Statistics for Business and Economics
8th Edition

Chapter 3

Probability

Chapter Goals

After completing this chapter, you should be able to:

- Explain basic probability concepts and definitions
- Use a Venn diagram or tree diagram to illustrate simple probabilities
- Apply common rules of probability
- Compute conditional probabilities
- Determine whether events are statistically independent
- Use Bayes' Theorem for conditional probabilities

3.1

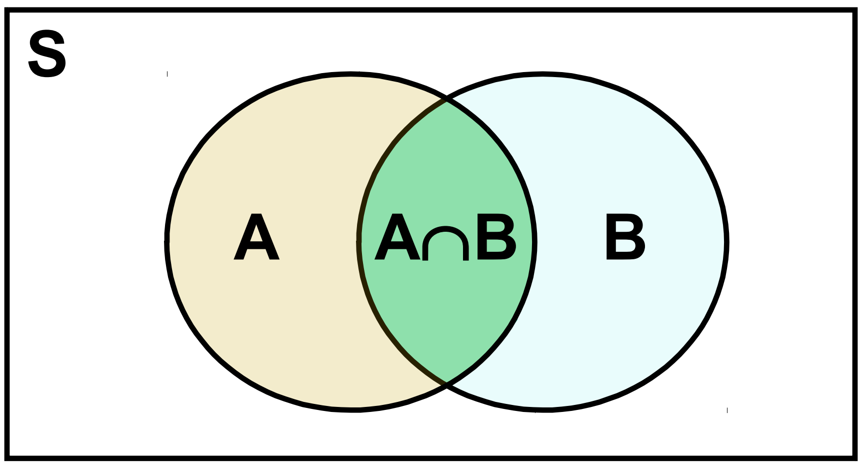
Important Terms

- Random Experiment – a process leading to an uncertain outcome
- Basic Outcome – a possible outcome of a random experiment
- Sample Space (S) – the collection of all possible outcomes of a random experiment
- Event (E) – any subset of basic outcomes from the sample space

Important Terms

(continued)

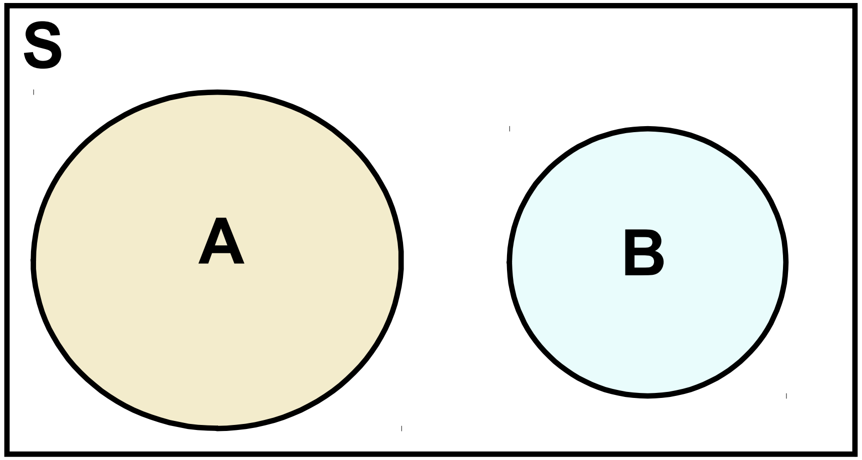
- Intersection of Events – If A and B are two events in a sample space S, then the intersection, A ∩ B, is the set of all outcomes in S that belong to both A and B



Important Terms

(continued)

