Statistics for Business and Economics

8th Edition

Chapter 7

Estimation: Single Population

Chapter Goals

After completing this chapter, you should be able to:

- Distinguish between a point estimate and a confidence interval estimate
- Construct and interpret a confidence interval estimate for a single population mean using both the Z and t distributions
- Form and interpret a confidence interval estimate for a single population proportion
- Create confidence interval estimates for the variance of a normal population
- Determine the required sample size to estimate a mean or proportion within a specified margin of error

Confidence Intervals

Contents of this chapter:

- Confidence Intervals for the Population Mean, μ
  - when Population Variance σ^2 is Known
  - when Population Variance σ^2 is Unknown
- Confidence Intervals for the Population Proportion, P (large samples)
- Confidence interval estimates for the variance of a normal population
- Finite population corrections
- Sample-size determination

7.1

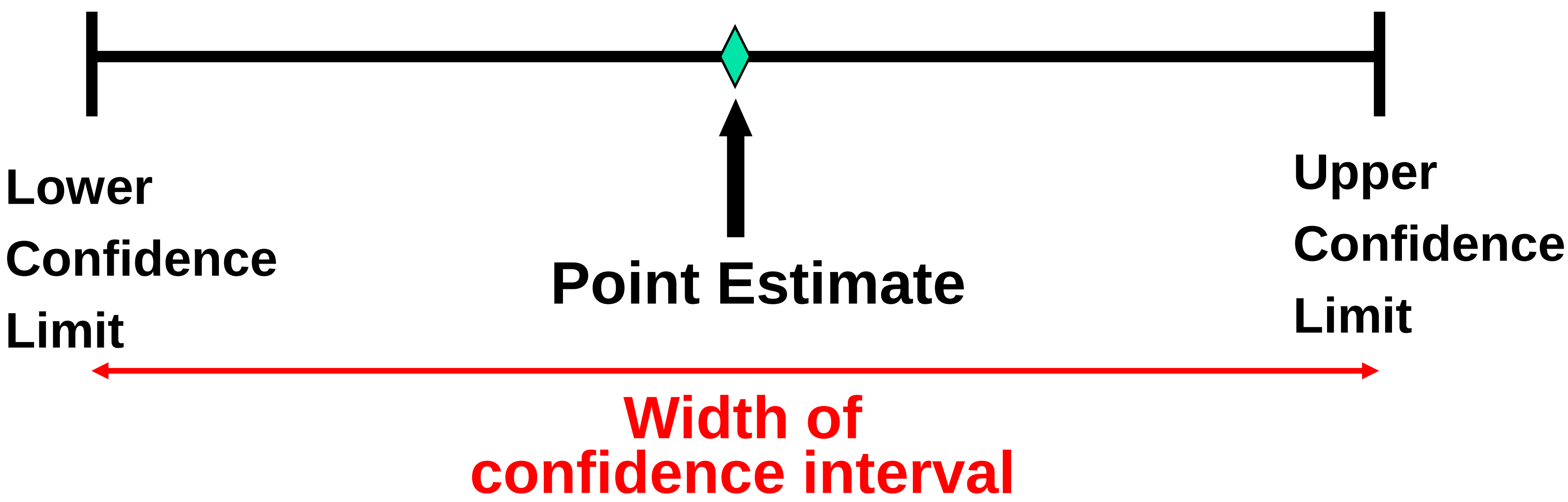
Properties of Point Estimators

- An estimator of a population parameter is
  - a random variable that depends on sample information . . .
  - whose value provides an approximation to this unknown parameter

- A specific value of that random variable is called an estimate

Point and Interval Estimates

- A point estimate is a single number,
- a confidence interval provides additional information about variability



Point Estimates

We can estimate a Population Parameter …		with a Sample Statistic (a Point Estimate)
Mean	μ	$\bar{x}$
Proportion	P	$\hat{p}$

Unbiasedness

- A point estimator $\hat{\theta}$ is said to be an unbiased estimator of the parameter θ if its expected value is equal to that parameter:

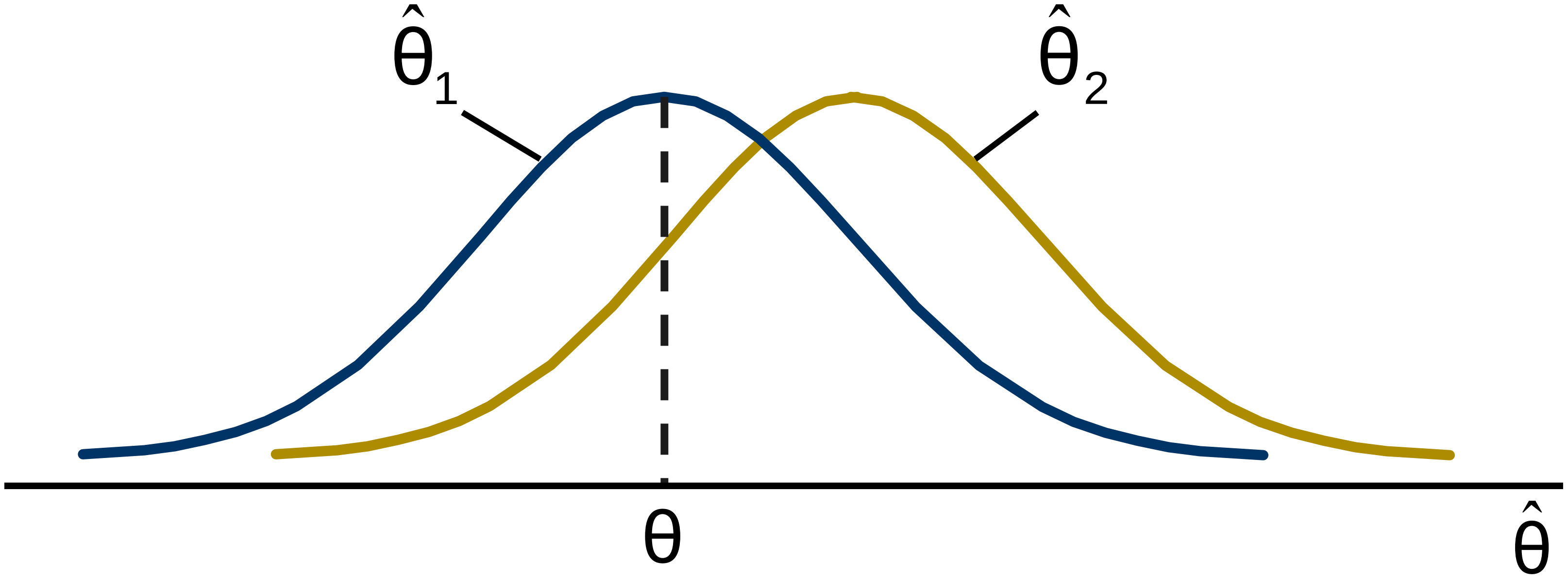
$$E(\hat{\theta}) = \theta$$

- Examples:
  - The sample mean $\bar{x}$ is an unbiased estimator of μ
  - The sample variance s^2 is an unbiased estimator of σ^2
  - The sample proportion $\hat{p}$ is an unbiased estimator of P

Unbiasedness

(continued)

- $\hat{\theta}_1$ is an unbiased estimator, $\hat{\theta}_2$ is biased:



Bias

- Let $\hat{\theta}$ be an estimator of θ

- The bias in $\hat{\theta}$ is defined as the difference between its mean and θ

$$\text{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta$$

- The bias of an unbiased estimator is 0

Most Efficient Estimator

- Suppose there are several unbiased estimators of θ
- The most efficient estimator or the minimum variance unbiased estimator of θ is the unbiased estimator with the smallest variance
- Let $\hat{\theta}_1$ and $\hat{\theta}_2$ be two unbiased estimators of θ, based on the same number of sample observations. Then,
  - $\hat{\theta}_1$ is said to be more efficient than $\hat{\theta}_2$ if $\text{Var}(\hat{\theta}_1) < \text{Var}(\hat{\theta}_2)$
  - The relative efficiency of $\hat{\theta}_1$ with respect to $\hat{\theta}_2$ is the ratio of their variances:

$$\text{Relative Efficiency} = \frac{\text{Var}(\hat{\theta}_2)}{\text{Var}(\hat{\theta}_1)}$$

