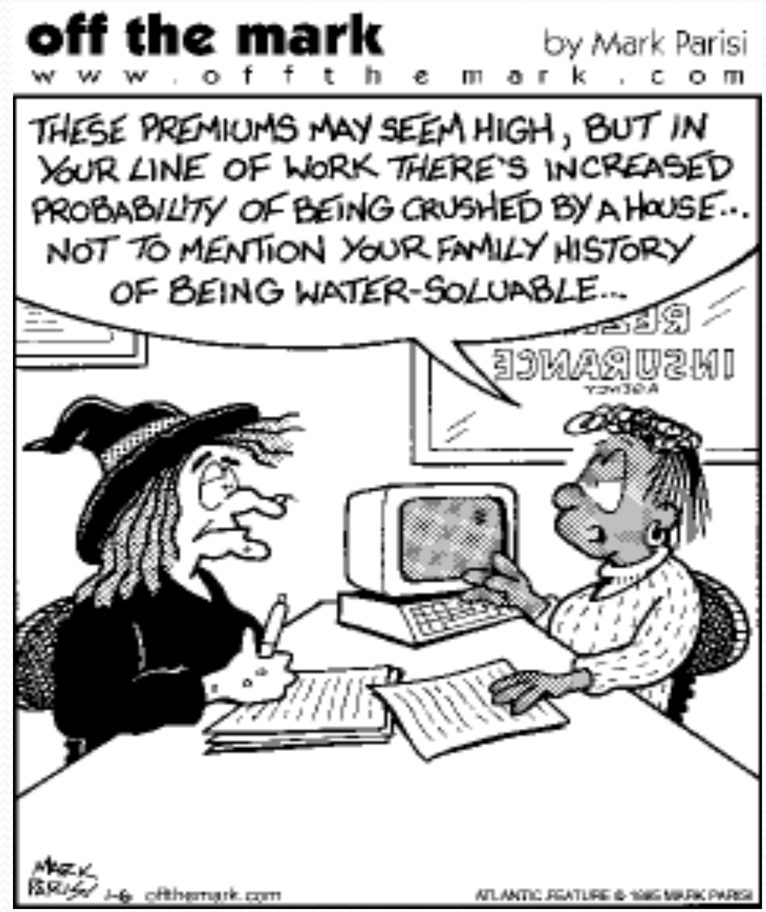
Probability



Lecture Goals

After completing this lecture, you should be able to:

- Explain basic probability concepts and definitions
- Use a Venn diagram or tree diagram to illustrate simple probabilities
- Apply common rules of probability
- Compute conditional probabilities
- Determine whether events are statistically independent

Definitions I

- An **experiment** is an occurrence we observe whose result is uncertain.
  - Example: Throw a pair of dice and then add the numbers facing up or flip a coin and record the result.

- An **outcome** is some specific aspect of the experiment that we observe.
  - Examples
    - In the throwing of a pair of dice and then adding the numbers facing up, any number from 2 to 12 eg. 7.
    - In the flipping a coin, an outcome would be heads or tails.

Definitions II

- The **sample space** for an experiment is the set of all possible outcomes. The sample space is denoted by S or U.
  - Example: In the throwing of a pair of dice and then adding the numbers facing up, the sample space is the set of all numbers from 2 to 12: S = {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

- An **event** is any subset of a sample space. Denoted by A or B or some letter assigned to that event. The outcomes in the event are called the **favourable** outcomes.

Example - Sample Space & Events

A coin is tossed three times, and the outcomes are noted. What is the event that heads comes up at least twice?

- **Step 1**: define the sample space for the three tosses of the coin
- **Step 2**: from the sample space determine the event that heads comes up at least twice

What is Probability?

