CONCORDIA UNIVERSITY

Department of Mathematics & Statistics

Course	Number	Sections
Mathematics	205	All
Examination	**Date**	**Duration**
Midterm Test	05 March, 2016	1 h 30 min
Special Instructions:	Only approved calculators are allowed **Show all your work for full marks**	

1. (10 marks): **a.** Sketch the graph of the function

$$f(x) = \begin{cases} 1+\sqrt{4-x^2} & -2 \le x \le 2 \\ 3-x & 2 < x \end{cases}$$

on the interval $[-2,6]$, and find the definite integral $\int_{-2}^{6} f(x)\,dx$ in terms of area (do **not** antidifferentiate).

b. Use the Fundamental Theorem of Calculus to calculate the derivative of $F(x) = \int_0^{1-x^2} (1-t)\,e^{-t^2}\,dt$, and determine g whether F is increasing or decreasing at $x = 1$.

2. (6 marks): Find $h(x)$ if $h'(x) = \sec^2(x)\sqrt{1+\tan(x)} + \dfrac{2x}{x^2+1}$ and $h(0) = 4$.

3. (10 marks): Calculate the following indefinite integrals

$$\text{(a)} \quad \int \frac{x+1}{x^3+4x}\,dx \qquad \text{(b)} \quad \int \frac{\ln^2(x)}{x^2}\,dx$$

4. (12 marks): Evaluate the following definite integrals (*do not approximate*):

$$\text{(a)} \quad \int_0^{\frac{\ln 3}{2}} \frac{e^{2x}}{e^{4x}+9}\,dx \qquad \text{(b)} \quad \int_0^{\pi/2} \cos^5(x)\,dx$$

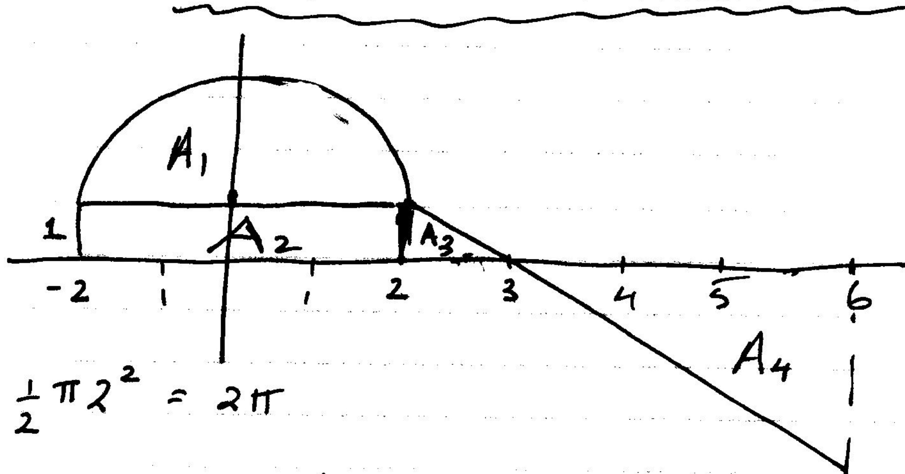
5. (6 marks): Find the area of the region bounded by the graphs of $f(x) = 4 + 2x - x^2$ and $g(x) = x^2 - 2x - 2$.

6. (6 marks): Find the average value of the function $f = \dfrac{x}{1+2x}$ on the interval $[0,4]$ (do not approximate).

Bonus. (3 marks): Given that $\int_0^{\pi} [f(x) + f''(x)] \sin x\,dx = 2$, and $f(\pi) = 1$, find $f(0)$.

Solutions to Math 205 Midterm March/16

#1 a)



$A_1 = \frac{1}{2}\pi 2^2 = 2\pi$

$A_2 = 4\times 1 = 4 \qquad A_3 = \frac{1}{2}1\cdot 1 = \frac{1}{2} \qquad A_4 = -\frac{1}{2}3\times 3 = -\frac{9}{2}$

Net area $= 2\pi + 4 + \frac{1}{2} - \frac{9}{2} = \underline{2\pi}$ Ans

b) Let $u = 1-x^2$

$$F'(x) = \frac{d}{du}\int_0^u (1-t)e^{-t^2}dt \;\frac{du}{dx}$$

$$= (1-u)e^{-u^2}\frac{du}{dx} = (1-(1-x^2))e^{-(1-x^2)^2}(-2x)$$

$$= \underline{-2x^3 e^{-(1-x^2)^2}} \quad \underline{\text{Ans}} \qquad F'(2) = -2, \text{ decreasing}$$