Confidence Interval Estimation

- How much uncertainty is associated with a point estimate of a population parameter?
- An interval estimate provides more information about a population characteristic than does a point estimate
- Such interval estimates are called confidence interval estimates

Confidence Interval Estimate

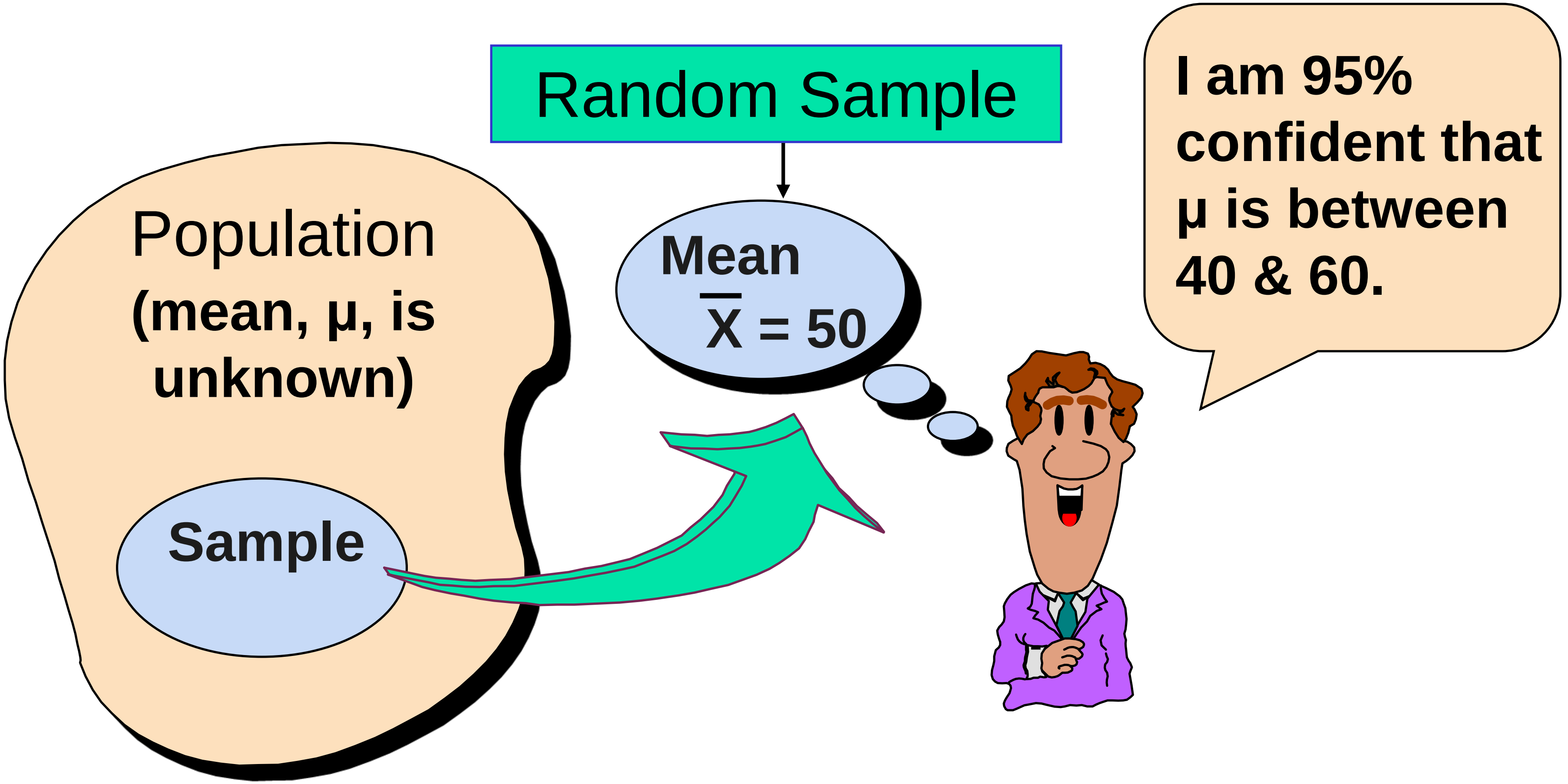
- An interval gives a range of values:
  - Takes into consideration variation in sample statistics from sample to sample
  - Based on observation from 1 sample
  - Gives information about closeness to unknown population parameters
  - Stated in terms of level of confidence
    - Can never be 100% confident

Confidence Interval and Confidence Level

- If $P(a < \theta < b) = 1 - \alpha$ then the interval from a to b is called a $100(1 - \alpha)\%$ confidence interval of θ.

- The quantity $100(1 - \alpha)\%$ is called the confidence level of the interval
  - α is between 0 and 1
  - In repeated samples of the population, the true value of the parameter θ would be contained in $100(1 - \alpha)\%$ of intervals calculated this way.
  - The confidence interval calculated in this manner is written as $a < \theta < b$ with $100(1 - \alpha)\%$ confidence

Estimation Process



Confidence Level, (1-α)

(continued)

- Suppose confidence level = 95%
- Also written $(1 - \alpha) = 0.95$
- A relative frequency interpretation:
  - From repeated samples, 95% of all the confidence intervals that can be constructed of size n will contain the unknown true parameter
- A specific interval either will contain or will not contain the true parameter
  - No probability involved in a specific interval

General Formula

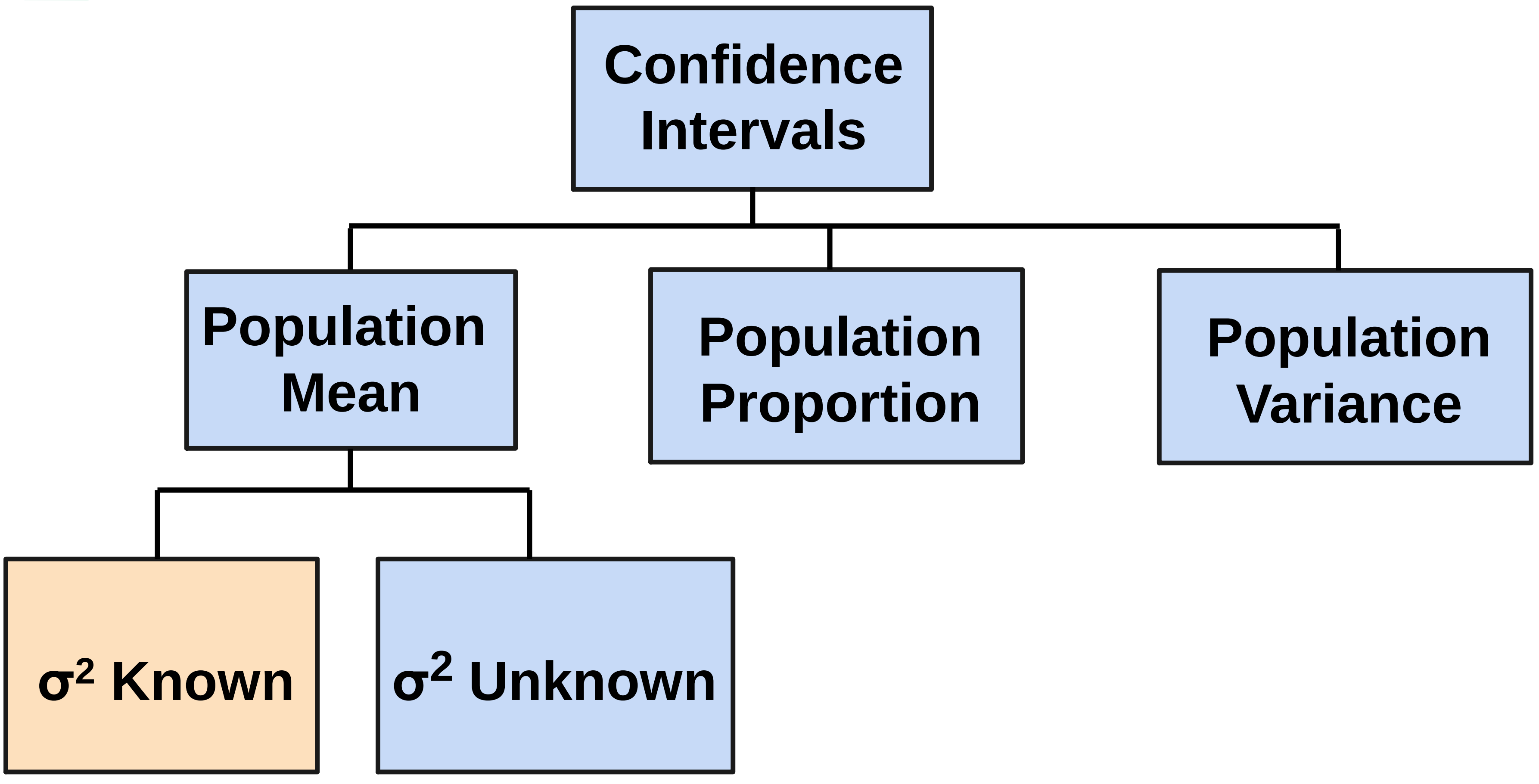
- The general form for all confidence intervals is:

$$\hat{\theta} \pm ME$$

Point Estimate ± Margin of Error

- The value of the margin of error depends on the desired level of confidence

Confidence Intervals



(From normally distributed populations)