Probability has been described as the following :

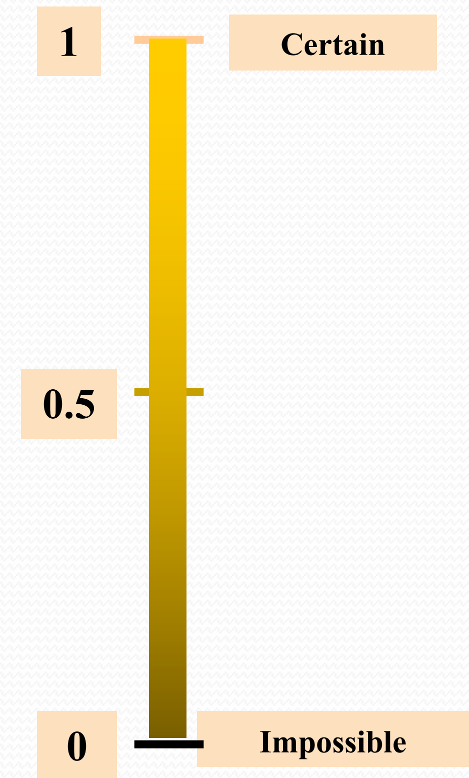
- A measure of **uncertainty**
- A measure of the **strength of belief** in the occurrence of an uncertain event
- A measure of the degree of **chance or likelihood of occurrence** of an uncertain event
- Measured by a number between 0 and 1 (or between 0% and 100%)

Probability (Cont'd)

- Probability is a numerical measure of the likelihood of an event occurring

- The formula to calculate probability is:

$$\frac{\#\text{ of favourable outcomes}}{\#\text{ of possible outcomes}}$$



Assessing Probability

- There are three approaches to assessing the probability of an uncertain event:

    1. classical probability

    2. relative frequency probability

    3. subjective probability

Classical Probability

- Assumes all outcomes in the sample space are equally likely to occur

Classical probability of event A:

$$P(A) = \frac{N_A}{N} = \frac{\text{number of outcomes that satisfy the event A}}{\text{total number of outcomes in the sample space}}$$

- Requires a count of the outcomes in the sample space

Permutations and Combinations

The number of possible orderings

- The total number of possible ways of arranging x objects in order is

$$x! = x(x-1)(x-2)\ldots(2)(1)$$

- x! is read as "x factorial"
- 0! = 1 by definition

Permutations and Combinations

(continued)

Permutations: the number of possible arrangements when x objects are to be selected from a total of n objects and arranged in order [with (n – x) objects left over]

$$P_x^n = n(n-1)(n-2)\ldots(n-x+1)$$

$$= \frac{n!}{(n-x)!}$$

Permutations and Combinations

(continued)

- Combinations: the number of possible combinations when x objects are to be selected from a total of n objects and order is not important

$$C_k^n = \frac{P_x^n}{x!} = \frac{n!}{x!(n-x)!}$$

Permutations and Combinations: Example

How many orderings of the letters A, B, C, D are possible?

- Solution: The number of possible orderings is

$$x! = 4 \times 3 \times 2 \times 1 = 24$$

- The orderings are

ABCD	ABDC	ACBD	ACDB	ADBC	ADCB
BACD	BADC	BCAD	BCDA	BDAC	BDCA
CABD	CADB	CBAD	CBDA	CDAB	CDBA
DABC	DACB	DBAC	DBCA	DCAB	DCBA

Permutations and Combinations: Example

Suppose that two letters are to be selected from A, B, C, D and arranged in order. How many permutations are possible?

- Solution: The number of permutations, with

  $n = 4$ and $x = 2$, is $$P_2^4 = \frac{4!}{(4-2)!} = 12$$

- The permutations are

  AB AC AD BA BC BD

  CA CB CD DA DB DC

Permutations and Combinations Example

(continued)

Suppose that two letters are to be selected from A, B, C, D. How many combinations are possible (i.e., order is not important)?

