COMP 361/5611 - Elementary Numerical Methods
Assignment 1 - Due Friday, January 29, 2016

(This assignment does not require any knowledge of the Lecture Notes.)

Problem 1. (25%) Compute the *Harmonic sum* , $\sum_{k=1}^{N} \frac{1}{k}$, using single precision (*i.e., float*), for $N = 10^n$, with $n = 1, 2, 3, \cdots, 8$, (or higher). Mathematically, *i.e.*, when using *exact* arithmetic, this sum is known to *diverge* as N tends to infinity, *i.e.*, the sum can be made arbitrarily large by taking N sufficiently large. Explain the observed behavior of the *numerically* computed sums. Similarly compute the sum $\sum_{k=1}^{N} \frac{1}{3^k}$, and report your findings. Mathematically this sum is known to *converge* as N tends to infinity.

Problem 2. (25%) For given $x^{(0)}$, say, $x^{(0)} = 1.0$, compute the sequence $x^{(1)}$, $x^{(2)}$, $x^{(3)}$, $\cdots$, $x^{(N)}$, up to a suitably large value of N, *e.g.*, $N = 20$, or higher where necessary, using the recurrence relation

$$x^{(k+1)} = f(x^{(k)}), \qquad k = 0, 1, 2, 3, \cdots,$$

where $f(x) = \frac{2x^3+5}{3x^2}$. Describe in a few words the observed behavior of the sequence. In particular, does the sequence approach a limiting value? If yes, then do you recognize what this limiting value is? Does the limiting value depend on $x^{(0)}$? Also compute the sequence with $f(x) = cx(1-x)$, for five cases, namely, $c = 0.95$, $c = 1.55$, $c = 2.0$, $c = 3.6$, and $c = 3.98$. For each of these cases choose the value of $x^{(0)}$ to lie between 0 and 1, for example, $x^{(0)} = 0.1$. In each case describe the observed behavior of the sequence.

Problem 3. (50%) Consider the approximate integration of a function $f(x)$ over the interval $[0, 1]$. Let M be a positive integer, and let $h = 1/M$, and $x_k = kh$, for $k = 0, 1, 2, 3, \cdots, M$. Thus $x_0 = 0$ and $x_M = 1$. Then

$$\int_0^1 f(x)dx \;\approx\; h\Big[f(x_0) + f(x_1) + f(x_2) + \;\; \cdots \;\; + f(x_{M-2}) + f(x_{M-1})\Big].$$

Another approximate integration formula is

$$\int_0^1 f(x)dx \;\approx\; \frac{h}{3}\Big[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\cdots+2f(x_{M-2})+4f(x_{M-1})+f(x_M)\Big].$$

This method, which requires M to be even, is known as *Simpson's Rule.*

Implement these two methods in programs, and use each of these to approximately integrate the function $f(x) = \sin(\pi x)$ over the interval $[0, 1]$, using successively the following values of M: $M = 2, 4, 8, 16, \cdots$. For each of these values of M print the error, *i.e.*, the absolute value of the difference between the approximation and the known exact value of the integral. Make sure to represent π to sufficient precision in your program!

Furthermore, for each of the two methods, determine the smallest value of M for which the error is less than 10^{-7}. How many function evaluations are required in each of these two cases?

Can you describe the observed behavior of the error? In particular, for each of the two methods, can you say approximately how the error depends on h ? More specifically, it is known that the errors will be approximately proportional to h^p, where p is an integer that depends on the method. Can you tell from the numerical results what p is for each of the two methods? Also can you explain what happens to the error when M gets "very large".