7.2

Confidence Interval Estimation for the Mean (σ^2 Known)

- Assumptions
  - Population variance σ^2 is known
  - Population is normally distributed
  - If population is not normal, use large sample
- Confidence interval estimate:

$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

(where $z_{\alpha/2}$ is the normal distribution value for a probability of $\alpha/2$ in each tail)

Confidence Limits

- The confidence interval is

$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

- The endpoints of the interval are

$$UCL = \bar{x} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$ Upper confidence limit

$$LCL = \bar{x} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$ Lower confidence limit

Margin of Error

- The confidence interval,

$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

- Can also be written as $\bar{x} \pm ME$

  where ME is called the margin of error

$$ME = z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

- The interval width, w, is equal to twice the margin of error

Reducing the Margin of Error

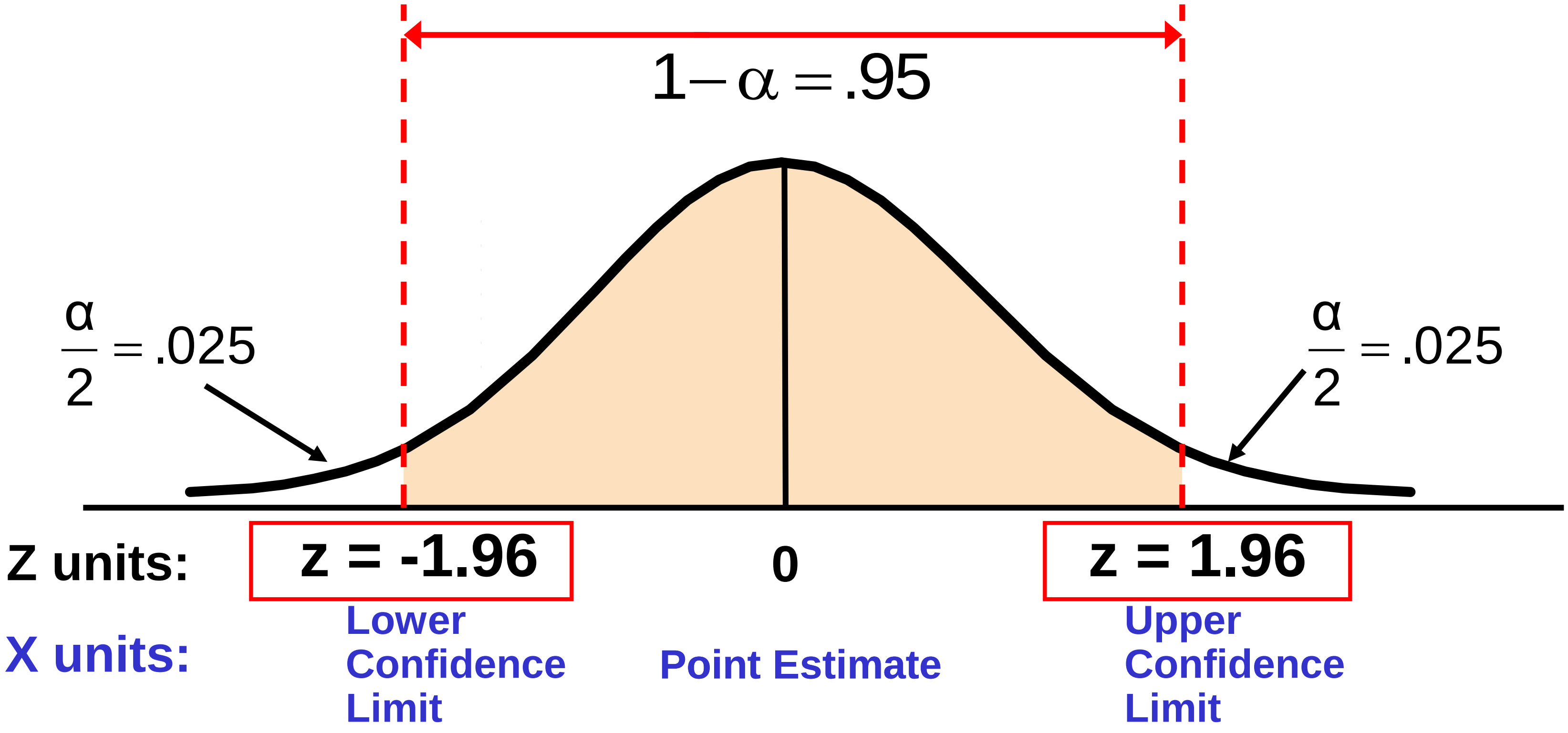
$$ME = z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

The margin of error can be reduced if

- the population standard deviation can be reduced ($\sigma\downarrow$)
- The sample size is increased ($n\uparrow$)
- The confidence level is decreased, $(1 - \alpha)\downarrow$

Finding $z_{\alpha/2}$

- Consider a 95% confidence interval:



- Find $z_{.025} = \pm 1.96$ from the standard normal distribution table

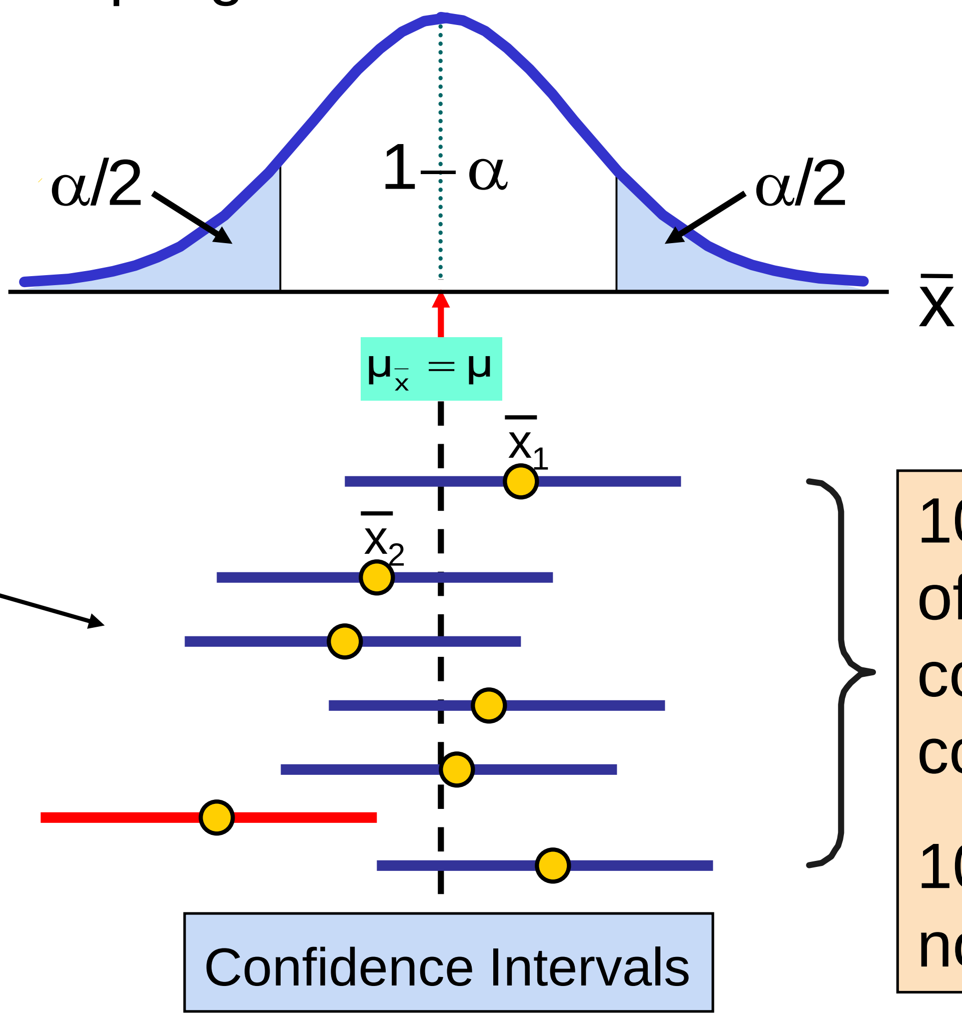
Common Levels of Confidence

- Commonly used confidence levels are 90%, 95%, 98%, and 99%

Confidence Level	***Confidence Coefficient,*** $1-\alpha$	$Z_{\alpha/2}$ ***value***
80%	**.80**	**1.28**
90%	**.90**	**1.645**
95%	**.95**	**1.96**
98%	**.98**	**2.33**
99%	**.99**	**2.58**
99.8%	**.998**	**3.08**
99.9%	**.999**	**3.27**