- Solution: The number of combinations is

$$C_2^4 = \frac{4!}{2!(4-2)!} = 6$$

- The combinations are

AB (same as BA)	BC (same as CB)
AC (same as CA)	BD (same as DB)
AD (same as DA)	CD (same as DC)

Assessing Probability

Three approaches (continued)

2. relative frequency probability

- the limit of the proportion of times that an event A occurs in a large number of trials, *n*

$$P(A) = \frac{n_A}{n} = \frac{\text{number of events in the population that satisfy event A}}{\text{total number of events in the population}}$$

3. subjective probability

an individual opinion or belief about the probability of occurrence

Expressing Probability

- Probability can be expressed in three ways:
  - Fraction
  - Proportion
  - Percentage

Fraction

$$p(heads) = \frac{1}{2} = 0.5 = 50\%$$

Percentage

Proportion

- Conventionally, probability is usually akin to proportion but all three ways are accepted

Axioms of Probability

- If A is an event, then $0 \le P(A) \le 1$
  - Probability is ALWAYS positive and ALWAYS less than or equal to 1

- $P(S) = 1$
  - The probabilities of all the outcomes sum to 1

- The probability of an event is the sum of the probabilities of the individual outcomes in the event of interest

$$P(A) = \sum_{A} P(O_i)$$

- If the probability of an event is zero, then the event is impossible

How to Calculate to Probability of an Event

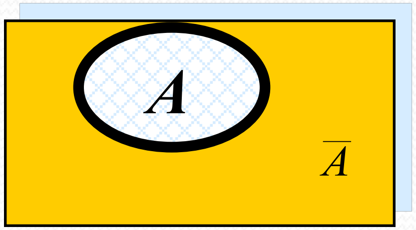
- **Step 1**: List all events in the sample space
- **Step 2**: Assign a probability to each event
- **Step 3**: Determine which event(s) is/are of interest
- **Step 4**: Sum the probabilities of the events of interest

Example – Calculating Probability

- In three tosses of a coin, calculate the probability of the event of at least two heads.

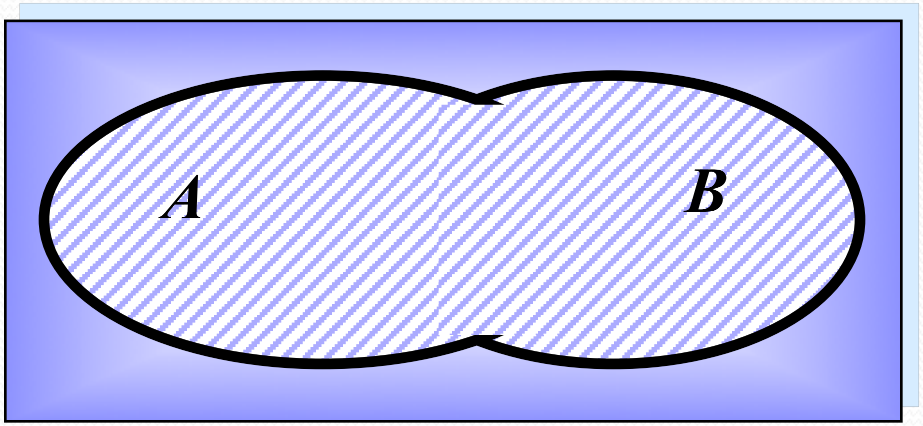
Revision of Set Theory

- A **set** is a collection of elements or objects of interest
  - Empty set (denoted by $\varnothing$)
    - a set containing no elements
  - Universal Set (denoted by U or S)
    - a set containing all possible elements
  - Subset Set (denoted by $\subset$ or $\subseteq$)
    - a set containing some or all possible elements of another set
  - Complement (Not). The complement of A is a set containing all elements not in A. This is denoted by A^c, $\overline{A}$ or A'



Probability Rules

- Union (Or)
  - a set containing all elements in either A, or B or both A and B
    - **Either**
    - **Or**
    - **At least**
    - **At most**