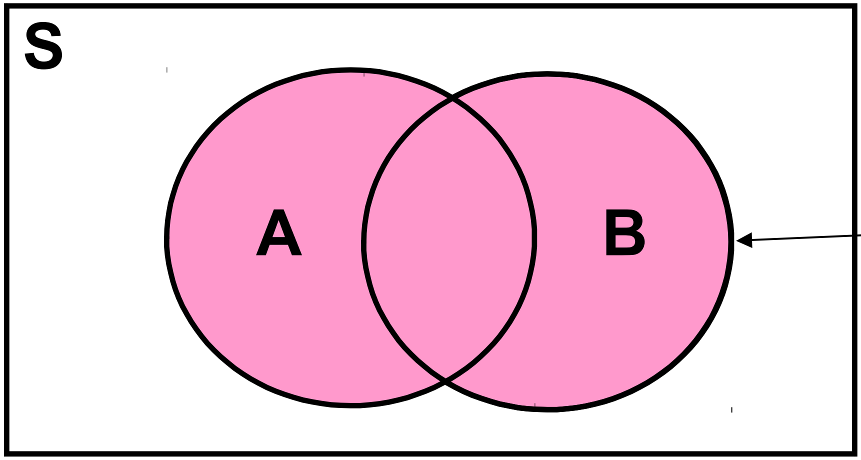
- A and B are Mutually Exclusive Events if they have no basic outcomes in common
  - i.e., the set A ∩ B is empty



Important Terms

(continued)

- Union of Events – If A and B are two events in a sample space S, then the union, A ∪ B, is the set of all outcomes in S that belong to either A or B



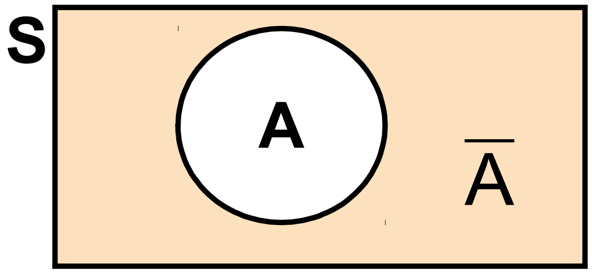
The entire shaded area represents A ∪ B

Important Terms

(continued)

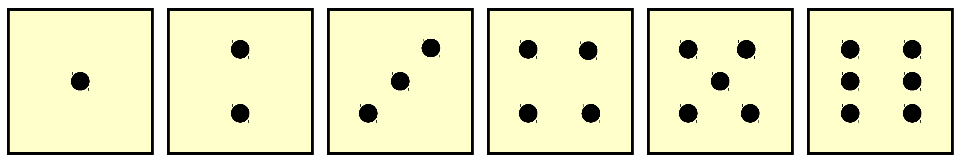
- Events E_1, E_2, …,E_k are Collectively Exhaustive events if $E_1 \cup E_2 \cup \ldots \cup E_k = S$
  - i.e., the events completely cover the sample space

- The Complement of an event A is the set of all basic outcomes in the sample space that do not belong to A. The complement is denoted $\bar{A}$



Examples

Let the Sample Space be the collection of all possible outcomes of rolling one die:



S = [1, 2, 3, 4, 5, 6]

Let A be the event "Number rolled is even"

Let B be the event "Number rolled is at least 4"

Then

A = [2, 4, 6] and B = [4, 5, 6]

Examples

(continued)

S = [1, 2, 3, 4, 5, 6] **A = [2, 4, 6]** **B = [4, 5, 6]**

Complements:

$$\overline{A} = [1, 3, 5] \qquad \overline{B} = [1, 2, 3]$$

Intersections:

$$A \cap B = [4, 6] \qquad \overline{A} \cap B = [5]$$

Unions:

$$A \cup B = [2, 4, 5, 6]$$

$$A \cup \overline{A} = [1, 2, 3, 4, 5, 6] = S$$

Examples

(continued)