Intervals and Level of Confidence

Sampling Distribution of the Mean



Intervals extend from

$$LCL = \bar{x} - z\frac{\sigma}{\sqrt{n}}$$

to

$$UCL = \bar{x} + z\frac{\sigma}{\sqrt{n}}$$

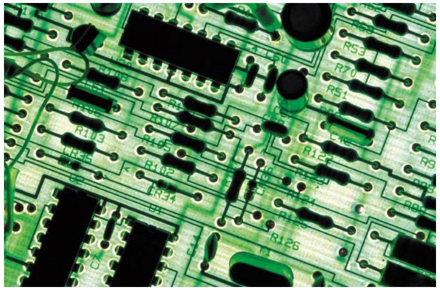
100(1-α)% of intervals constructed contain μ;

100(α)% do not.

Example

- A sample of 11 circuits from a large normal population has a mean resistance of 2.20 ohms. We know from past testing that the population standard deviation is 0.35 ohms.

- Determine a 95% confidence interval for the true mean resistance of the population.



Example

(continued)

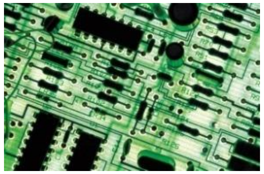
- A sample of 11 circuits from a large normal population has a mean resistance of 2.20 ohms. We know from past testing that the population standard deviation is .35 ohms.
- Solution:

$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

$$= 2.20 \pm 1.96\,(.35/\sqrt{11})$$

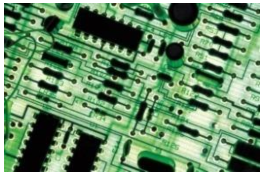
$$= 2.20 \pm .2068$$

$$1.9932 < \mu < 2.4068$$



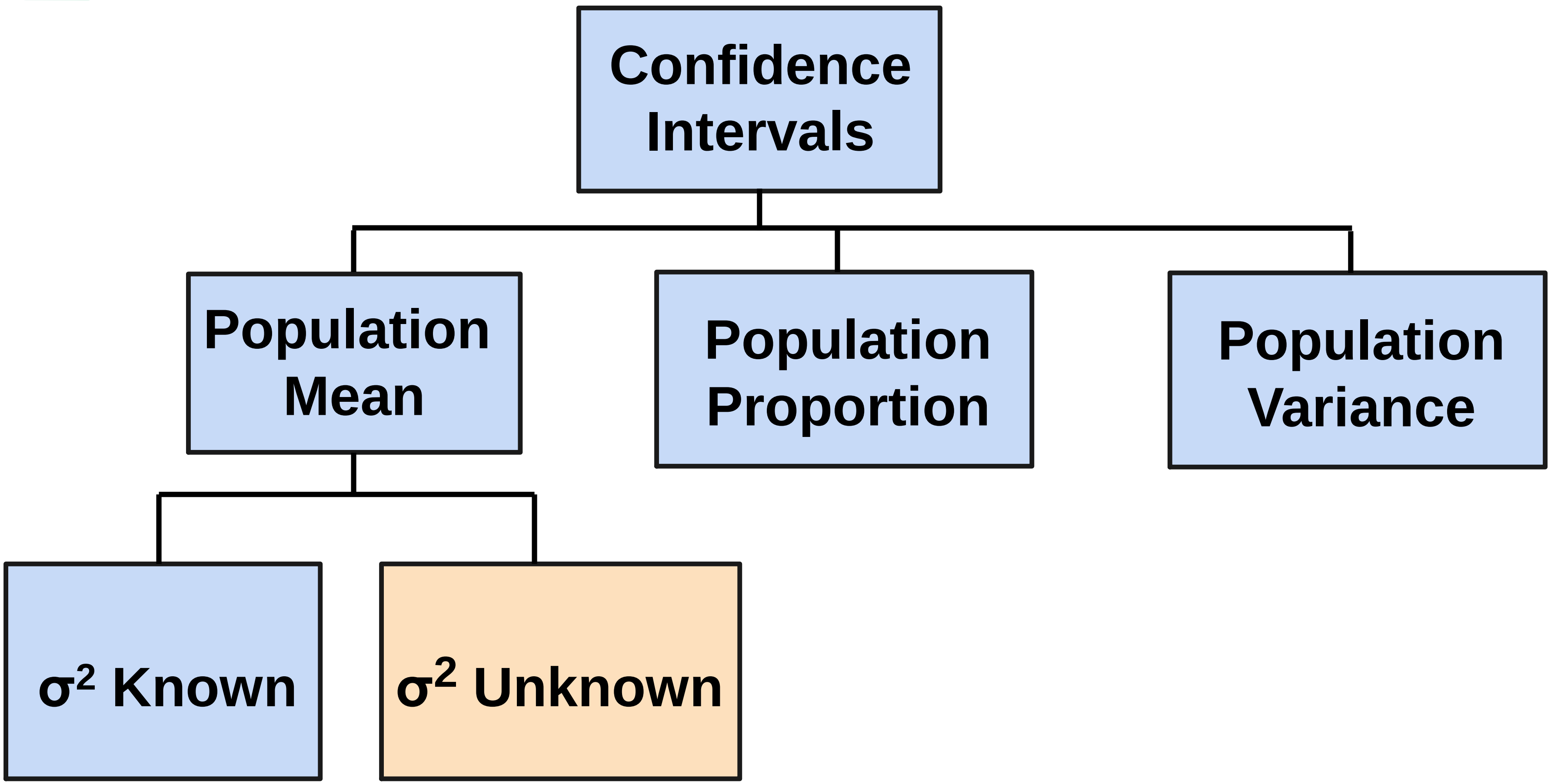
Interpretation

- We are 95% confident that the true mean resistance is between 1.9932 and 2.4068 ohms
- Although the true mean may or may not be in this interval, 95% of intervals formed in this manner will contain the true mean



Confidence Interval Estimation for the Mean (σ^2 Unknown)

7.3



(From normally distributed populations)

Student's t Distribution

- Consider a random sample of n observations
  - with mean $\bar{x}$ and standard deviation s
  - from a normally distributed population with mean μ

