

Examination	Instructor	Date	Time	Pages
Midterm	M. Bertola	October 2015	1hr 15min	1

**Special Instructions:** Explain carefully but not verbosely your steps!  
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**Problem 1** (10 pts). (a) Write the negation of the sentence

$$\forall \epsilon > 0 \exists N_0 \in \mathbb{N} : \forall n, m \geq N_0 (|x_n - x_m| < \epsilon) \vee (x_n x_m = 0)$$

(b) Prove that for any sets  $A$  and  $B$  we have  $A = (A \cap B) \cup (A \setminus B)$ .

**Problem 2** (10 pts). Use the “squeeze” theorem to show

$$\lim_{n \rightarrow \infty} \sqrt[n]{5^n + 2^n - \sin(n^2)} = 5$$

You can use without proof the fact that, for every  $n \in \mathbb{N}$ , the function  $f(x) = \sqrt[n]{x}$  is increasing.

**Problem 3** (10 pts). The Fibonacci sequence is defined by induction as follows

$$x_1 := 1, \quad x_2 := 1, \quad x_n = x_{n-1} + x_{n-2}$$

The first few ones are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, ... Prove that  $\forall k \in \mathbb{N}$ , the element  $x_{3k}$  is even.

**Problem 4** (10 pts). Prove that  $\sqrt{3} + \sqrt{2}$  is not rational. Prove that for any  $a, b \in \mathbb{Q}$  the number  $x = a\sqrt{3} + b\sqrt{2}$  is not rational, except for the case  $a = b = 0$ . You do not need to prove that  $\sqrt{2}$  and  $\sqrt{3}$  (separately) are not rational. Explain carefully all your logic.

**Problem 5** (10 pts). Let  $f$  and  $g$  be two functions from a set  $X$  to the interval  $[1, \infty)$ . Let  $h(x) = f(x)g(x)$  be the product function and  $r(x) = \frac{f(x)}{g(x)}$  be the quotient function. Recall that the sup or inf of a function means the sup or inf of its image. Prove that; (1)  $\sup h \leq (\sup f)(\sup g)$ ; (2)  $\sup r \leq \frac{\sup f}{\inf g}$ .

**Problem 6** (10 pts). Show that the set of the roots of polynomials with integer coefficients is countable.

**Problem 7** (Bonus, 5pts). Suppose that  $(x_n)_{n \in \mathbb{N}}$  is a convergent sequence of real numbers. Prove that it satisfies the following property (**Cauchy**)

$$\forall \epsilon > 0 \exists N_0 \in \mathbb{N} : \forall n, m \geq N_0 |x_n - x_m| < \epsilon.$$

**Solution of Problem 1**

(a)

$$\exists \epsilon > 0 : \forall N_0 \in \mathbb{N} \exists n, m \geq N_0 : (|x_n - x_m| \geq \epsilon) \wedge (x_n x_m \neq 0)$$

(b) Let  $\alpha = (x \in A)$  and  $\beta = (x \in B)$ . Then

$$(A \cap B) \cup (A \setminus B)$$

correspond to the truth of  $(\alpha \wedge \beta) \vee (\alpha \wedge \neg\beta)$ . Thus the proof follows if we prove the equivalence  $\alpha \leftrightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \neg\beta)$ . This is best done with a **table of truth values** as in the exercises. (I won't do it here)

**Alternatively:** If  $x \in A \cap B$  then  $x \in A$ ; if  $x \in A \setminus B$  then  $x \in A$ . Thus the RHS is contained in  $A$ . Viceversa, if  $x \in A$  then it either belongs to  $B$  or not. If it does then  $x \in A \cap B$ , if it doesn't then  $x \in A \setminus B$ . This proves the reverse inclusion and concludes the proof.

**Solution of Problem 2**

Note that  $-1 \leq \sin(n^2) \leq 1$  and hence  $\sin(n^2) \leq 5^n$ . Similarly  $1 \leq 2^n \leq 5^n$ . Thus

$$0 \leq 2^n + \sin(n^2) \leq 2 \cdot 5^n.$$

$$5 = \sqrt[n]{5^n} \leq \sqrt[n]{5^n + 2^n - \sin(n^2)} \leq \sqrt[n]{5^n + 2 \cdot 5^n} = 5 \sqrt[n]{3}$$

Noticing that  $\lim_{n \rightarrow \infty} 3^{\frac{1}{n}} = 1$  and using the theorem on the product of limits, we have from the squeeze theorem

$$5 = \lim_{n \rightarrow \infty} \sqrt[n]{5^n} \leq \lim_{n \rightarrow \infty} \sqrt[n]{5^n + 2^n - \sin(n^2)} \leq 5 \lim_{n \rightarrow \infty} 3^{\frac{1}{n}} = 5$$

**Solution of Problem 3**

By induction:  $x_3 = 1 + 1 = 2$  is even. Suppose that  $x_{3k}$  is even: then

$$x_{3(k+1)} = x_{3k+3} = x_{3k+2} + x_{3k+1} = x_{3k+1} + x_{3k} + x_{3k+1} = x_{3k} + 2x_{3k+1}$$

Thus  $x_{3(k+1)}$  is the sum of two even numbers, hence even. (It should be noted that the sequence is constituted of integers, but I take this as self-evident, but it can be observed immediately by induction since  $x_{n+1} = x_n + x_{n-1}$  is an integer if  $x_\ell$ ,  $\ell \leq n$  are).

