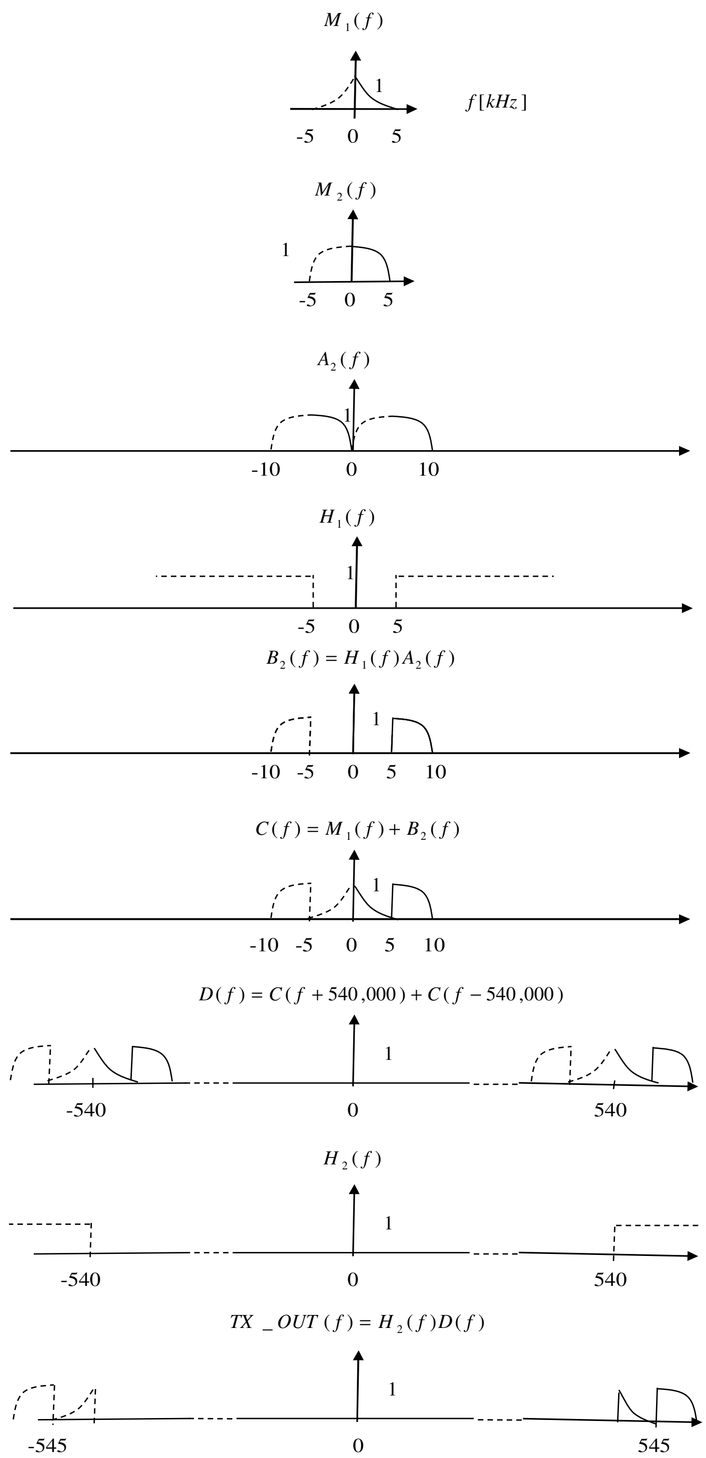
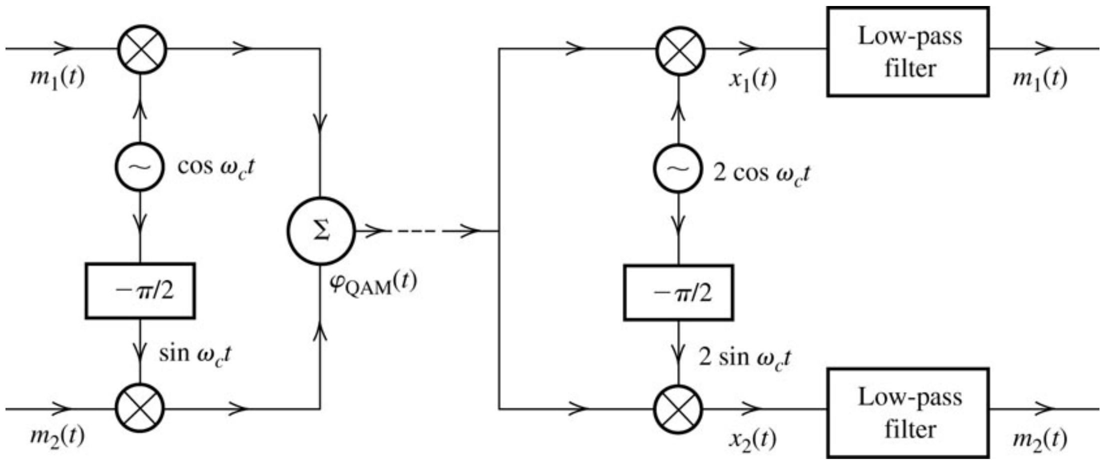
Solution 1a:



The signal $tx\_out(t)$ is spread between 540 to 550 kHz and therefore the bandwidth of this signal is 10 kHz and is centered at $f_c = 545$ kHz.