- Then the variable

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

follows the Student's t distribution with (n - 1) degrees of freedom

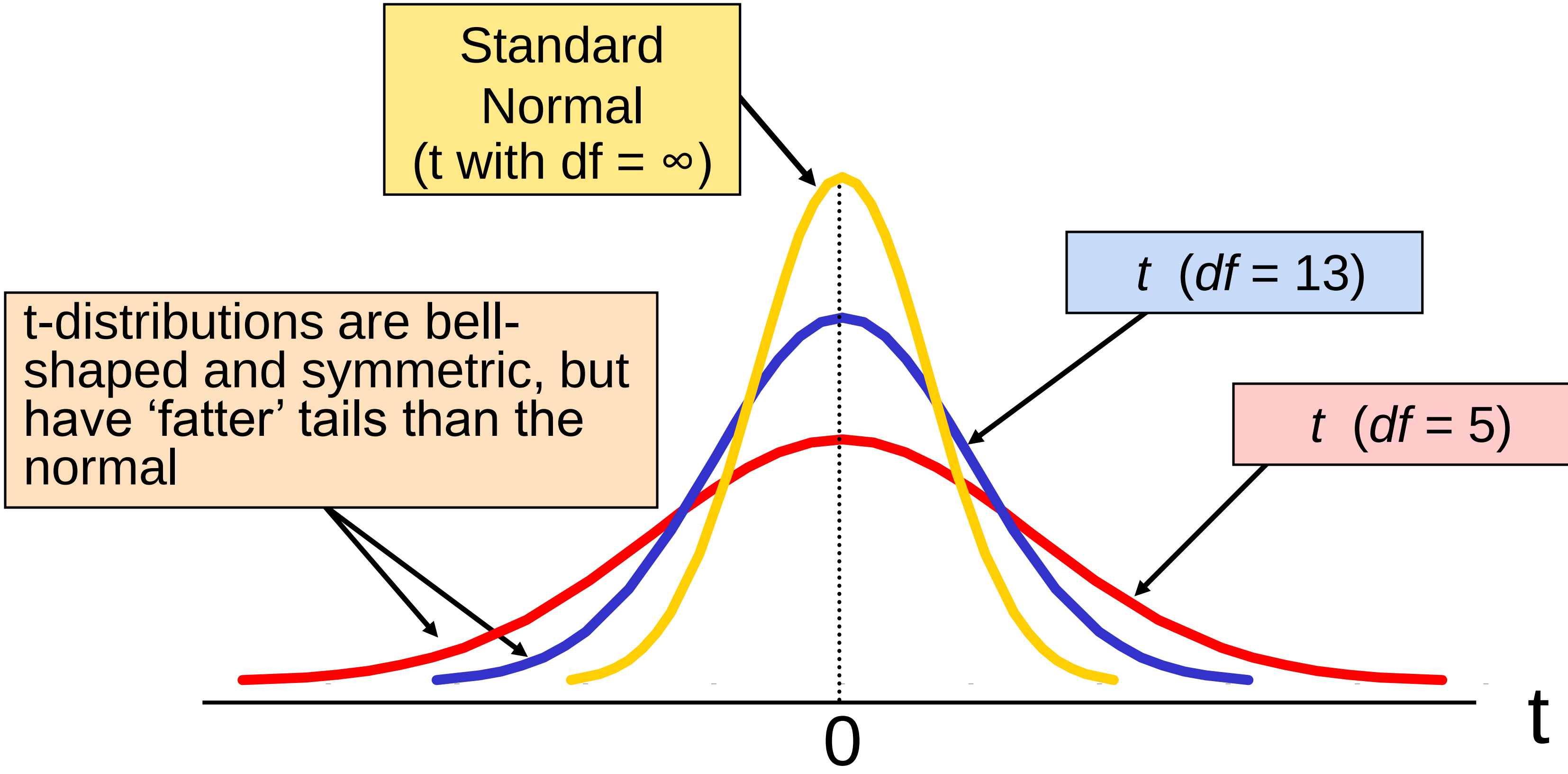
Student's t Distribution

- The t is a family of distributions
- The t value depends on degrees of freedom (d.f.)
  - Number of observations that are free to vary after sample mean has been calculated

$$d.f. = n - 1$$

Student's t Distribution

Note: t ⟶ Z as n increases

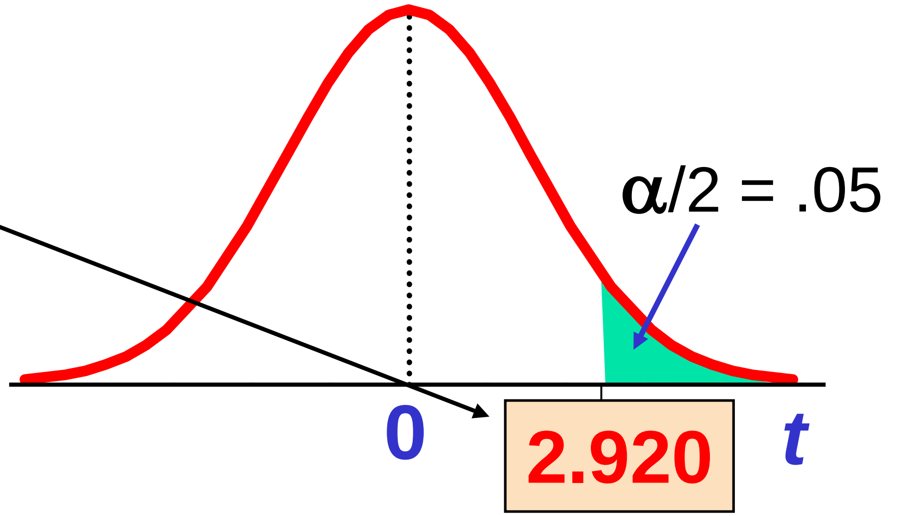


Student's t Table

df	Upper Tail Area .10	**.05**	.025
1	3.078	6.314	12.706
2	1.886	**2.920**	4.303
3	1.638	2.353	3.182

The body of the table contains t values, not probabilities

Let: n = 3
df = n - 1 = 2
α = .10
α/2 =.05



t distribution values

With comparison to the Z value

Confidence Level	t (10 d.f.)	t (20 d.f.)	t (30 d.f.)	Z
.80	1.372	1.325	1.310	1.282
.90	1.812	1.725	1.697	1.645
.95	2.228	2.086	2.042	1.960
.99	3.169	2.845	2.750	2.576

Note: t $\longrightarrow$ Z as n increases

Confidence Interval Estimation for the Mean (σ^2 Unknown)

- If the population standard deviation σ is unknown, we can substitute the sample standard deviation, s
- This introduces extra uncertainty, since s is variable from sample to sample
- So we use the t distribution instead of the normal distribution

Confidence Interval Estimation for the Mean (σ^2 Unknown)

(continued)

- Assumptions
  - Population standard deviation is unknown
  - Population is normally distributed
  - If population is not normal, use large sample
- Use Student's t Distribution
- Confidence Interval Estimate:

$$\bar{x} \pm t_{n-1,\alpha/2} \frac{s}{\sqrt{n}}$$

where $t_{n-1,\alpha/2}$ is the critical value of the t distribution with n-1 d.f. and an area of $\alpha/2$ in each tail:

$$P(t_{n-1} > t_{n-1,\alpha/2}) = \alpha/2$$

Margin of Error

- The confidence interval,

$$\bar{x} \pm t_{n-1,\alpha/2} \frac{s}{\sqrt{n}}$$

- Can also be written as $\bar{x} \pm ME$

where ME is called the margin of error:

$$ME = t_{n-1,\alpha/2} \frac{s}{\sqrt{n}}$$

Example

A random sample of n = 25 has $\bar{x}$ = 50 and s = 8. Form a 95% confidence interval for μ

