# Assignment 1

1. A shipping firm sells $q$ cases of wine from Chile to consumers in Canada. They receive a price according to the demand function $p_s = a_1 - \frac{1}{3}q$, while paying $p_b = a_2 + \frac{1}{6}q$ *per unit* for the wine. Further, transportation costs them $\gamma$ per case shipped. Assume that $a_1, a_2$ and $\gamma$ are positive.

   (a) Express the firm's profit as a function of cases sold.

   (b) Assuming that $a_1 - a_2 - \gamma > 0$, find the number of cases sold that maximize profit. What happens if $a_1 - a_2 - \gamma \leq 0$?

   (c) Suppose the Chilean government imposes a tax on wine exports of $\tau$ per case. Rewrite firm's profit and solve for optimal $q$ in this case.

   (d) Imagine the Chilean government only cares about tax revenue. What is the optimal $\tau$?

   **Solution** This question is designed to get you used to setting up a simple economic problem mathematically.

   (a) Profits are: $\pi(q) = \text{(revenue-costs)} = p_s q - (p_b q + \gamma q) = a_1 q - \frac{1}{3}q^2 - a_2 q - \frac{1}{6}q^2 - \gamma q$

   (b) Optimal $q^*$ satisfies $\pi'(q^*) = 0$. So $q^* = a_1 - a_2 - \gamma$. Of course if this is negative, then optimal output is negative which makes no sense. In this case the firm would not operate.

   (c) With the tax, profits are $\pi(q, \tau) = p_s q - ([p_b + \tau]q + \gamma q) = a_1 q - \frac{1}{3}q^2 - a_2 q - \frac{1}{6}q^2 - \gamma q - \tau q$. Optimal $q^*$ satisfies $\pi'(q^*) = 0$. So $q^*(\tau) = a_1 - a_2 - \gamma - \tau$.

   (d) Government tax revenue is simply $R(\tau) = q^*(\tau) \times \tau = a_1\tau - a_2\tau - \gamma\tau - \tau^2$. Revenue is maximized at $R'(\tau^*) = 0$, so that $\tau^* = \frac{a_1 - a_2 - \gamma}{2}$. Question: How do we know that all these optimal points we've found are actually maxima?

2. A consumer has $\$m$ and a utility function defined over two goods $u(c_1, c_2)$. Prices of goods 1 and 2 are $p_1$ and $p_2$ respectively.

   (a) Describe the consumer's budget constraint in terms of the inner product of two vectors.

   (b) Let $u(c_1, c_2) = \alpha_1 c_1^{\frac{1}{\alpha_1}} + \alpha_2 c_2^{\frac{1}{\alpha_2}}$. Describe the utility function as a set.

   **Solution**

   (a) Consumption is the vector $c = (c_1, c_2)$ and prices are $p = (p_1, p_2)$. The budget constraint is $m = p \cdot c = p_1 c_1 + p_2 c_2$, assuming they always use up their entire budget.

   (b) The relation is characterized by the set $F = \{(u, c_1, c_2) : u = \alpha_1 c_1^{\frac{1}{\alpha_1}} + \alpha_2 c_2^{\frac{1}{\alpha_2}}\}$, which is a subset of $\mathbb{R}^3$ when utility and consumption are real numbers and the utility function maps $\mathbb{R}^2 \to \mathbb{R}$.

3. Consider the market for cheese. Denote the quantity traded $q$ and the price $p$. Supply of cheese is determined by optimal firm behaviour and characterized by $p_s = q$. Consumer demand is $p_d = 16 - 9q + q^2$.

   (a) Solve for the potential equilibria in this market. Illustrate with a diagram.

   (b) Are all solutions found in part (a) "reasonable"? Discuss briefly.

## Solution

(a) An equilibrium occurs when there is no excess demand in the market (i.e supply=demand). In this case that means $q = 16 - 9q + q^2$, which is satisfied when $q = 2, 8$. Thus we have two potential equilibria.