Solution 1b:

Single sideband signals are hard to generate since as shown in part "a" of this problem ideal filters are required. To transmit two signals $m_1(t)$ and $m_2(t)$, we can use Quadrature Amplitude Modulation (QAM) which do not require any filter at all and therefore it is practical.



QAM Transmitter **QAM Receiver**

The QAM transmitter output is $\varphi_{QAM}(t) = m_1(t)\cos(2\pi f_c t) + m_2(t)\sin(2\pi f_c t)$ and the carrier frequency is $f_c = 545 \quad kHz$. To find the bandwidth and center frequency of $\varphi_{QAM}(t)$, we should examine two signals $m_1(t)\cos(2\pi \times 545t)$ and $m_2(t)\sin(2\pi \times 545t)$. The base-band bandwidth of $m_1(t)$ and $m_2(t)$ is 5kHz and as we know the bandwidth of each of the Double –Side-Band modulated signals $m_1(t)\cos(2\pi \times 545t)$ and $m_2(t)\sin(2\pi \times 545t)$ would be double of that which is 10kHz. The center frequency of each of these signals would be the carrier frequency of 545kHz.

Therefore, the transmitted signal $\varphi_{QAM}(t) = m_1(t)\cos(2\pi \times 545t) + m_2(t)\sin(2\pi \times 545t)$ will have center frequency of 545kHz with bandwidth of 10kHz.

To recover $m_1(t)$ and $m_2(t)$, coherent QAM receiver as shown above could be used. This can be proved by calculating $x_1(t)$ and $x_2(t)$ based on the structure of the receiver.

$$x_1(t) = 2\varphi_{QAM}(t)\cos(2\pi \times 545t) = 2[m_1(t)\cos(2\pi \times 545t) + m_2(t)\sin(2\pi \times 545t)]\cos(2\pi \times 545t)$$

$$= m_1(t) + m_1(t)\cos(2\pi \times 1090t) + m_2(t)\sin(2\pi \times 1090t)$$

$$x_2(t) = 2\varphi_{QAM}(t)\sin(2\pi \times 545t) = 2[m_1(t)\cos(2\pi \times 545t) + m_2(t)\sin(2\pi \times 545t)]\sin(2\pi \times 545t)$$

$$= m_2(t) - m_2(t)\cos(2\pi \times 1090t) + m_1(t)\sin(2\pi \times 1090t)$$

The last two terms in both equations are bandpass signals centered at $2f_c = 1090kHz$ which are suppressed by the lowpass filter and therefore, the output of these filters will recover $m_1(t)$ and $m_2(t)$. Note that these low-pass filters are not ideal.

Solution 2a:

Nyquist sampling frequency is double the maximum frequency of $M_1(f)$ and $M_2(f)$ which is $2\times 5 = 10kHz$. We should sample with 150% of that which is 15 kHz.

Let's check the condition:

$$\mu^2 >> \frac{m_p^2}{\overline{m^2(t)}} \Rightarrow 50^2 >> \frac{1^2}{20\times 10^{-3}} \Rightarrow 2500 >> 50$$

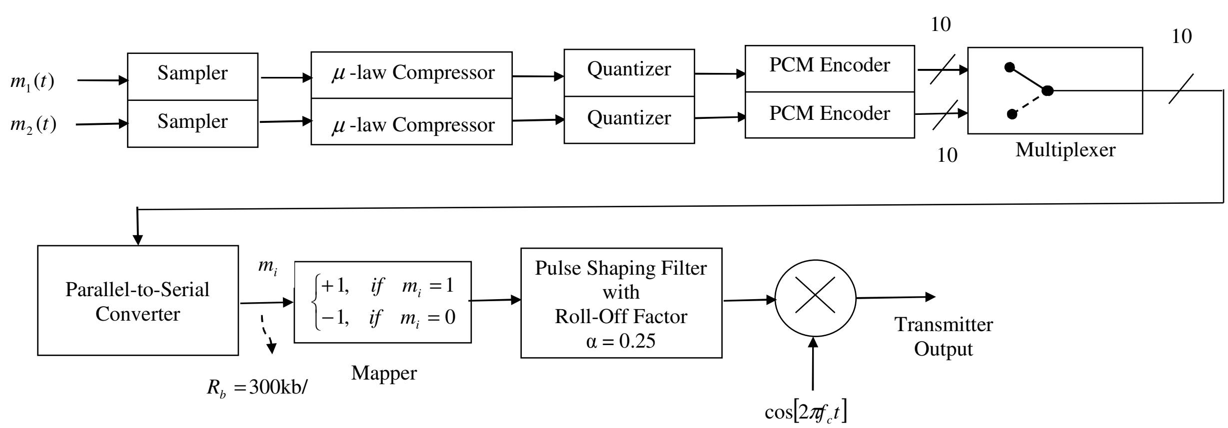
Therefore, we can use the following formula:

$$\frac{S_o}{N_o} = \frac{3L^2}{[\ln(1+\mu)]^2} \Rightarrow 10^{4.8} = \frac{3L^2}{[\ln(1+50)]^2} \Rightarrow L = 570.22$$

L should be power of 2 and larger than 570. Therefore, $2^n = 1024$ and we need at least $n = 10$ bits for the quantizer. Note that for each signal $n = 10$ bits.

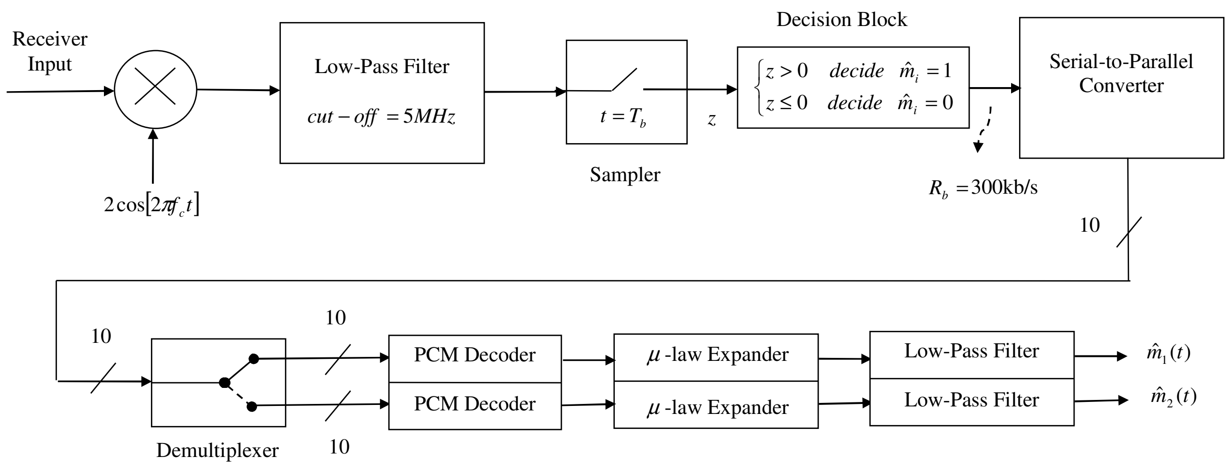
The sampling frequency is $f_s = 15000Hz$ and each sample is $n = 10$ bits per signal, therefore the bit rate for bit-by-bit serial transmission is $R_b = 2\times f_s \times n = 2\times 15000\times 10 = 300 \quad kbits/\sec$.

The bandwidth of BPSK modulator is $BW = (1+\alpha)R_b = (1+0.2)\times 300 = 360 \quad kHz$.



Solution 2b:

The block diagram of the receiver is shown below:



The received signal is a BPSK modulated signal with bandwidth of 360 kHz and carrier frequency of $f_c = 5MHz$. This signal is multiplied by $2\cos[2\pi f_c t]$.

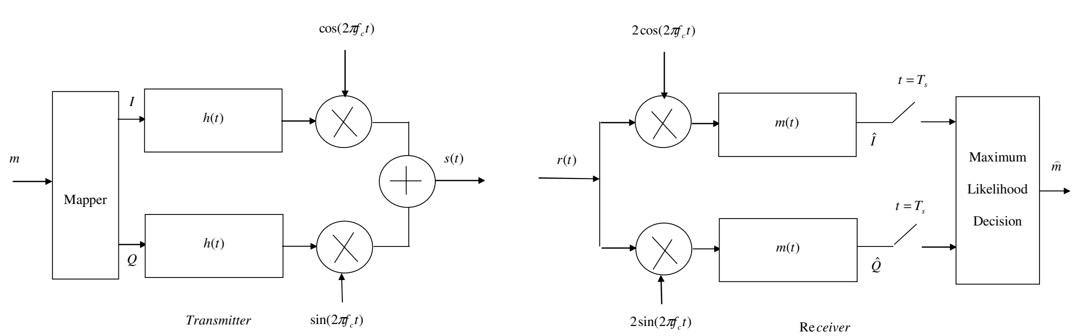
At the output of the mixer, there will be the baseband signal and a high frequency signal centered at 10MHz. The high frequency signal will be removed by the low pass filter.