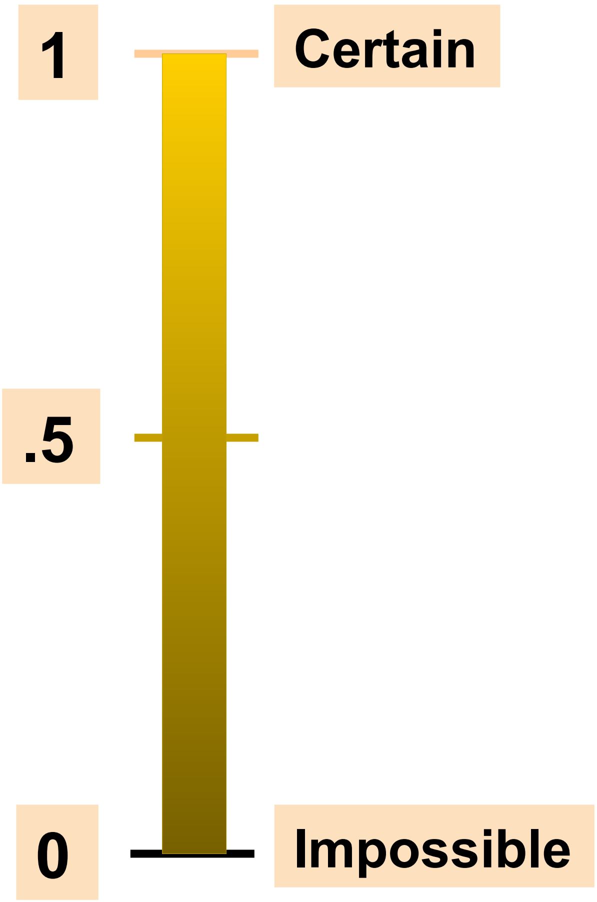
S = [1, 2, 3, 4, 5, 6]	A = [2, 4, 6]	B = [4, 5, 6]

- Mutually exclusive:
  - A and B are not mutually exclusive
    - The outcomes 4 and 6 are common to both
- Collectively exhaustive:
  - A and B are not collectively exhaustive
    - $A \cup B$ does not contain 1 or 3

Probability and Its Postulates

- Probability – the chance that an uncertain event will occur (always between 0 and 1)

$0 \le P(A) \le 1$ **For any event A**



Assessing Probability

- There are three approaches to assessing the probability of an uncertain event:

1. classical probability
2. relative frequency probability
3. subjective probability

Classical Probability

- Assumes all outcomes in the sample space are equally likely to occur

Classical probability of event A:

$$P(A) = \frac{N_A}{N} = \frac{\text{number of outcomes that satisfy the event A}}{\text{total number of outcomes in the sample space}}$$

- Requires a count of the outcomes in the sample space

Counting the Possible Outcomes

- Use the Combinations formula to determine the number of combinations of n items taken k at a time

$$C_k^n = \frac{n!}{k!(n-k)!}$$

- where
  - $n! = n(n-1)(n-2)\ldots(1)$
  - $0! = 1$ by definition

Permutations and Combinations

The number of possible orderings

- The total number of possible ways of arranging x objects in order is

$$x! = x(x-1)(x-2)\ldots(2)(1)$$

- x! is read as “x factorial”

Permutations and Combinations

(continued)

Permutations: the number of possible arrangements when x objects are to be selected from a total of n objects and arranged in order [with (n – x) objects left over]

$$P_x^n = n(n-1)(n-2)\ldots(n-x+1)$$

$$= \frac{n!}{(n-x)!}$$

Permutations and Combinations

(continued)

- Combinations: The number of combinations of x objects chosen from n is the number of possible selections that can be made

$$C_k^n = \frac{P_x^n}{x!}$$

$$= \frac{n!}{x!(n-x)!}$$

Permutations and Combinations Example

Suppose that two letters are to be selected from A, B, C, D and arranged in order. How many permutations are possible?

- Solution The number of permutations, with

  $n = 4$ and $x = 2$, is $P^4_2 = \dfrac{4!}{(4-2)!} = 12$

- The permutations are

  AB AC AD BA BC BD

  CA CB CD DA DB DC

Permutations and Combinations Example

(continued)

Suppose that two letters are to be selected from A, B, C, D. How many combinations are possible (i.e., order is not important)?