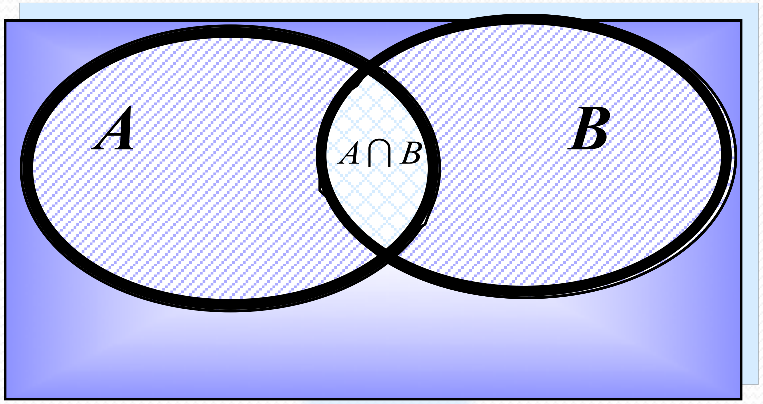


$A \cup B$

- The general addition rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Probability Rules

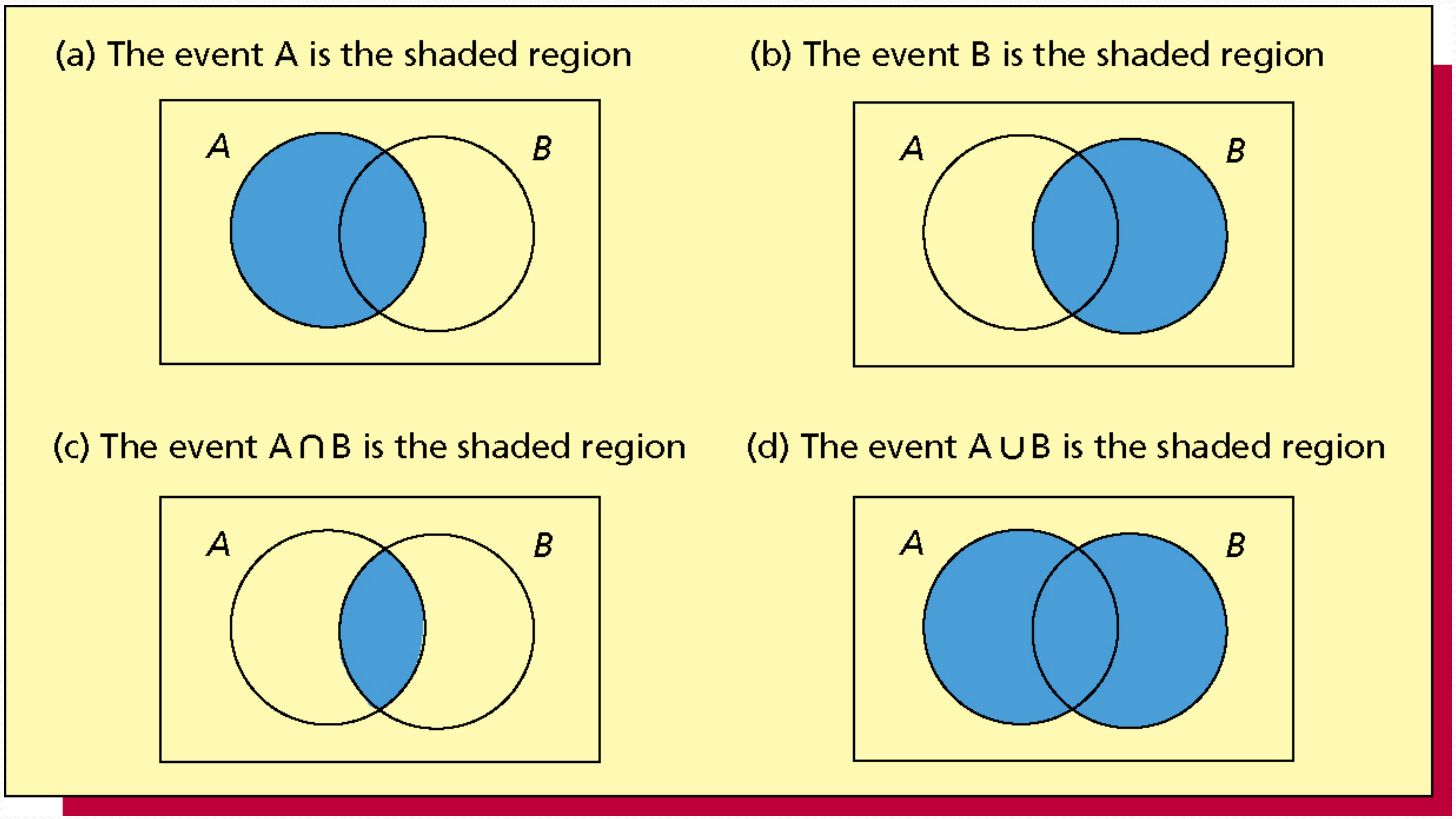
- Intersection (And)
  - a set containing all elements in both A and B
    - **Joint occurrence**
    - **Happening at the same time**
    - **Simultaneously**
    - **Both**