- d.f. = n – 1 = 24, so $t_{n-1,\alpha/2} = t_{24,.025} = 2.0639$

The confidence interval is

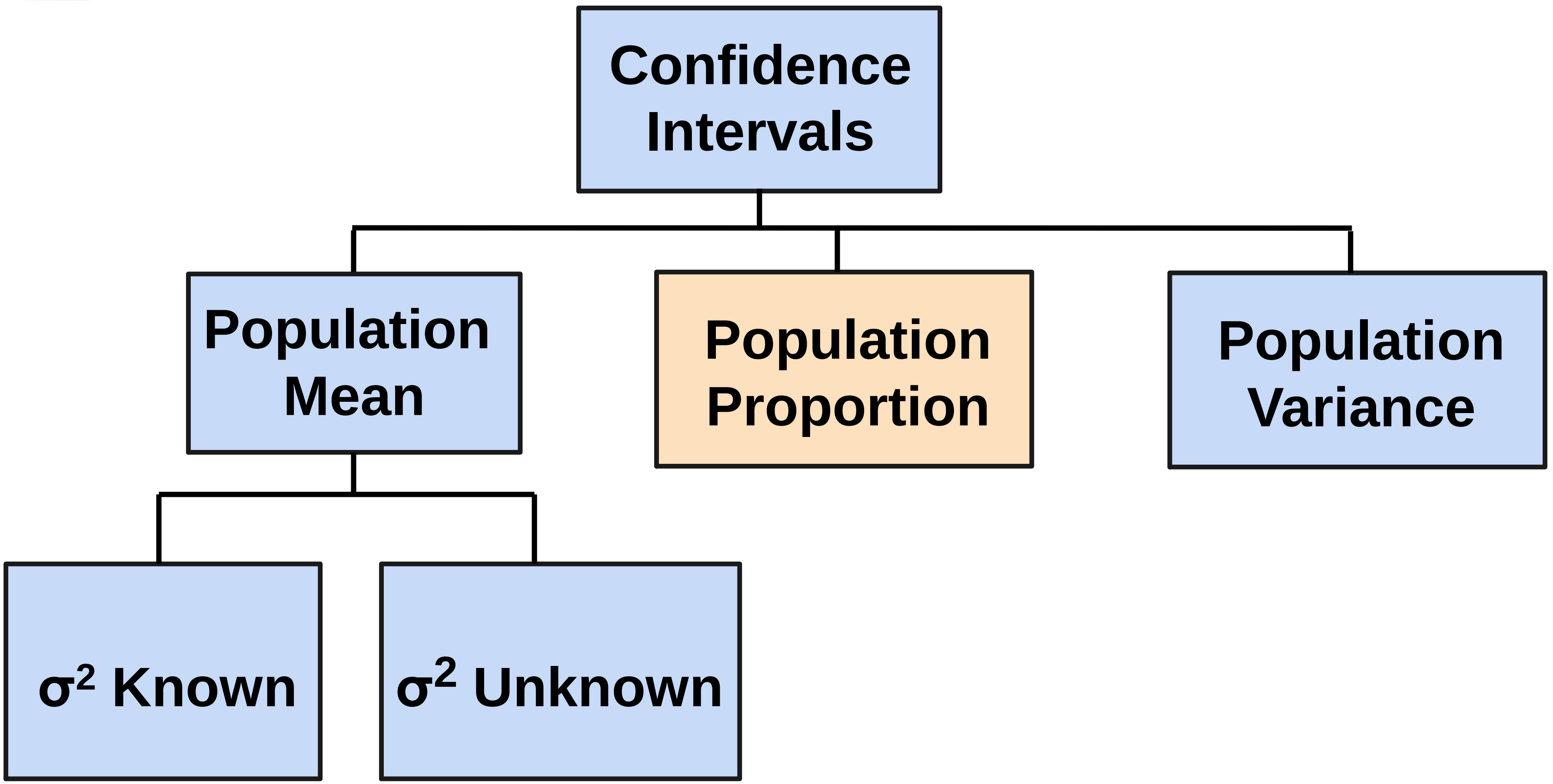
$$\bar{x} \pm t_{n-1,\alpha/2} \frac{s}{\sqrt{n}}$$

$$50 \pm (2.0639)\frac{8}{\sqrt{25}}$$

$$46.698 < \mu < 53.302$$

Confidence Interval Estimation for Population Proportion

7.4



Confidence Interval Estimation for Population Proportion

- An interval estimate for the population proportion (P) can be calculated by adding an allowance for uncertainty to the sample proportion ($\hat{p}$)

Confidence Intervals for the Population Proportion

- Recall that the distribution of the sample proportion is approximately normal if the sample size is large, with standard deviation

$$\sigma_P = \sqrt{\frac{P(1-P)}{n}}$$

- We will estimate this with sample data:

$$\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Confidence Interval Endpoints

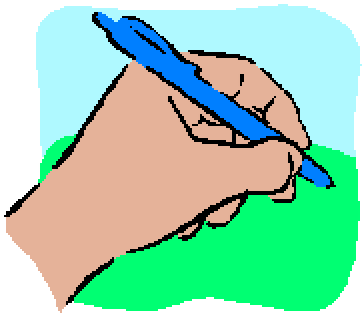
- The confidence interval for the population proportion is given by

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

- where
  - $z_{\alpha/2}$ is the standard normal value for the level of confidence desired
  - $\hat{p}$ is the sample proportion
  - n is the sample size
  - $nP(1-P) > 5$

Example

- A random sample of 100 people shows that 25 are left-handed.
- Form a 95% confidence interval for the true proportion of left-handers



Example

(continued)

- A random sample of 100 people shows that 25 are left-handed. Form a 95% confidence interval for the true proportion of left-handers.

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$\frac{25}{100} \pm 1.96\sqrt{\frac{.25(.75)}{100}}$$

$$0.1651 < P < 0.3349$$

Interpretation

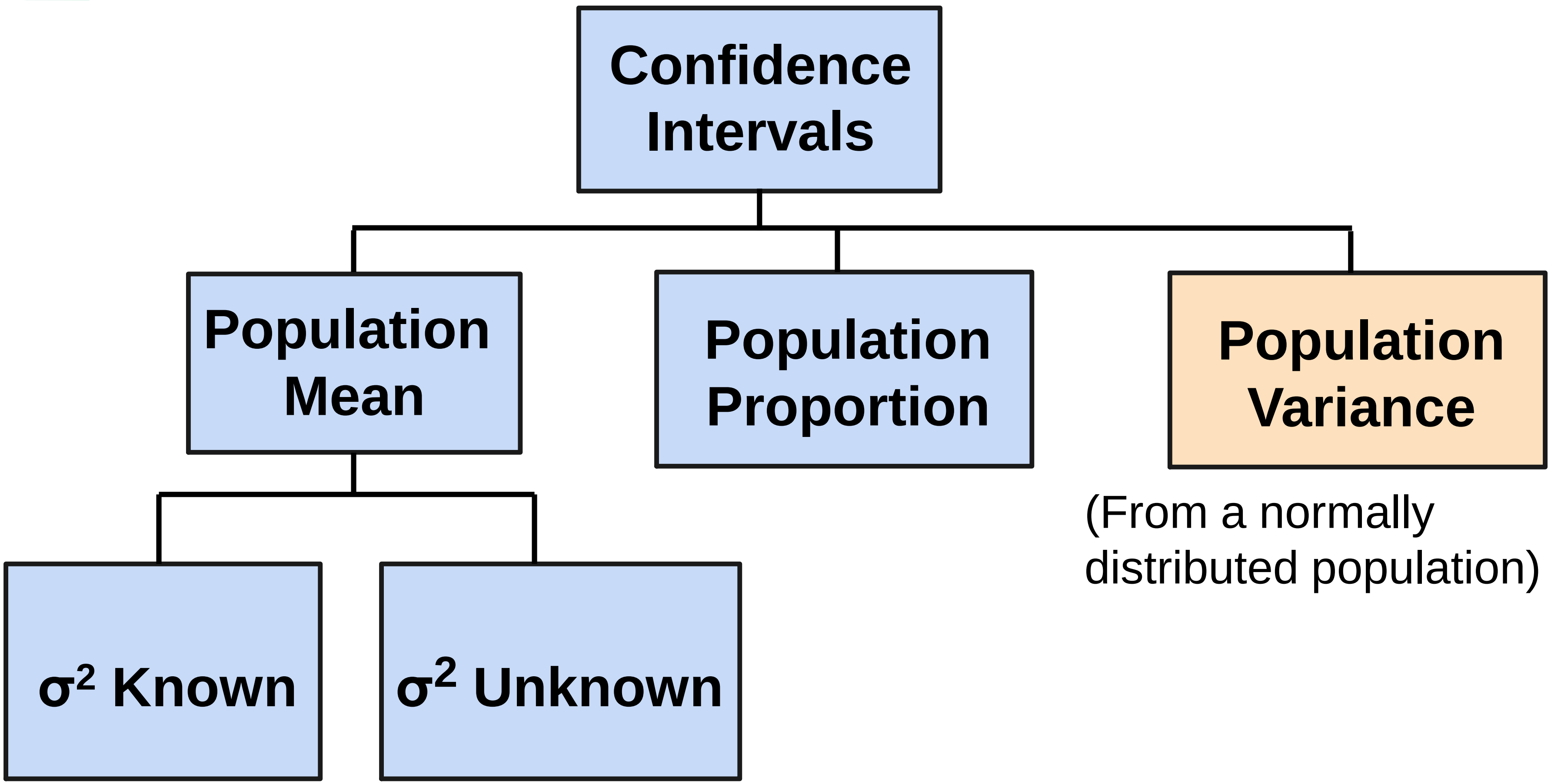
- We are 95% confident that the true proportion of left-handers in the population is between

  16.51% and 33.49%.

- Although the interval from 0.1651 to 0.3349 may or may not contain the true proportion, 95% of intervals formed from samples of size 100 in this manner will contain the true proportion.