- Solution The number of combinations is

$$C_2^4 = \frac{4!}{2!(4-2)!} = 6$$

- The combinations are

AB (same as BA)	BC (same as CB)
AC (same as CA)	BD (same as DB)
AD (same as DA)	CD (same as DC)

Assessing Probability

Three approaches (continued)

2. relative frequency probability
   - the limit of the proportion of times that an event A occurs in a large number of trials, *n*

$$P(A) = \frac{n_A}{n} = \frac{\text{number of events in the population that satisfy event A}}{\text{total number of events in the population}}$$

3. subjective probability

an individual opinion or belief about the probability of occurrence

Probability Postulates

1. If A is any event in the sample space S, then

$$0 \leq P(A) \leq 1$$

2. Let A be an event in S, and let O_i denote the basic outcomes. Then

$$P(A) = \sum_{A} P(O_i)$$

(the notation means that the summation is over all the basic outcomes in A)

3. P(S) = 1

3.3 Probability Rules

- The Complement rule:

$$P(\overline{A}) = 1 - P(A) \qquad \text{i.e., } P(A) + P(\overline{A}) = 1$$

- The Addition rule:
  - The probability of the union of two events is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

A Probability Table

Probabilities and joint probabilities for two events A and B are summarized in this table:

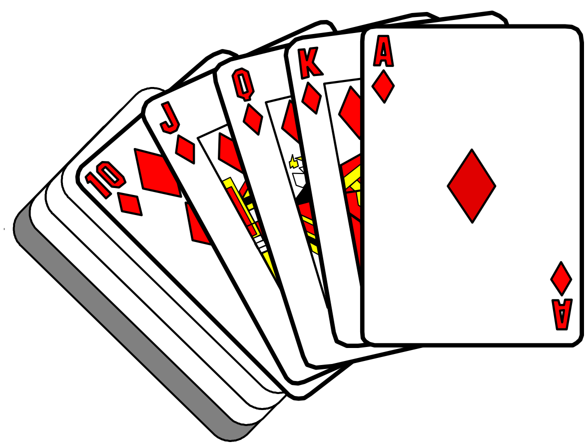
	B	$\overline{B}$	
A	$P(A \cap B)$	$P(A \cap \overline{B})$	$P(A)$
$\overline{A}$	$P(\overline{A} \cap B)$	$P(\overline{A} \cap \overline{B})$	$P(\overline{A})$
	$P(B)$	$P(\overline{B})$	$P(S) = 1.0$

Addition Rule Example

Consider a standard deck of 52 cards, with four suits: ♥ ♣ ♦ ♠

Let event A = card is an Ace

Let event B = card is from a red suit



Addition Rule Example

(continued)

$$P(\text{Red} \cup \text{Ace}) = P(\text{Red}) + P(\text{Ace}) - P(\text{Red} \cap \text{Ace})$$

$$= 26/52 + 4/52 - 2/52 = 28/52$$

Don't count the two red aces twice!

Type	Color: Red	Color: Black	Total
Ace	2	2	4
Non-Ace	24	24	48
Total	26	26	52

Conditional Probability

- A conditional probability is the probability of one event, given that another event has occurred:

$$P(A\,|\,B)=\frac{P(A\cap B)}{P(B)}$$

→ **The conditional probability of A given that B has occurred**

$$P(B\,|\,A)=\frac{P(A\cap B)}{P(A)}$$

→ **The conditional probability of B given that A has occurred**

Conditional Probability Example

- **Of the cars on a used car lot, 70% have air conditioning (AC) and 40% have a CD player (CD). 20% of the cars have both.**

- What is the probability that a car has a CD player, given that it has AC ?

  i.e., we want to find P(CD | AC)

Conditional Probability Example

(continued)