$A \cap B$

- General Multiplication Rule: $P(A \cap B) = P(A \mid B)P(B)$

All Together

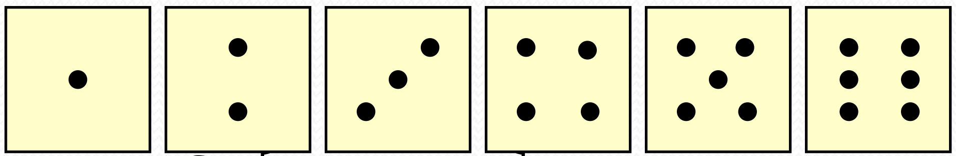


Example – Union & Intersection

- Let A = event randomly selected student does not abstain from alcohol
  - $P(A) = 0.75$
- Let B = event randomly selected student ever tried marijuana
  - $P(B) = 0.38$
- Let $P(A \cap B) = 0.37$
- Calculate the probability of A union B

Example – All Types of Events

- Let the sample space be the collection of all possible outcomes of rolling one die:



S = [1, 2, 3, 4, 5, 6]

Let A be the event "Number rolled is even"
Let B be the event "Number rolled is at least 4"
Then:
A = [2, 4, 6] and B = [4, 5, 6]

What elements are in:

1. The complement of A
2. The complement of B
3. $A \cap B$
4. $A \cup B$
5. $A \cup A'$

Definitions III

- Events E_1, E_2, … E_k are **Collectively Exhaustive** events if $E_1 \cup E_2 \cup \ldots \cup E_k = S$
  - i.e., the events completely cover the sample space

A Probability Table

Probabilities and joint probabilities for two events A and B are summarized in this table:

	B	$\overline{B}$	
A	$P(A \cap B)$	$P(A \cap \overline{B})$	$P(A)$
$\overline{A}$	$P(\overline{A} \cap B)$	$P(\overline{A} \cap \overline{B})$	$P(\overline{A})$
	$P(B)$	$P(\overline{B})$	$P(S) = 1.0$

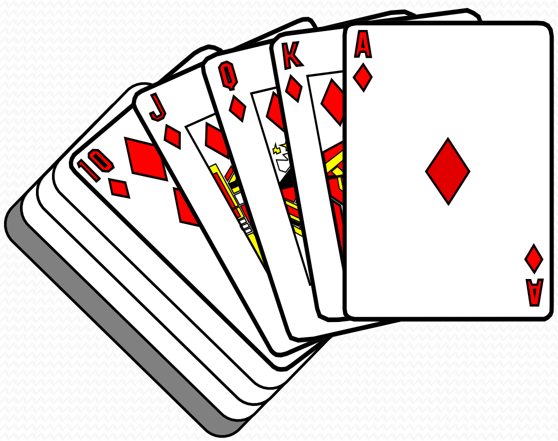
Addition Rule Example

Consider a standard deck of 52 cards, with four suits:

♥ ♣ ♦ ♠

Let event A = card is an Ace

Let event B = card is from a red suit



Addition Rule Example

(continued)

P(Red U Ace) = P(Red) + P(Ace) - P(Red ∩ Ace)

= 26/52 + 4/52 - 2/52 = 28/52

Type	Color		Total
	Red	Black	
Ace	2	2	4
Non-Ace	24	24	48
Total	26	26	52

Don't count the two red aces twice!

Types of Probabilities

- **Marginal (simple) Probability**: this is the probability that one event will occur (Ex. the probability of event A)

- If S is the sample space and *A* is the event of interest, then the marginal total is the probability of the event occurring

$$marginal\ \ probability = \frac{n(A)}{n(S)}$$

Marginal Probability Example

P(Ace)

$$= P(\text{Ace and Red}) + P(\text{Ace and Black}) = \frac{2}{52} + \frac{2}{52} = \frac{4}{52}$$

Type	Color		Total
	Red	Black	
Ace	2	2	4
Non-Ace	24	24	48
Total	26	26	52

Types of Probabilities

- **Joint Probability**: this is the probability that two or more events will occur at the same time (Ex. the probability of A and B)

- It is the intersection of two events : P(A ∩ B)

$$joint \;\; probability = \frac{n(A \cap B)}{n(S)}$$

  - Or

$$joint \;\; probability = \frac{n(A \text{ and } B)}{n(S)}$$

Joint Probability Example

P(Red and Ace)

$$= \frac{\text{number of cards that are red and ace}}{\text{total number of cards}} = \frac{2}{52}$$

Type	Color		Total
	Red	Black	
Ace	2	2	4
Non-Ace	24	24	48
Total	26	26	52

Joint Probabilities Using Contingency Table

Event	Event: B_1	Event: B_2	Total
A_1	$P(A_1 \text{ and } B_1)$	$P(A_1 \text{ and } B_2)$	$P(A_1)$
A_2	$P(A_2 \text{ and } B_1)$	$P(A_2 \text{ and } B_2)$	$P(A_2)$
Total	$P(B_1)$	$P(B_2)$	1

Joint Probabilities

Marginal (Simple) Probabilities

Types of Probabilities

- A **conditional probability** is the probability of one event, given that another event has already occurred:

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$$

The conditional probability of A given that B has occurred

$$P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)}$$

The conditional probability of B given that A has occurred

Where P(A and B) = joint probability of A and B

P(A) = marginal probability of A

P(B) = marginal probability of B

Conditional Probability Example

- Of the cars on a used car lot, 70% have air conditioning (AC) and 40% have a CD player (CD). 20% of the cars have both.

- What is the probability that a car has a CD player, given that it has AC ?

i.e., we want to find P(CD | AC)

Conditional Probability Example

(continued)

- Of the cars on a used car lot, 70% have air conditioning (AC) and 40% have a CD player (CD). 20% of the cars have both.

	CD	No CD	Total
AC	.2	.5	.7
No AC	.2	.1	.3
Total	.4	.6	1.0

$$P(CD \mid AC) = \frac{P(CD \cap AC)}{P(AC)} = \frac{.2}{.7} = .2857$$

Conditional Probability Example

(continued)

- Given AC, we only consider the top row (70% of the cars). Of these, 20% have a CD player. 20% of 70% is 28.57%.

	CD	No CD	Total
AC	.2	.5	.7
No AC	.2	.1	.3
Total	.4	.6	1.0

$$P(CD \mid AC) = \frac{P(CD \cap AC)}{P(AC)} = \frac{.2}{.7} = .2857$$

Example – Types of Probabilities

a) If a person is selected at random, what is the probability that (s)he gets a dealer who provides good service?

b) If a person randomly selects a dealer, what is the probability that (s)he gets dealer who was in business for less than 10 years and also provides good service?

c) If a person randomly selects a dealer who was in business for more than 10 years, what is the probability that (s)he gets one that provides good service?