#2 Let $u = \tan x \Rightarrow du = \sec^2 x\, dx$

$$\Rightarrow \int \sec^2 x\sqrt{1+\tan x}\,dx = \int\sqrt{1+u}\,du.$$

Now let $v = \sqrt{1+u} \Rightarrow v^2 = 1+u \Rightarrow u = v^2-1 \Rightarrow du = 2v\,dv$

$$\Rightarrow \int\sqrt{1+u}\,du = \int v\,2v\,dv = 2\int v^2 dv = \frac{2}{3}v^3 = \frac{2}{3}(1+u)^{3/2}$$

$$= \frac{2}{3}(1+\tan x)^{3/2}$$

$$\Rightarrow h(x) = \frac{2}{3}(1+\tan x)^{3/2} + \ln(x^2+1) + C$$

$$h(0) = 4 = \frac{2}{3}\cdot 1 + \ln 1 + C \Rightarrow C = \frac{10}{3}$$

#3a $\displaystyle\int \frac{x+1}{x^3+4x}\,dx = \int \frac{x+1}{x(x^2+4)}\,dx$

$$\frac{x+1}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4} \Rightarrow x+1 = (A+B)x^2 + Cx + 4A$$
$$\Rightarrow C=1,\ A=\tfrac{1}{4},\ B=-\tfrac{1}{4}$$

$$= \frac{1}{4}\int \frac{1}{x}\,dx + \int \frac{-\frac{1}{4}x+1}{x^2+4}\,dx = \frac{1}{4}\ln|x| - \frac{1}{8}\int \frac{2x}{x^2+4}\,dx + \frac{1}{2}\int \frac{2}{x^2+4}\,dx$$

$$= \underline{\frac{1}{4}\ln|x| - \frac{1}{8}\ln(x^2+4) + \frac{1}{2}\tan^{-1}\frac{x}{2} + C}$$

#3b Let $u = (\ln x)^2$ & $dv = x^{-2}\,dx$

$\Rightarrow du = \dfrac{2\ln x}{x}$ $\quad v = -\dfrac{1}{x}$

Integration by parts $\Rightarrow$

$$\int \frac{\ln^2 x}{x^2}\,dx = -\frac{1}{x}(\ln x)^2 + 2\int \frac{\ln x}{x^2}\,dx$$

Now let $U = \ln x$, $dV = x^{-2}dx$

$\Rightarrow dU = \dfrac{1}{x}\,dx$, $\quad V = -\dfrac{1}{x}$

$$\Rightarrow \int \frac{\ln x}{x^2}\,dx = -\frac{1}{x}\ln x - \int\left(-\frac{1}{x}\right)\frac{1}{x}\,dx$$
$$= -\frac{\ln x}{x} + \int x^{-2}\,dx = -\frac{\ln x}{x} - \frac{1}{x}$$

Hence

$$\underline{\int \frac{\ln^2 x}{x^2}\,dx = -\frac{1}{x}(\ln x)^2 - 2\frac{\ln x}{x} - \frac{2}{x} + C} \qquad \text{ans}$$

MATH 205 Winter 2016 Midterm - Solutions

4 ⓐ $\displaystyle\int_0^{\ln 3/2} \frac{e^{2x}}{e^{4x}+9}\,dx$

$$\left\{\begin{array}{l} u = e^{2x} \\ du = 2e^{2x}\,dx \\ u_1 = e^{2 0} = 1 \\ u_2 = e^{2\frac{\ln 3}{2}} = e^{\ln 3} = 3 \end{array}\right.$$

//

$$\frac{1}{2}\int_1^3 \frac{du}{u^2+9} = \frac{1}{2}\cdot\frac{1}{9}\int_1^3 \frac{du}{\left(\frac{u}{3}\right)^2+1}$$

$$\begin{array}{l} s = \frac{u}{3} \\ ds = \frac{1}{3}\,du \\ s_1 = 1/3 \quad s_2 = 1 \end{array}$$

$$= \frac{1}{6}\arctan s\Big|_{1/3}^{1}$$

$$= \frac{1}{6}\left[\arctan 1 - \arctan\frac{1}{3}\right] = \underline{\underline{\frac{1}{6}\left[\frac{\pi}{4} - \arctan\frac{1}{3}\right]}}$$

$$\simeq 0.007727$$

#4 ⓑ

$$\int_0^{\pi/2} \cos^5(x)\,dx = \int_0^{\pi/2} \cos^4(x)\cos x\,dx =$$

$$= \int_0^{\pi/2}\left[1-\sin^2 x\right]^2 \cos x\,dx$$

$$\left\{\begin{array}{l} u = \sin x \\ du = \cos x\,dx \\ u_1 = \sin 0 = 0 \\ u_2 = \sin\frac{\pi}{2} = 1 \end{array}\right.$$

$$= \int_0^1 \left[1-u^2\right]^2 du = \int_0^1 (1-2u^2+u^4)\,du$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5\Big|_0^1 = 1 - 2/3 + \frac{1}{5} = \underline{\underline{8/15}}$$