- **Of the cars on a used car lot, 70% have air conditioning (AC) and 40% have a CD player (CD). 20% of the cars have both.**

	CD	No CD	Total
AC	.2	.5	.7
No AC	.2	.1	.3
Total	.4	.6	1.0

$$P(CD \mid AC) = \frac{P(CD \cap AC)}{P(AC)} = \frac{.2}{.7} = .2857$$

Conditional Probability Example

(continued)

- **Given AC, we only consider the top row (70% of the cars). Of these, 20% have a CD player. 20% of 70% is 28.57%.**

	CD	No CD	Total
AC	.2	.5	.7
No AC	.2	.1	.3
Total	.4	.6	1.0

$$P(CD \mid AC) = \frac{P(CD \cap AC)}{P(AC)} = \frac{.2}{.7} = .2857$$

Multiplication Rule

- Multiplication rule for two events A and B:

$$P(A \cap B) = P(A|B)P(B)$$

- also

$$P(A \cap B) = P(B|A)P(A)$$

Multiplication Rule Example

P(Red ∩ Ace) = P(Red| Ace)P(Ace)

$$= \left(\frac{2}{4}\right)\left(\frac{4}{52}\right) = \frac{2}{52}$$

$$= \frac{\text{number of cards that are red and ace}}{\text{total number of cards}} = \frac{2}{52}$$

Type	Color		Total
	Red	Black	
Ace	2	2	4
Non-Ace	24	24	48
Total	26	26	52

Statistical Independence

- Two events are statistically independent if and only if:

$$P(A \cap B) = P(A)P(B)$$

  - Events A and B are independent when the probability of one event is not affected by the other event

- If A and B are independent, then

$$P(A \mid B) = P(A) \qquad \textbf{if } P(B)>0$$

$$P(B \mid A) = P(B) \qquad \textbf{if } P(A)>0$$

Statistical Independence

(continued)

- For multiple events:

$E_1, E_2, \ldots, E_k$ are statistically independent if and only if:

$$P(E_1 \cap E_1 \cap \ldots \cap E_1) = P(E_1)P(E_2)\ldots P(E_k)$$

Statistical Independence Example

- **Of the cars on a used car lot, 70% have air conditioning (AC) and 40% have a CD player (CD). 20% of the cars have both.**

	CD	No CD	Total
AC	.2	.5	.7
No AC	.2	.1	.3
Total	.4	.6	1.0

- **Are the events AC and CD statistically independent?**

Statistical Independence Example

(continued)

	CD	No CD	Total
AC	.2	.5	.7
No AC	.2	.1	.3
Total	.4	.6	1.0

$P(AC \cap CD) = 0.2$

$P(AC) = 0.7$

$P(CD) = 0.4$

$P(AC)P(CD) = (0.7)(0.4) = 0.28$

$P(AC \cap CD) = 0.2 \neq P(AC)P(CD) = 0.28$

So the two events are **not** statistically independent

Bivariate Probabilities

Outcomes for bivariate events:

	B_1	B_2	. . .	B_k
A_1	$P(A_1 \cap B_1)$	$P(A_1 \cap B_2)$	. . .	$P(A_1 \cap B_k)$
A_2	$P(A_2 \cap B_1)$	$P(A_2 \cap B_2)$	. . .	$P(A_2 \cap B_k)$
. . .	. . .	. . .	. . .	. . .
A_h	$P(A_h \cap B_1)$	$P(A_h \cap B_2)$	. . .	$P(A_h \cap B_k)$

Joint and Marginal Probabilities

- The probability of a joint event, $A \cap B$:

$$P(A \cap B) = \frac{\text{number of outcomes satisfying A and B}}{\text{total number of elementary outcomes}}$$

- Computing a marginal probability:

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + \cdots + P(A \cap B_k)$$

  - Where B_1, B_2, …, B_k are k mutually exclusive and collectively exhaustive events