	Quality of Service After Warranty		Total
	Good	Bad	
≥ 10	16	4	20
< 10	10	20	30
Total	26	24	50

Multiplication Rule

- Multiplication rule for two events A and B:

$$P(A \cap B) = P(A|B)P(B)$$

- also

$$P(A \cap B) = P(B|A)P(A)$$

Multiplication Rule Example

P(Red ∩ Ace) = P(Red| Ace)P(Ace)

$$= \left(\frac{2}{4}\right)\left(\frac{4}{52}\right) = \frac{2}{52}$$

$$= \frac{\text{number of cards that are red and ace}}{\text{total number of cards}} = \frac{2}{52}$$

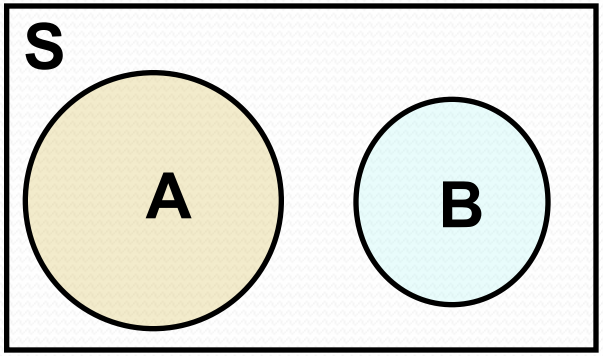
Type	Color		Total
	Red	Black	
Ace	2	2	4
Non-Ace	24	24	48
Total	26	26	52

Definitions IV

- **Mutually Exclusive** Events
  - events which have no elements in common
  - They have no intersection, $P(A \cap B) = 0$
  - they cannot occur at the same time

- If A and B are mutually exclusive then

$$P(A \cup B) = P(A) + P(B)$$



Definitions V

- **Statistical Independence**
  - events are independent if the occurrence of one event does not affect the probability of occurrence of other event(s)

- Two events are said to be statistically independent **<u>if and only if</u>**

$$P(A \cap B) = P(A) * P(B)$$

Conditional Probability & Independence

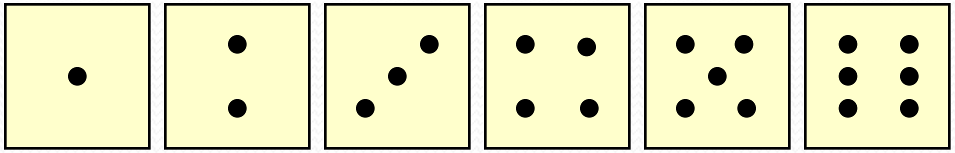
- IF and ONLY if (iff) the events are independent, then the conditional probability is equal to

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) \times P(B)}{P(B)} = P(A)$$

- Similarly, $P(B|A) = P(B)$

Example – Independent Events I

- A fair die is tossed. Let A be the event that an even number is obtained and let B be the event that a number less than or equal to 4 is obtained. Show that A and B are independent.



Example – Independent Events II

- Let S be the sample space for the toss of three fair coins.
  - S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}
- Let A be the event that heads comes up exactly once;
  - A = {HTT, THT, TTH}
- and let B be the event that the first coin comes up heads;
  - B = {HHH, HHT, HTH, HTT}.

