b(w)) \bmod 3 \neq 0\}$
   Soln. The solution is not straightforward to obtain directly, while it is easy to find a DFA for the language, and then convert to a regular expression. We obtain:
   $((aa(ba)^*(a+bb)+ab)+b(ba)*(a+bb))^* + aab + aa + a + bb + b$

   (c) all strings which contain no runs of a's of length greater than two (a run is a substring that is as long as possible while consisting of the same symbol from Σ).
   Soln. $(b + ab + aab)^\star(a + aa + \lambda)$

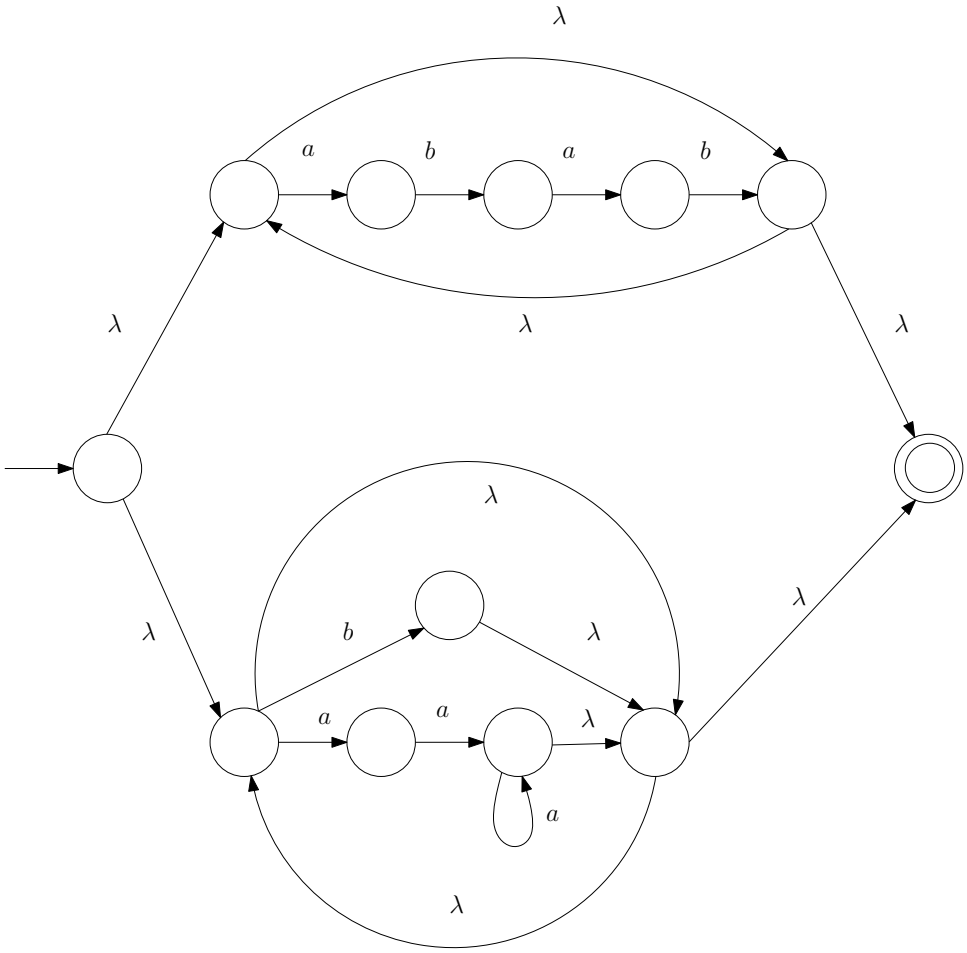
2. (10 marks) Use the construction to convert regular expressions to NFAs to draw NFAs that accepts the following languages:

   (a) $(b(ab)^* + bab)^*ba$
   Soln. Observe that $bab \in L(b(ab)^*)$, therefore the given regular expression can be simplified as $(b(ab)^*)^*ba$. The corresponding NFA is given below.



(b) $((abab)^* + (aaa^* + b)^*)$

Soln.