The baseband signal will go to the sampler which will sample the signal at every $T_b = \frac{1}{300000}[\text{sec}]$. Decision will be made at the output of the sampler and bits with bit rate of $R_b = 300{,}000 bits/\sec$ will be produced.

Then serial to parallel converter will group the bits 10-bit by 10-bit which is one sample at sampling frequency of 30kHz. Then, the demultiplexer produces two streams of 10-bit samples each with sampling frequency of 15 kHz. These samples are passed through PCM decoder and μ-law expander (μ=50).

Output of the expanders are filtered by a low-pass filter to remove ripples. The estimates of original signals will be produced at the output of these filters.

Solution 3a:



If the outputs of the mapper are I and Q, we have:

$$\hat{I} = \{r(t)\times 2\cos(2\pi f_c t)\} * m(t) = \{2s(t)\cos(2\pi f_c t)\} * m(t) =$$

$$= 2\{[Ih(t)\cos(2\pi f_c t) + Qh(t)\sin(2\pi f_c t)]\cos(2\pi f_c t)\} * m(t) =$$

$$= 2\{[Ih(t)\cos(2\pi f_c t)\cos(2\pi f_c t) + Qh(t)\sin(2\pi f_c t)\cos(2\pi f_c t)]\} * m(t) =$$

$$= 2\{\{I\cos(2\pi f_c t)\cos(2\pi f_c t) + Q\sin(2\pi f_c t)\cos(2\pi f_c t)\}h(t)\} * m(t) =$$

$$= \{\{I[1 + \cos(4\pi f_c t)] + Q[\sin(4\pi f_c t)]\}h(t)\} * m(t) =$$
$$= \{I \times h(t)\} * m(t) + \{I\{\cos(4\pi f_c t) + \sin(4\pi f_c t)\}\times h(t)\} * m(t)$$

We know that $h(t) \overset{FT}{\Rightarrow} H(f)$ is a low pass filter. On the other hand, $m(t) \overset{FT}{\Rightarrow} M(f)$ is matched to $h(t) \overset{FT}{\Rightarrow} H(f)$ and therefore it is also low pass filter with the same shape as the pulse shaping filter. We have $M(f) = H(f)$.

As shown above, $\hat{I}$ has two components. Considering lowpass nature of $m(t)$, the second component is zero since the high frequency component $I\{\cos(4\pi f_c t) + \sin(4\pi f_c t)\}$ will be removed by filter $m(t)$. Therefore, $\hat{I} = I \times h(t) * m(t)$ and similarly we can show that $\hat{Q} = Q \times h(t) * m(t)$.

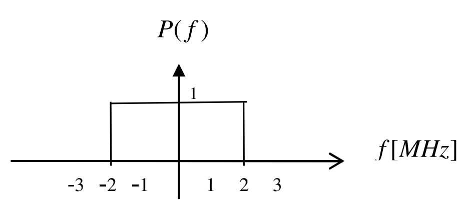
The pass-band transmitter and receiver are equivalent to the following two baseband transmitters and the receivers, since we also have:

$$\hat{I} = I \times h(t) * m(t) \quad \text{and} \quad \hat{Q} = Q \times h(t) * m(t)$$

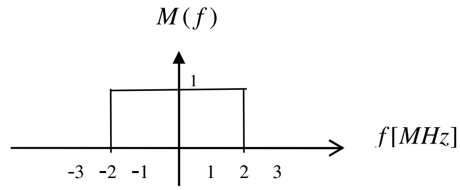
In fact, we have two transmitter/receiver pairs, one for I-channel and the other one for Q-channel.

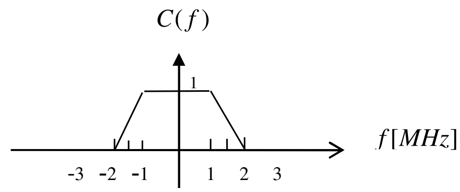
Solution 3b:

Pulse shaping filter $P(f)$ is,

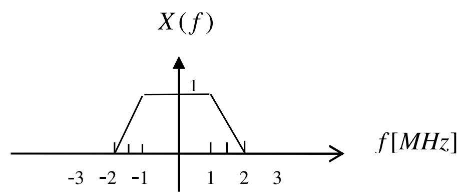


The matched filter has transfer function of $M(f) = P(f)$





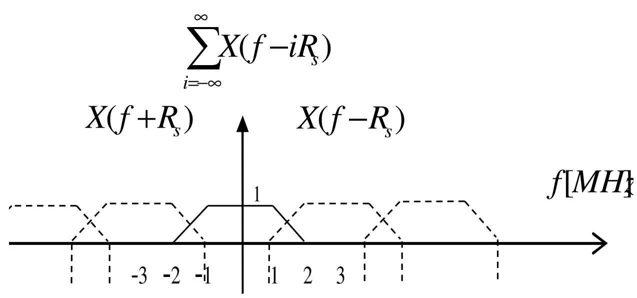
The overall frequency domain response of the system is $X(f) = P(f)C(f)M(f)$ as shown below:



According to Nyquist theorem, there is no intersymbol interference if

$$\sum_{i=-\infty}^{\infty} X(f - iR_s) = \text{constant}$$

In this equation, R_s is the symbol rate. As shown below, the summation will have a constant



value of "1" if $R_s = 3Msym/\sec$.

Since the modulation scheme is 8-ary, the bit rate of the system for no ISI is:'
$R_b = (\log_2 8)R_s = 3 \times 3 = 9Mbit/\sec$

According to Nyquist theorem when there is no ISI for rate $R_s = \frac{1}{T_s}$, we have:

$$x(nT_s) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}, \text{ for } n \text{ integer}$$

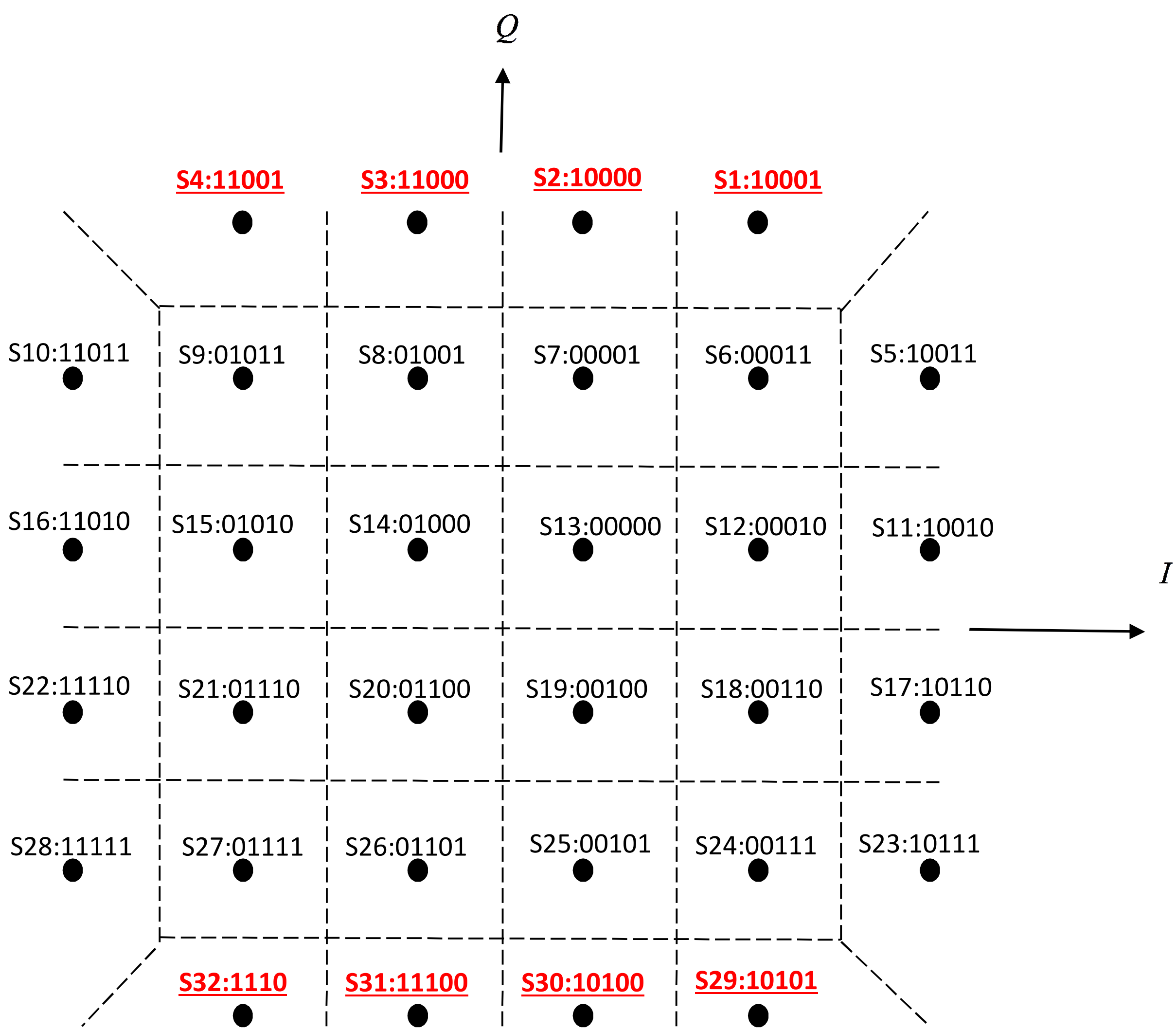
Therefore, for $T_s^{'} = 2T_s$, the following is satisfied:

$$x(mT_s^{'}) = \begin{cases} 1 & m = 0 \\ 0 & m \neq 0 \end{cases}, \text{ for } m \text{ integer}$$

In this case the new bit rate would be $R_b^{'} = R_b / 2 = 4.5 Mbit/\sec$. Essentially, for $R_b^{'} = R_b / n$ where $n = 2,3,4,5,...$ which is a positive integer.

Solution 4a:

Decision boundaries for a maximum likelihood receiver is shown using dashed lines in the constellation diagram below.



The constellation points can be divided into 3 categories from probability of error point of view and therefore:

$$P(symbol \quad error) = \frac{1}{32}\sum_{i=1}^{32} P(error|S_i) = \frac{1}{32}\left(8P(error|S_1) + 8P(error|S_2) + 16P(error|S_6)\right)$$

Based on Union bound we have:

$$P(error|S_1) = 2Q\left(\frac{\Delta/2}{\sigma}\right), \quad P(error|S_2) = 3Q\left(\frac{\Delta/2}{\sigma}\right), \quad P(error|S_6) = 4Q\left(\frac{\Delta/2}{\sigma}\right)$$

$$P(symbol \quad error) = \frac{1}{32}\left(8\times 2Q\left(\frac{\Delta/2}{\sigma}\right) + 8\times 3Q\left(\frac{\Delta/2}{\sigma}\right) + 16\times 4Q\left(\frac{\Delta/2}{\sigma}\right)\right) = \frac{104}{32}Q\left(\frac{\Delta/2}{\sigma}\right)$$

$$P(symbol \quad error) = \frac{13}{4}Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)$$

Now we find the relation of average signal energy E_s with Δ.

The constellation points can be divided into 5 categories from signal energy point of view and therefore:

$$Average \quad Signal \quad Energy = E_s = \frac{1}{32}\sum_{i=1}^{32} E_i = \frac{1}{32}\left(8E_1 + 8E_2 + 4E_6 + 8E_7 + 4E_{13}\right)$$

$$E_1 = \left(\frac{3\Delta}{2}\right)^2 + \left(\frac{5\Delta}{2}\right)^2 = \frac{34}{4}\Delta^2, \; E_2 = \left(\frac{\Delta}{2}\right)^2 + \left(\frac{5\Delta}{2}\right)^2 = \frac{26}{4}\Delta^2, \; E_6 = \left(\frac{3\Delta}{2}\right)^2 + \left(\frac{3\Delta}{2}\right)^2 = \frac{18}{4}\Delta^2,$$

$$E_7 = \left(\frac{\Delta}{2}\right)^2 + \left(\frac{3\Delta}{2}\right)^2 = \frac{10}{4}\Delta^2, \; E_7 = \left(\frac{\Delta}{2}\right)^2 + \left(\frac{\Delta}{2}\right)^2 = \frac{2}{4}\Delta^2$$

$$E_s = \frac{1}{32}\left(8\times 34 + 8\times 26 + 4\times 18 + 8\times 10 + 4\times 2\right)\frac{\Delta^2}{4} = \frac{40}{8}\Delta^2 \Rightarrow \Delta = \sqrt{\frac{E_s}{5}}$$

Therefore: $$P_E(32QAM) = P(symbol \quad error) = \frac{13}{4}Q\left(\frac{\sqrt{\frac{E_s}{5}}}{\sqrt{2N_0}}\right) = \frac{13}{4}Q\left(\sqrt{\frac{E_s}{10N_0}}\right)$$

Since for 32QAM we cannot use Gray coding and $E_s = (\log_2 32)E_b = 5E_b$, therefore:

$$P_E(32QAM) = \frac{13}{4}Q\left(\sqrt{\frac{5P_r}{10N_0R_b}}\right) = \frac{13}{4}Q\left(\sqrt{\frac{P_r}{2N_0R_b}}\right)$$

Solution 4b:

The mapping of 5-bit numbers to signal constellation points are shown in the constellation diagrams. Only 8 points shown by red/undelined have two bit difference with adjacent vertical points. The rest of the points have 1-bit distance.