Marginal Probability Example

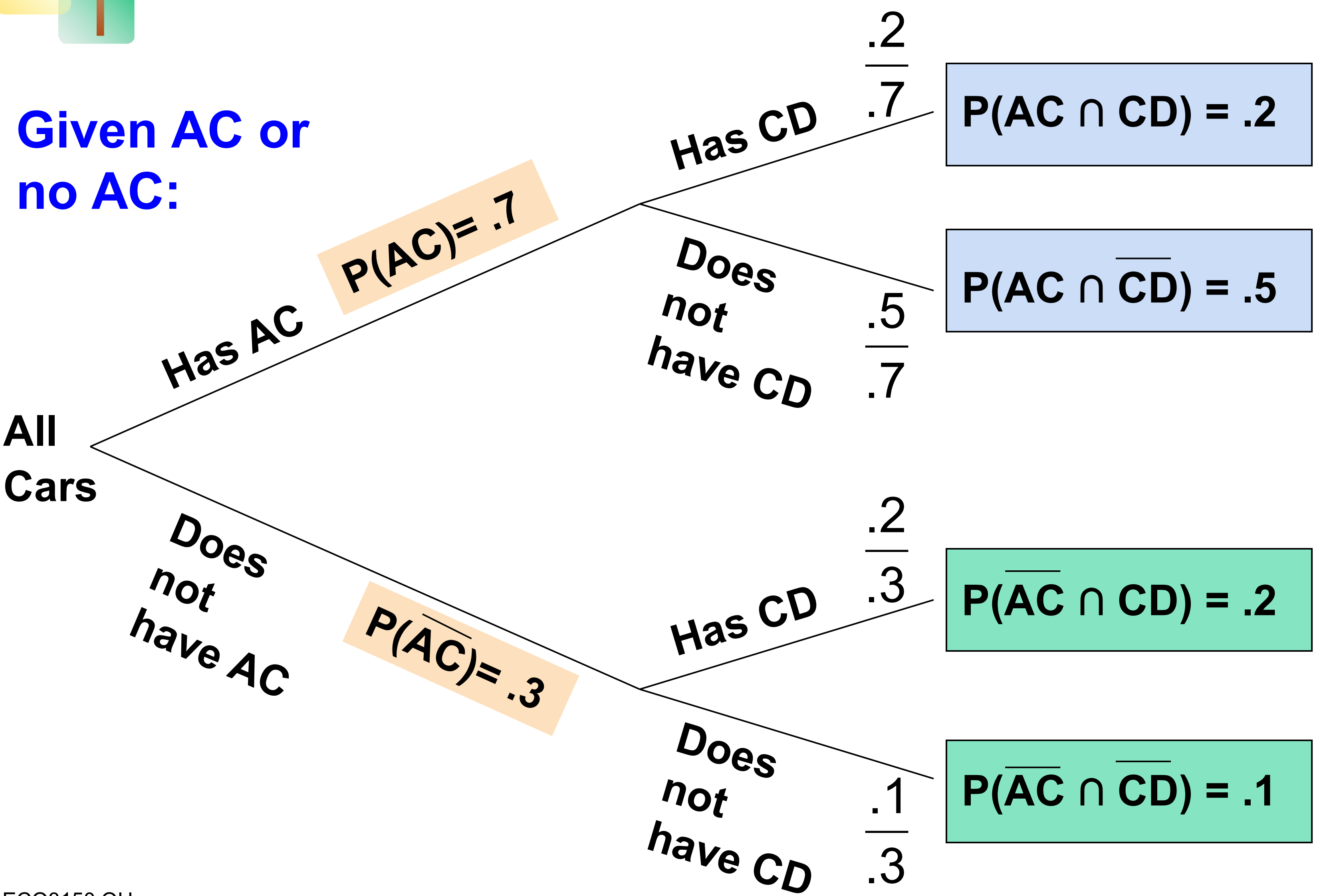
P(Ace)

$$= P(Ace \cap Red) + P(Ace \cap Black) = \frac{2}{52} + \frac{2}{52} = \frac{4}{52}$$