3. (10 marks) Consider the right linear grammar G below:

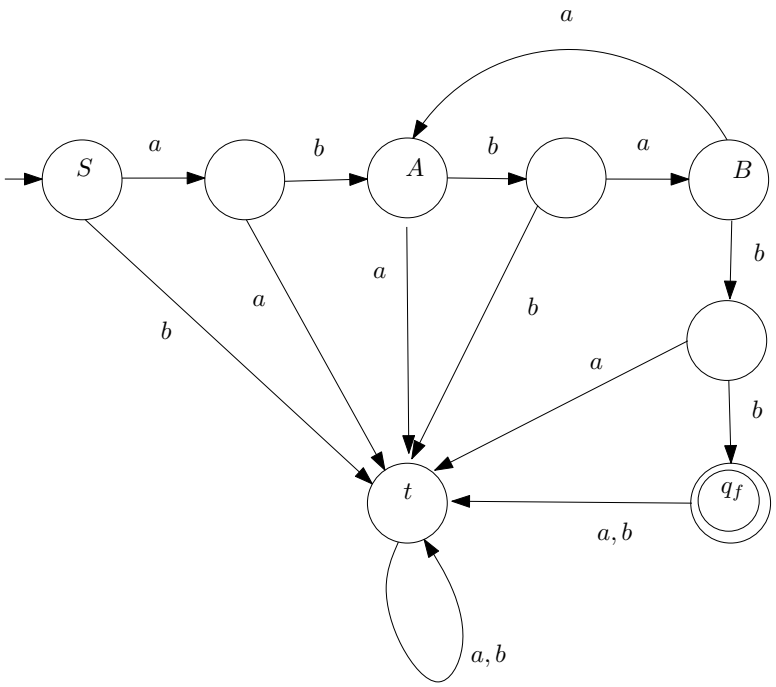
   $S \longrightarrow abA$

   $A \longrightarrow baB$

   $B \longrightarrow aA \mid bb$

   (a) Construct a dfa M that accepts the language generated by the grammar.

   Soln.



(b) Construct a left-linear grammar for the same language.

Soln. To obtain a left-linear grammar, we can construct the FA M' that accepts $reverse(L(M))$ given in (a) above, simply by reversing the direction of the arrows, and by changing the status of the initial and final states. Next, we derive a right-linear grammar G_1 from M' so that $L(G_1) = L(M') = reverse(L(G))$ (where G was the right-linear grammar given in the question):

$S \to bE$
$E \to bD$
$D \to aC$
$C \to bB$
$B \to aD \mid bA$
$A \to aF$
$F \to \lambda$

This can easily be simplified to G_2 where $L(G_2) = L(G_1)$:

$S \to bbD$
$D \to abB$
$B \to aD \mid ba$

Next we convert G_2 to a left-linear grammar G_3 simply by reversing the right-hand sides of all productions so that $L(G_3) = reverse(L(G_2)) = reverse(reverse(L(G))) = L(G)$ as required:

$S \to Dbb$
$D \to Bba$
$B \to Da \mid ab$

4. (10 marks) Given a language L, we define *Suffix(L)* $= \{v \mid \exists w \in L, w = uv\}$. That is, *Suffix(L)* is the language containing all suffixes of strings in L. Given a DFA $M = (Q, \Sigma, q_0, F, \delta)$ such that $L(M) = L$, give a procedure to find a DFA M' such that $L(M') =$ *Suffix(L)*.

Soln. Let $M = (Q, \Sigma, q_0, F, \delta)$ be a DFA such that $L(M) = L$. Assume without loss of generality that all states in Q are reachable from q_0, if not, all unreachable states and any

transitions to and from them can be removed without affecting the language accepted by the DFA. We build an NFA $M' = (Q \cup \{q_0'\}, \Sigma, q_0', F, \delta')$ where $\delta'(q,a) = \delta(q,a)$ for all $q \in Q, a \in \Sigma$, and $q \in \delta'(q_0', \lambda)$ for all $q \in Q$. That is, M' adds a new initial state q' and has λ-transitions to all states in Q, otherwise, the transitions are exactly the same as in M.