Therefore: $\frac{1}{\log_2 32} P(32QAM) < P_b(32QAM) < \frac{2}{\log_2 32} P(32QAM)$

$$\frac{13}{5\times 4} Q\left(\sqrt{\frac{P_r}{2N_0 R_b}}\right) < P_b(32QAM) < \frac{2\times 13}{5\times 4} Q\left(\sqrt{\frac{P_r}{2N_0 R_b}}\right)$$

Answer: $\frac{13}{20} Q\left(\sqrt{\frac{P_r}{2N_0 R_b}}\right) < P_b(32QAM) < \frac{13}{10} Q\left(\sqrt{\frac{P_r}{2N_0 R_b}}\right)$

The mapper is defined as shown in the following table:

	S1 10001	S2 10000	S3 11000	S4 11001	S5 10011	S6 00011	S7 00001	S8 01001	S9 01011	.	.	.	S29 10101	S30 10100	S31 11100	S32 11101
I	3	1	-1	-3	7	5	3	1	-1	·	·	·	3	1	-1	-3
Q	5	5	5	5	3	3	3	3	-3	·	·	·	-5	-5	-5	-5

Decision block has I_r and Q_r as input which is output of the samplers. Decision block will find $\hat{I}$ and $\hat{Q}$ which are estimates of I and Q, respectively. Decision block works as follows:

Step 1: Estimate the points of the constellation which have vertical and/or horizontal decision boundaries: as follows: If $0 < I_r < 2d$ and $0 < Q_r < 2d$, then estimate S13:00000. Else If $2d < I_r < 4d$ and $0 < Q_r < 2d$, then estimate S12:00010.Else If $4d < I_r$ and $0 < Q_r < 2d$, then estimate S11:10010. Else continue the same for following 21 points: S14, S15, S16, S17, S18, S19, S20, S21, S22, S24, S25, S26, S27, S6, S7, S8, S9, S2, S3, S30, S31

Step 2: If the decision was not made in step 1, calculate the phase $\theta = \tan^{-1}\left(\frac{Q_r}{I_r}\right)$ and make decision as follows: If $0 < \theta < \pi/4$, decide S5. Else If $\pi/4 < \theta < \pi/2$, decide S1. Else continue the same for S4, S10, S28, S32, S23, S29.

Solution 5a:

16QAM: $P_E(MQAM)=4(\frac{\sqrt{M}-1}{\sqrt{M}})Q\left(\sqrt{\left(\frac{3}{M-1}\right)\frac{E_s}{N_0}}\right)$

$$P_B(Gray)=\frac{P_E}{\log_2 M}=\frac{4}{\log_2 M}(\frac{\sqrt{M}-1}{\sqrt{M}})Q\left(\sqrt{\left(\frac{3}{M-1}\right)\frac{E_s}{N_0}}\right)=\frac{4}{\log_2 16}(\frac{\sqrt{16}-1}{\sqrt{16}})Q\left(\sqrt{\left(\frac{3}{16-1}\right)\frac{E_s}{N_0}}\right)$$

$$10^{-5}=\frac{3}{4}Q\left(\sqrt{\left(\frac{1}{5}\right)\frac{E_s}{N_0}}\right)=\frac{3}{4}Q\left(\sqrt{\frac{4E_b}{5N_0}}\right)\Rightarrow Q\left(\sqrt{\frac{4E_b}{5N_0}}\right)=\frac{4}{3}\times10^{-5}\Rightarrow\sqrt{\frac{4E_b}{5N_0}}=4.2\Rightarrow E_b=2.205\times10^{-9}$$

$P_r=E_bR_b=2.205\times10^{-9}\times20\times10^{6}=44.1\times10^{-3}$ Watts

16PSK: $P_E(MPSK)=2Q\left(\sqrt{\frac{2E_s}{N_0}}\sin\frac{\pi}{M}\right)$

$$P_B(Gray)=\frac{P_E}{\log_2 M}=\frac{2}{\log_2 M}Q\left(\sqrt{\frac{2E_s}{N_0}}\sin\frac{\pi}{M}\right)=\frac{2}{\log_2 16}Q\left(\sqrt{\frac{2E_s}{N_0}}\sin\frac{\pi}{16}\right)$$

$$10^{-5}=\frac{1}{2}Q\left(\sqrt{\frac{2E_s}{N_0}}\sin\frac{\pi}{16}\right)=\frac{1}{2}Q\left(\sqrt{\frac{8E_b}{N_0}}\sin\frac{\pi}{16}\right)\Rightarrow Q\left(\sqrt{\frac{8E_b}{N_0}}\sin\frac{\pi}{16}\right)=2\times10^{-5}\Rightarrow\sqrt{\frac{8E_b}{N_0}}\sin\frac{\pi}{16}=4.15$$