(b) The first case, $q = 2$, occurs on a part of the demand curve which is downward sloping. The second case occurs at a point where the demand curve is upward sloping. This is strange and does not arise from standard consumer maximizing behaviour. We generally would disregard such an equilibria since upward sloping demand violates a number of standard economic assumptions. Draw the supply and demand functions and see what the equilibria look like.

4. Solve the following:

   (a)

$$\begin{bmatrix} 27 & 44 & 51 \\ 35 & 39 & 62 \\ 33 & 50 & 47 \end{bmatrix} + \begin{bmatrix} 25 & 42 & 48 \\ 33 & 40 & 66 \\ 35 & 48 & 50 \end{bmatrix}$$

   (b) Solve $AB$ and $BA$, where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

   (c) Compute $(A+B)^T$, for $A$ and $B$ below:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} B = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$$

   Check that $(A+B)^T = A^T + B^T$.

## Solution

(a)

$$\begin{bmatrix} 27 & 44 & 51 \\ 35 & 39 & 62 \\ 33 & 50 & 47 \end{bmatrix} + \begin{bmatrix} 25 & 42 & 48 \\ 33 & 40 & 66 \\ 35 & 48 & 50 \end{bmatrix} = \begin{bmatrix} 52 & 86 & 99 \\ 68 & 79 & 128 \\ 68 & 98 & 97 \end{bmatrix}$$

(b)

$$AB = \begin{bmatrix} 3 & 0 \\ 3 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

Note that both products are defined only when they are square matrices. However, even when defined, $AB \neq BA$.

(c) Regardless of the method,

$$(A+B)^T = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$$

5. Suppose a firm produces three types of output, using two types of input. Its output quantities are given by the column vector

$$q = \begin{bmatrix} 15,000 \\ 27,000 \\ 13,000 \end{bmatrix}$$

and the unit prices of these are given by the row vector

$$p = \begin{bmatrix} 10 & 12 & 5 \end{bmatrix}.$$

The amounts of inputs it uses are given by the column vector

$$z = \begin{bmatrix} 11,000 \\ 30,000 \end{bmatrix}$$

and the input prices by the row vector

$$w = \begin{bmatrix} 20 & 8 \end{bmatrix}.$$

Write an expression for and solve firm profits.

## Solution

(a) Firm profits are $\Pi = pq - wz = 79,000$.

6. The market for tea is described by the following supply and demand functions:

$$D_t = 100 - 5p_t + 3p_c$$
$$S_t = -10 + 2p_t$$

and the market for coffee by:

$$D_c = 120 - 8p_c + 2p_t$$
$$S_c = -20 + 5p_c,$$

where $p_t$ and $p_c$ are the prices of tea and coffee respectively. Set up the problem using matrix notation and solve for the equilibrium prices of tea and coffee.

**Solution** To solve, set supply equal to demand in each market, giving a system of two equations and two unknowns (prices): $Ap = b$ or

$$\begin{bmatrix} 7 & -3 \\ -2 & 13 \end{bmatrix} \begin{bmatrix} p_t \\ p_c \end{bmatrix} = \begin{bmatrix} 110 \\ 140 \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} p_t \\ p_c \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ -2 & 13 \end{bmatrix}^{-1} \begin{bmatrix} 110 \\ 140 \end{bmatrix} = \begin{bmatrix} 13/85 & 3/85 \\ 2/85 & 7/85 \end{bmatrix} \begin{bmatrix} 110 \\ 140 \end{bmatrix} = \begin{bmatrix} 21.76 \\ 14.12 \end{bmatrix}$$

7. Find $A^{-1}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

## Solution

$$A^{-1} = \begin{bmatrix} -3/2 & -2 & 5/2 \\ 1/2 & 1 & -1/2 \\ 1/2 & 0 & -1/2 \end{bmatrix}$$