We claim that $L(M') = Suffix(L)$. We first show that $w \in Suffix(L)$ implies that $w \in L(M')$. Suppose $w \in Suffix(L)$. Then there is a string $u \in \Sigma^*$ such that $uw \in L$. Let $\delta^*(q_0, u) = q$. Then since $uw \in L$, we have $\delta^*(q, w) \in F$. But $q \in \delta'^*(q_0', \lambda)$ and since there is a path from q to a final state in M with label w, the same holds true in M', which implies that $\delta'^*(q_0', w) \cap F \neq \emptyset$, that is $w \in L(M')$. Next we show that $w \in L(M')$ implies that $w \in Suffix(L)$. Suppose $w \in L(M')$. There there is a path in M' labelled w from q_0' to a final state $q_f \in F$. Let the state immediately after q_0' in this path be q. From the construction, it is clear that $\delta(q, w) = q_f$. Since q is reachable from q_0 by assumption, there exists a string u such that $\delta^*(q_0, u) = q$. Then $\delta^*(q_0, uw) = q_f$, which implies that $w \in Suffix(L)$. This proves the claim.

Finally, M' can be converted using the subset construction to a DFA M'' such that $L(M'') = L(M') = Suffix(L)$,.

5. (10 marks) Prove or disprove the following statements.

   (a) If L_1 and L_2 are non-regular languages, then $L_1 \cup L_2$ is also non-regular.
   Soln. False. Counter-example: $L_1 = \{a^n b^n \mid n \geq 1\}$. and $L_2 = \overline{L_1}$. Then L_1 and L_2 are both non-regular, but $L_1 \cup L_2 = (a+b)^\star$ which is regular.

   (b) If L_1 is a regular language and L_2 is a non-regular language, then $L_1 L_2$ is regular.
   Soln. False. Counter-example: Let $L_1 = a^\star$ and $L_2 = \{a^n b^n \mid n \geq 0\}$. Then L_1 is regular, and L_2 is non-regular but $L_1 L_2 = a^\star b^\star$ which is regular.

6. (20 marks) Which of the following languages is regular? Prove your answer.

   (a) $L = \{a^n b^l : n \leq l \leq 2n\}$.
   Soln. Non-regular. Suppose instead that L is regular and let m be the constant of the pumping lemma. Choose $w = a^m b^{2m}$. Clearly $w \in L$ and $|w| = 3m > m$. Let $w = xyz$ with $|xy| \leq m$ and $|y| \geq 1$. Then $y = a^i$ with $1 \leq i \leq m$. Consider $xz = a^{m-i} b^{2m}$. Clearly $xz \notin L$ since $2m > 2(m-1) \geq 2(m-i)$. Therefore L is not regular.

   (b) $L = \{a^n b^l : n \geq 100 \text{ and } l \leq 100\}$
   Soln. Regular. L can be expressed as the regular expression $a^{100} a^\star (\lambda + b + b^2 + \cdots + b^{100})$.

   (c) $L = \{a^n b^l : n/l \text{ is an integer }\}$.
   Soln. Non-regular. Suppose instead that L is regular and let m be the constant of the pumping lemma. Choose $w = a^{(m+1)!} b^{m+1}$. Clearly $w \in L$ and $|w| \geq m$. Let $w = xyz$ with $|xy| \leq m$ and $|y| \geq 1$. Then $y = a^i$ with $1 \leq i \leq m$. Consider $xz = a^{(m+1)!-i} b^{m+1}$. Observe that while $m+1$ divides $(m+1)!$, it does not divide i since $1 \leq i \leq m$. Therefore $m+1$ does not divide $(m+1)! - i$, which implies $xz \notin L$. Therefore L is not regular.

   (d) $L = \{ww : w \in (a+b)^\star\}$.
   Soln. Non-regular. Suppose instead that L is regular and let m be the constant of the pumping lemma. Choose $w = a^m b^m a^m b^m$. Clearly $w \in L$ and $|w| \geq m$. Let $w = xyz$ with $|xy| \leq m$ and $|y| \geq 1$. Then $y = a^i$ with $1 \leq i \leq m$. Consider $xz = a^{m-i} b^m a^m b^m$. If xz is of odd length, then clearly $xz \notin L$. If instead xz is of even length, then i must be even and $xz = w_1 w_2$ where $w_1 = a^{m-i} b^m a^{i/2}$ while $w_2 = a^{m-i/2} b^m$. Clearly, $|w_1| = |w_2| = 2m - i/2 = |w|/2$, so $xz \notin L$. Therefore L is not regular.