7.5

Confidence Interval Estimation for the Variance



Confidence Intervals for the Population Variance

- Goal: Form a confidence interval for the population variance, σ^2

  - The confidence interval is based on the sample variance, s^2

  - Assumed: the population is normally distributed

Confidence Intervals for the Population Variance

(continued)

The random variable

$$\chi^2_{n-1} = \frac{(n-1)s^2}{\sigma^2}$$

follows a chi-square distribution with (n – 1) degrees of freedom

Where the chi-square value $\chi^2_{n-1,\alpha}$ denotes the number for which

$$P(\chi^2_{n-1} > \chi^2_{n-1,\alpha}) = \alpha$$

Confidence Intervals for the Population Variance

(continued)

The 100(1 - α)% confidence interval for the population variance is given by

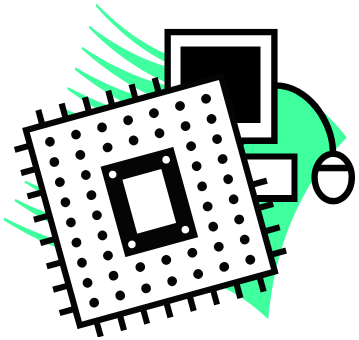
$$\text{LCL} = \frac{(n-1)s^2}{\chi^2_{n-1,\,\alpha/2}}$$

$$\text{UCL} = \frac{(n-1)s^2}{\chi^2_{n-1,\,1-\alpha/2}}$$

Example

You are testing the speed of a batch of computer processors. You collect the following data (in Mhz):

Sample size	**17**
Sample mean	**3004**
Sample std dev	**74**



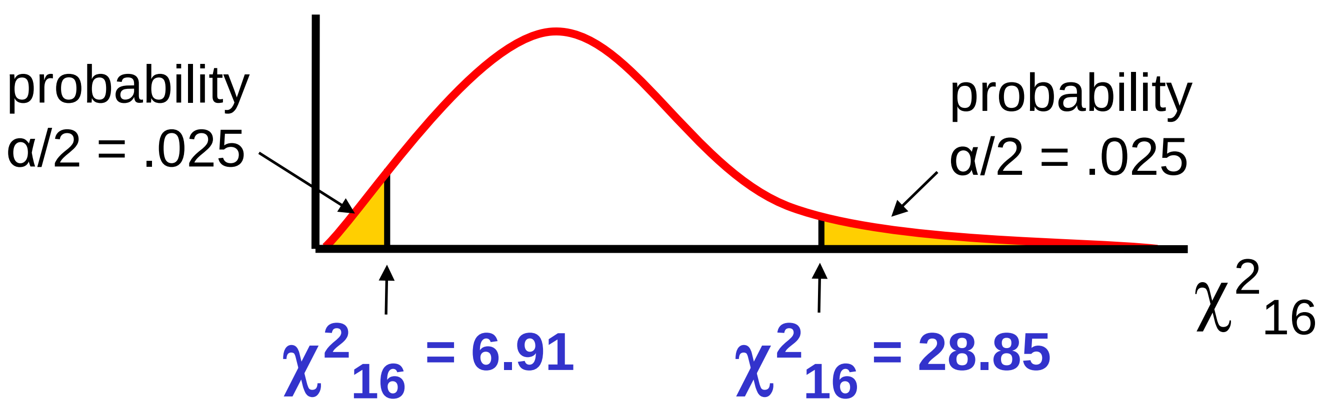
Assume the population is normal.
Determine the 95% confidence interval for σ_x^2

Finding the Chi-square Values

- n = 17 so the chi-square distribution has (n – 1) = 16 degrees of freedom
- α = 0.05, so use the the chi-square values with area 0.025 in each tail:

$$\chi^2_{n-1,\,\alpha/2} = \chi^2_{16\,,\,0.025} = 28.85$$

$$\chi^2_{n-1,\,1-\alpha/2} = \chi^2_{16\,,\,0.975} = 6.91$$



Calculating the Confidence Limits

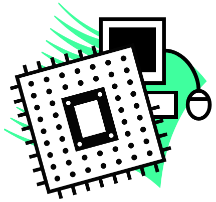
- The 95% confidence interval is

$$\frac{(n-1)s^2}{\chi^2_{n-1,\,\alpha/2}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{n-1,\,1-\alpha/2}}$$

$$\frac{(17-1)(74)^2}{28.85} < \sigma^2 < \frac{(17-1)(74)^2}{6.91}$$

$$3037 < \sigma^2 < 12680$$

Converting to standard deviation, we are 95% confident that the population standard deviation of CPU speed is between 55.1 and 112.6 Mhz



Confidence Interval Estimation: Finite Populations

7.6

- If the sample size is more than 5% of the population size (and sampling is without replacement) then a finite population correction factor must be used when calculating the standard error

Finite Population Correction Factor

- Suppose sampling is without replacement and the sample size is large relative to the population size
- Assume the population size is large enough to apply the central limit theorem
- Apply the finite population correction factor when estimating the population variance

$$\text{finite population correction factor} = \frac{N-n}{N-1}$$

Estimating the Population Mean

- Let a simple random sample of size n be taken from a population of N members with mean μ
- The sample mean is an unbiased estimator of the population mean μ
- The point estimate is:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Finite Populations: Mean

- If the sample size is more than 5% of the population size, an unbiased estimator for the variance of the sample mean is

$$\hat{\sigma}^2_{\bar{x}} = \frac{s^2}{n}\left(\frac{N-n}{N-1}\right)$$

- So the 100(1-α)% confidence interval for the population mean is

$$\bar{x} \pm t_{n-1,\alpha/2}\hat{\sigma}_{\bar{x}}$$

Estimating the Population Total

- Consider a simple random sample of size n from a population of size N
- The quantity to be estimated is the population total $N\mu$
- An unbiased estimation procedure for the population total $N\mu$ yields the point estimate $N\bar{x}$

Estimating the Population Total

- An unbiased estimator of the variance of the population total is

$$N^2\hat{\sigma}^2_{\bar{x}} = N^2 \frac{s^2}{n}\left(\frac{N-n}{N-1}\right)$$

- A 100(1 - α)% confidence interval for the population total is

$$N\bar{x} \pm t_{n-1,\alpha/2} N\hat{\sigma}_{\bar{x}}$$

Confidence Interval for Population Total: Example

A firm has a population of 1000 accounts and wishes to estimate the value of the total population balance

A sample of 80 accounts is selected with average balance of $87.60 and standard deviation of $22.30

Find the 95% confidence interval estimate of the total balance

Example Solution

$$N = 1000, \quad n = 80, \quad \bar{x} = 87.6, \quad s = 22.3$$

$$N^2\hat{\sigma}^2_{\bar{x}} = N^2 \frac{s^2}{n}\frac{(N-n)}{N-1} = (1000)^2 \frac{(22.3)^2}{80}\frac{920}{999} = 5724559.6$$

$$N\hat{\sigma}_{\bar{x}} = \sqrt{5724559.6} = 2392.6$$

$$N\bar{x} \pm t_{79,0.025} N\hat{\sigma}_{\bar{x}} = (1000)(87.6) \pm (1.9905)(2392.6)$$

$$82837.53 < N\mu < 92362.47$$

The 95% confidence interval for the population total balance is \$82,837.53 to \$92,362.47

Estimating the Population Proportion: Finite Population

- Let the true population proportion be P
- Let $\hat{p}$ be the sample proportion from n observations from a simple random sample
- The sample proportion, $\hat{p}$, is an unbiased estimator of the population proportion, P