Type	Color		Total
	Red	Black	
Ace	2	2	4
Non-Ace	24	24	48
Total	26	26	52

Using a Tree Diagram

Given AC or no AC:



Odds

- The odds in favor of a particular event are given by the ratio of the probability of the event divided by the probability of its complement
- The odds in favor of A are

$$\text{odds} = \frac{P(A)}{1-P(A)} = \frac{P(A)}{P(\overline{A})}$$

Odds: Example

- Calculate the probability of winning if the odds of winning are 3 to 1:

$$\text{odds} = \frac{3}{1} = \frac{P(A)}{1 - P(A)}$$

  - Now multiply both sides by 1 – P(A) and solve for P(A):

$$3 \times (1 - P(A)) = P(A)$$
$$3 - 3P(A) = P(A)$$
$$3 = 4P(A)$$
$$P(A) = 0.75$$

Overinvolvement Ratio

- The probability of event A_1 conditional on event B_1 divided by the probability of A_1 conditional on activity B_2 is defined as the overinvolvement ratio:

$$\frac{P(A_1|B_1)}{P(A_1|B_2)}$$

- An overinvolvement ratio greater than 1 implies that event A_1 increases the conditional odds ratio in favor of B_1:

$$\frac{P(B_1|A_1)}{P(B_2|A_1)} > \frac{P(B_1)}{P(B_2)}$$

3.5 Bayes' Theorem

Let A_1 and B_1 be two events. Bayes' theorem states that

$$P(B_1 | A_1) = \frac{P(A_1 | B_1) P(B_1)}{P(A_1)}$$

and

$$P(A_1 | B_1) = \frac{P(B_1 | A_1) P(A_1)}{P(B_1)}$$

- a way of revising conditional probabilities by using available or additional information

3.5

Bayes' Theorem

Bayes' theorem (alternative statement)

$$P(E_i \mid A) = \frac{P(A \mid E_i)P(E_i)}{P(A)}$$

$$= \frac{P(A \mid E_i)P(E_i)}{P(A \mid E_1)P(E_1) + P(A \mid E_2)P(E_2) + \ldots + P(A \mid E_k)P(E_k)}$$

- where:

  E_i = i^{th} event of k mutually exclusive and collectively exhaustive events

  A = new event that might impact $P(E_i)$

Bayes' Theorem Example

- A drilling company has estimated a 40% chance of striking oil for their new well.
- A detailed test has been scheduled for more information. Historically, 60% of successful wells have had detailed tests, and 20% of unsuccessful wells have had detailed tests.
- Given that this well has been scheduled for a detailed test, what is the probability that the well will be successful?

Bayes' Theorem Example

(continued)

- Let S = successful well

  U = unsuccessful well
- P(S) = .4 , P(U) = .6 (prior probabilities)
- Define the detailed test event as D
- Conditional probabilities:

  P(D|S) = .6 P(D|U) = .2
- Goal is to find P(S|D)

Bayes' Theorem Example

(continued)

Apply Bayes' Theorem:

$$P(S|D) = \frac{P(D|S)P(S)}{P(D|S)P(S) + P(D|U)P(U)}$$

$$= \frac{(.6)(.4)}{(.6)(.4) + (.2)(.6)}$$

$$= \frac{.24}{.24 + .12} = .667$$

So the revised probability of success (from the original estimate of .4), given that this well has been scheduled for a detailed test, is .667

Chapter Summary

- Defined basic probability concepts
  - Sample spaces and events, intersection and union of events, mutually exclusive and collectively exhaustive events, complements
- Examined basic probability rules
  - Complement rule, addition rule, multiplication rule
- Defined conditional, joint, and marginal probabilities
- Reviewed odds and the overinvolvement ratio
- Defined statistical independence
- Discussed Bayes' theorem