MATH 205 Winter 2016 Midterm - Solutions

#5

$g(x)=f(x)$ $\quad 2x^2-4x-6=0$

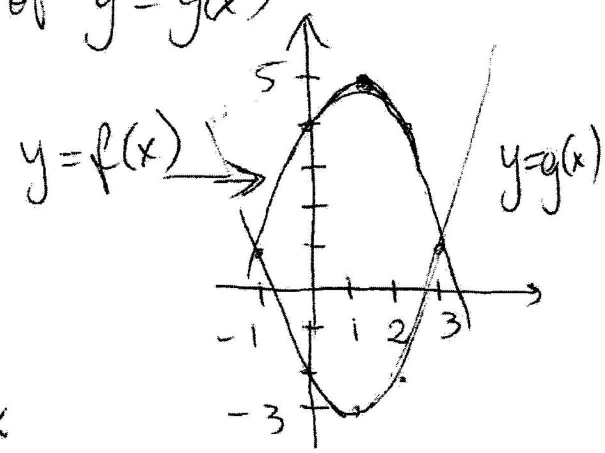
$x^2-2x-3=0$

$(x+1)(x-3)=0 \Rightarrow x=-1$ or $x=3$

$f(1)=5$ $(1,5)$ vertex

$g(1)=-3$ $(1,-3)$ vertex of $y=g(x)$

$f(x) \geq g(x)$ on $[-1,3]$



$$\text{Area} = \int_{-1}^{3} [f(x)-g(x)]\,dx$$

symmetry along $x=1$

$$= 2\int_{1}^{3} [4+2x-x^2-x^2+2x+2]\,dx$$

$$= 2\int_{1}^{3} (6+4x-2x^2)\,dx = 2\left[6\cdot 2 + 2x^2\Big|_1^3 - \frac{2}{3}x^3\Big|_1^3\right] = 2\left[12+18-2-18+\frac{2}{3}\right]$$

$$= 2\left[10+\frac{2}{3}\right] = 21\frac{1}{3} \text{ or } \frac{64}{3}$$

Math 205 Midterm Winter 2016 – Solutions

#6. Average (f) on $[a,b]$ $= \frac{1}{b-a}\int_a^b f(x)dx$

Here: $a=0 \quad b=4$

$$\int_0^4 f(x)\,dx = \int_0^4 \frac{x}{1+2x}\,dx$$

$\begin{cases} u = 1+2x \\ u-1 = 2x \\ \Rightarrow x = \frac{1}{2}[u-1] \\ du = 2\,dx \\ u_1 = 1 \quad u_2 = 9 \end{cases}$

$$= \int_1^9 \frac{\frac{1}{2}[u-1]}{u} \cdot \frac{1}{2}\,du$$

$$= \frac{1}{4}\int_1^9 \left[1-\frac{1}{u}\right] du = \frac{1}{4}\left[8 - \ln|u|\Big|_1^9\right]$$

$$= 2 - \frac{1}{4}(\ln 9 - \ln 1) = 2 - \frac{1}{4}\ln 9 \overset{\text{or}}{=}$$

$$= 2 - \frac{1}{2}\ln 3$$

$$\text{Ave}(f) = \frac{\int_0^4 f(x)\,dx}{4} = \frac{1}{4}\left[2 - \frac{1}{2}\ln 3\right]$$

$$= \underline{\underline{\frac{1}{2} - \frac{1}{8}\ln 3}} \simeq 0.36267$$

Math 205 Winter 2016 Midterm - Solutions

Bonus $\int_0^{\pi} [f(x)\sin x + f''(x)\sin x]\,dx = 2$

I: $\int_0^{\pi} f(x)\sin x\,dx$ $\left\{\begin{array}{ll} u=f(x) & dv=\sin x\,dx \\ du=f'(x)dx & v=-\cos x \end{array}\right.$

$= -f(x)\cos x \Big|_0^{\pi} + \underbrace{\int_0^{\pi} f'(x)\cos x\,dx}$

II: $\int_0^{\pi} f''(x)\sin x\,dx$ $\left\{\begin{array}{ll} u=\sin x & dv=f''(x)dx \\ du=\cos x\,dx & v=f'(x) \end{array}\right.$

$= f'(x)\sin x \Big|_0^{\pi} - \underbrace{\int_0^{\pi} f'(x)\cos x\,dx}$

$\mathrm{I} + \mathrm{II} = 2$ so

$2 = -f(\pi)\cos(\pi) + f(0)\cos 0 + f'(\pi)\underbrace{\sin\pi}_{=0} - f'(0)\underbrace{\sin 0}_{=0}$

$2 = -f(\pi)(-1) + f(0)\cdot 1$ but $f(\pi)=1$

$\Rightarrow 2 = 1 + f(0) \Rightarrow \underline{\underline{f(0) = 1}}$