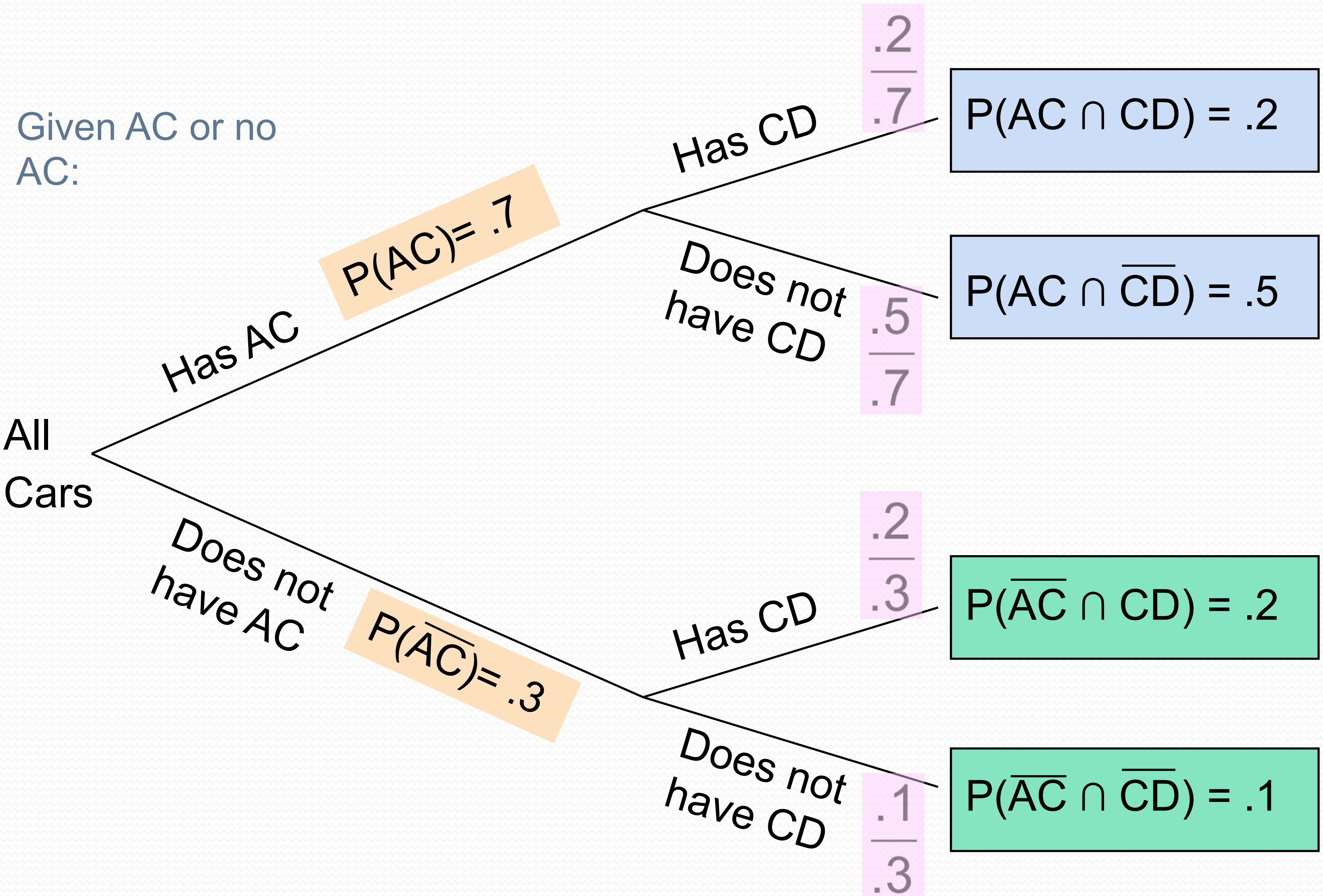
- Check these two events for independence.

Example – Independent Events III

- Using the example on slide 29, determine whether the events AC and CD are statistically independent.

Using a Tree Diagram

Given AC or no AC:



Lecture Summary

- Defined basic probability concepts
  - Sample spaces and events, intersection and union of events, mutually exclusive and collectively exhaustive events, complements
- Examined basic probability rules
  - Complement rule, addition rule, multiplication rule
- Defined conditional, joint, and marginal probabilities
- Defined statistical independence