$E_b=5.65\times10^{-9}\Rightarrow P_r=E_bR_b=5.65\times10^{-9}\times20\times10^{6}=113\times10^{-3}$ Watts

16FSK: $P_E(MFSK)=(M-1)Q\left(\sqrt{\frac{E_s}{N_0}}\right)$

$$P_B(Non-Gray)=\frac{M/2}{M-1}P_E=\frac{M}{2}Q\left(\sqrt{\frac{E_s}{N_0}}\right)=\frac{16}{2}Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

$$10^{-5}=8Q\left(\sqrt{\frac{E_s}{N_0}}\right)=8Q\left(\sqrt{\frac{4E_b}{N_0}}\right)\Rightarrow Q\left(\sqrt{\frac{4E_b}{N_0}}\right)=\frac{1}{8}\times10^{-5}\Rightarrow\sqrt{\frac{4E_b}{N_0}}=4..7$$

$E_b=5.52\times10^{-10}\Rightarrow P_r=E_bR_b=5.52\times10^{-10}\times20\times10^{6}=11.04\times10^{-3}$ Watts

$$BW(MQAM)=BW(MQAM)=(1+\beta)R_s=(1+\beta)\frac{R_b}{\log_2 M}$$

$BW(16QAM)=BW(16QAM)=(1+0)\frac{20\times10^{-6}}{\log_2 16}=5\times10^{6}$ Hz

$BW(MFSK)=\frac{MR_s}{2}=\frac{MR_b}{2\log_2 M}$ $\qquad$ $BW(16FSK)=\frac{16\times20\times10^{6}}{2\log_2 16}=40\times10^{6}$ Hz

Solution 5b:

The BW of 16PAM is the same as 16QAM and 16PSK which is 5MHz.

For finding the bit error rate of 16PAM, we have to use union bound since it is not available in the formula sheet. If Δ is the distance between the constellation points, we have

$$P_E(16PAM) = \frac{1}{16}(2\times p + 14\times 2p) = \frac{15}{8}p = \frac{15}{8}Q\left(\frac{\Delta}{2\sqrt{N_0/2}}\right)$$

$$E_s = \frac{1}{16}(2\times\frac{\Delta^2}{4} + 2\times\frac{9\Delta^2}{4} + 2\times\frac{25\Delta^2}{4} + 2\times\frac{49\Delta^2}{4} + 2\times\frac{81\Delta^2}{4} + 2\times\frac{121\Delta^2}{4} + 2\times\frac{169\Delta^2}{4} + 2\times\frac{225\Delta^2}{4})$$

$$E_s = \frac{340}{16}\Delta^2 = \frac{85}{4}\Delta^2 \qquad \Delta = 2\sqrt{\frac{E_s}{85}}$$

$$P_E(16PAM) = \frac{15}{8}Q\left(\frac{\Delta}{2\sqrt{N_0/2}}\right) = \frac{15}{8}Q\left(\sqrt{\frac{2E_s}{85N_0}}\right) = \frac{15}{8}Q\left(\sqrt{\frac{2\times 4E_b}{85N_0}}\right) = \frac{15}{8}Q\left(\sqrt{\frac{2\times 4P_r}{85N_0R_b}}\right)$$

Using Gray coding: $P_b(16PAM) = \frac{P_E(16PAM)}{4} = \frac{15}{32}Q\left(\sqrt{\frac{2\times 4P_r}{85N_0R_b}}\right) = 10^{-5}$

$$Q\left(\sqrt{\frac{2\times 4P_r}{85N_0R_b}}\right) = \frac{32}{15}\times 10^{-5} \Rightarrow 2.1333\times 10^{-5} \Rightarrow \sqrt{\frac{2\times 4P_r}{85N_0R_b}} = 5.06$$

$$P_r = \frac{5.06^2\times 85N_0R_b}{8} = \frac{5.06^2\times 85\times 10^{-10}\times 20\times 10^6}{8} = 544mWatts$$

MQAM and MPSK are two dimensional, MPAM is one-dimensional but MFSK is 16 dimensional.

	Received Power [mW]	Bandwidth [MHz]	Complexity
16PSK	113	5	2
16QAM	44	5	2
16FSK	11	40	16
16PAM	544	5	1

16FSK is the best in terms of received power (or performance) and it is the worst in terms of bandwidth and complexity.

Normally, requirement is a low bandwidth and therefore 16QAM and 16PSK are right choices for modulation scheme with low complexity and low bandwidth. The received power (or performance) is worse than 16FSK but can be normally resolved with coding.

Obviously, 16QAM is much better than 16PSK because of the good received power (or performance) and therefore the choice is always 16QAM. 16PAM is the worst in performance.