Confidence Intervals for Population Proportion: Finite Population

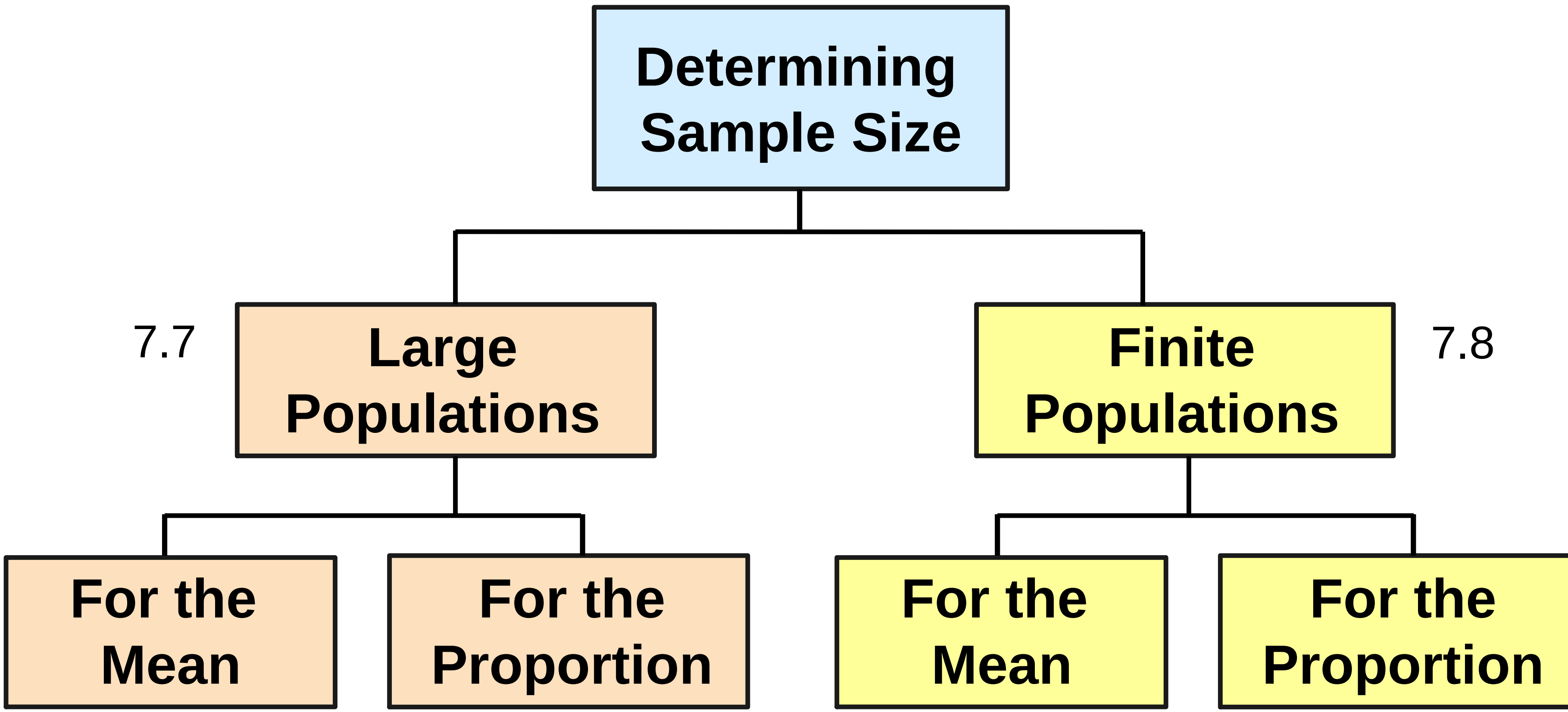
- If the sample size is more than 5% of the population size, an unbiased estimator for the variance of the population proportion is

$$\hat{\sigma}^2_{\hat{p}} = \frac{\hat{p}(1-\hat{p})}{n}\left(\frac{N-n}{N-1}\right)$$

- So the 100(1-α)% confidence interval for the population proportion is

$$\hat{p} \pm z_{\alpha/2}\hat{\sigma}_{\hat{p}}$$

Sample-Size Determination



Sample-Size Determination: Large Populations

Large Populations

For the Mean

(Known population variance)

$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

Margin of Error (sampling error)

$$ME = z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

Sample-Size Determination: Large Populations

(continued)

Large Populations

For the Mean

(Known population variance)

$$ME = z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

→ Now solve for n to get →

$$n = \frac{z_{\alpha/2}^2 \sigma^2}{ME^2}$$

Sample-Size Determination

(continued)

- To determine the required sample size for the mean, you must know:
  - The desired level of confidence (1 - α), which determines the $z_{\alpha/2}$ value
  - The acceptable margin of error (sampling error), ME
  - The population standard deviation, σ

Required Sample Size Example

If σ = 45, what sample size is needed to estimate the mean within ± 5 with 90% confidence?

$$n = \frac{z_{\alpha/2}^2 \sigma^2}{ME^2} = \frac{(1.645)^2 (45)^2}{5^2} = 219.19$$

So the required sample size is **n = 220**

(Always round up)

Sample Size Determination: Population Proportion

Large Populations

For the Proportion

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$ME = z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Margin of Error (sampling error)

Sample Size Determination: Population Proportion

(continued)

Large Populations

For the Proportion

$$ME = z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$\hat{p}(1-\hat{p})$ cannot be larger than 0.25, when $\hat{p} = 0.5$

→ Substitute 0.25 for $\hat{p}(1-\hat{p})$ and solve for n to get

→ $$n = \frac{0.25\, z_{\alpha/2}^2}{ME^2}$$

Sample Size Determination: Population Proportion

(continued)

- The sample and population proportions, $\hat{p}$ and P, are generally not known (since no sample has been taken yet)
- P(1 – P) = 0.25 generates the largest possible margin of error (so guarantees that the resulting sample size will meet the desired level of confidence)
- To determine the required sample size for the proportion, you must know:
  - The desired level of confidence (1 - α), which determines the critical $z_{\alpha/2}$ value
  - The acceptable sampling error (margin of error), ME
  - Estimate P(1 – P) = 0.25

Required Sample Size Example: Population Proportion

How large a sample would be necessary to estimate the true proportion defective in a large population within ±3%, with 95% confidence?

Required Sample Size Example

(continued)

Solution:

For 95% confidence, use $z_{0.025} = 1.96$

ME = 0.03

Estimate P(1 – P) = 0.25

$$n = \frac{0.25\, z_{\alpha/2}^2}{ME^2} = \frac{(0.25)(1.96)^2}{(0.03)^2} = \boxed{1067.11}$$

So use n = 1068

Sample-Size Determination: Finite Populations

7.8

Finite Populations

For the Mean

A finite population correction factor is added:

$$\text{Var}(\bar{X}) = \frac{\sigma^2}{n}\left(\frac{N-n}{N-1}\right)$$

1. Calculate the required sample size n_0 using the prior formula:

$$n_0 = \frac{z_{\alpha/2}^2 \sigma^2}{ME^2}$$

2. Then adjust for the finite population:

$$n = \frac{n_0 N}{n_0 + (N-1)}$$

Sample-Size Determination: Finite Populations

Finite Populations

For the Proportion

A finite population correction factor is added:

$$\mathrm{Var}(\hat{p}) = \frac{P(1\text{-}P)}{n}\left(\frac{N-n}{N-1}\right)$$

1. Solve for n:

$$n = \frac{NP(1-P)}{(N-1)\sigma_{\hat{p}}^2 + P(1-P)}$$

2. The largest possible value for this expression (if P = 0.25) is:

$$n = \frac{0.25(1-P)}{(N-1)\sigma_{\hat{p}}^2 + 0.25}$$

3. A 95% confidence interval will extend $\pm 1.96\,\sigma_{\hat{p}}$ from the sample proportion

Example: Sample Size to Estimate Population Proportion

How large a sample would be necessary to estimate within ±5% the true proportion of college graduates in a population of 850 people with 95% confidence?

Required Sample Size Example

(continued)

Solution:

- For 95% confidence, use $z_{0.025} = 1.96$
- ME = 0.05

$$1.96\,\sigma_{\hat{p}} = 0.05 \quad \Rightarrow \quad \sigma_{\hat{p}} = 0.02551$$

$$n_{max} = \frac{0.25N}{(N-1)\sigma_{\hat{p}}^2 + 0.25} = \frac{(0.25)(850)}{(849)(0.02551)^2 + 0.25} = 264.8$$

So use n = 265

Chapter Summary

- Introduced the concept of confidence intervals
- Discussed point estimates
- Developed confidence interval estimates
- Created confidence interval estimates for the mean (σ^2 known)
- Introduced the Student's t distribution
- Determined confidence interval estimates for the mean (σ^2 unknown)

Chapter Summary

(continued)

- Created confidence interval estimates for the proportion
- Created confidence interval estimates for the variance of a normal population
- Applied the finite population correction factor to form confidence intervals when the sample size is not small relative to the population size
- Determined required sample size to meet confidence and margin of error requirements