**Solution of Problem 4.**

We use that  $\sqrt{6}$  is not rational. Then suppose that  $x = a\sqrt{2} + b\sqrt{3}$  is rational (by contradiction):

$$\mathbb{Q} \ni (a\sqrt{2} + b\sqrt{3})^2 = 2a^2 + 3b^2 + 2ab\sqrt{6}$$

If  $ab \neq 0$  then we have

$$\sqrt{6} = \frac{x^2 - 2a^2 - 3b^2}{2ab}$$

and the right side is a rational expression involving rational numbers, hence rational. This contradicts that  $\sqrt{6}$  is irrational. The case  $a = 1 = b$  proves the first assertion of the exercise.

For the case  $a = 0 \neq b$  we have  $x = b\sqrt{3}$ . If  $x \in \mathbb{Q}$  then  $\sqrt{3} = \frac{x}{b} \in \mathbb{Q}$ , which is a contradiction (we can use that  $\sqrt{3} \notin \mathbb{Q}$ ).

Similarly for the case  $a \neq 0 = b$  we have  $x = a\sqrt{2}$ . If  $x \in \mathbb{Q}$  then  $\sqrt{2} = \frac{x}{a} \in \mathbb{Q}$ , which is a contradiction (we can use that  $\sqrt{2} \notin \mathbb{Q}$ ).

The case  $a = 0 = b$  yields clearly  $x = 0\sqrt{2} + 0\sqrt{3} = 0 \in \mathbb{Q}$ .

**Solution of Problem 5** Since  $f, g$  take values in  $[1, \infty)$  then  $\sup f \geq 1 \leq \sup g$  and also  $\inf f \geq 1 \leq \inf g$ . In particular all of these numbers are **positive**.

1. Now

$$f(x)g(x) \leq (\sup f)g(x) \leq (\sup f)(\sup g)$$

where the first is valid because  $g(x) > 0$  and the last inequality is valid only because  $\sup f > 0$ . In particular  $\sup f \sup g$  is an upper bound for the range of  $h(x) = f(x)g(x)$ . Thus  $\sup h \leq \sup f \sup g$ .

2. From  $g(x) \geq \inf g \geq 1 > 0$  we have  $0 < \frac{1}{g(x)} \leq \frac{1}{\inf g} \leq 1$ . Then

$$0 < r(x) = \frac{f(x)}{g(x)} \leq \frac{f(x)}{\inf g} \leq \frac{\sup f}{\inf g}$$

The inequalities are valid as stated because all the quantities are strictly positive. The RHS is an upper bound of the image of  $r(x)$  and hence the  $\sup r$  of the LHS yields  $\sup r \leq \frac{\sup f}{\inf g}$ .

**Solution of Problem 6.** This seems to have been a curved ball, but I can't fathom why.

Let

$$P_{n,\mathbb{Z}} := \{p : p(x) = \sum_{j=0}^n a_j x^j : \forall 0 \leq j \leq n \quad a_j \in \mathbb{Z} \quad a_n \neq 0\}$$

denote the set of all polynomials of degree  $n$  with integer coefficients. There is a bijection between  $P_{n,\mathbb{Z}}$  and the set

$$\phi_n(p) : P_{n,\mathbb{Z}} \rightarrow \underbrace{\mathbb{Z} \times \mathbb{Z} \cdots \mathbb{Z}}_{n\text{-times}} \times \mathbb{Z}^*$$

given by

$$\phi_n(p) = (a_0, \dots, a_{n-1}, a_n)$$

(any polynomial is uniquely determined by its coefficients). Here  $\mathbb{Z}^*$  denotes the nonzero integers: it is a subset of  $\mathbb{Z}$  and hence countable. Since the domain of  $\phi$  is a *finite cartesian product of countable sets*, by the eponymous theorem it is countable.

Thus, for each  $n \in \mathbb{N}$  the set  $P_{n,\mathbb{Z}}$  is countable. Thus the set of all polynomials with integer coefficients is

$$P_{\mathbb{Z}} := \bigcup_{n \in \mathbb{N}} P_{n,\mathbb{Z}}$$

is the *countable union of countable sets*, and hence countable.

For each  $p \in P_{\mathbb{Z}}$  denote  $\mathcal{R}(p) := \{x : p(x) = 0\}$  the roots of the polynomial; by the fundamental theorem of algebra (given in class as a reminder), it has at most  $\deg p$  elements, and it is thus **finite**.

Now the set of all roots of all polynomials with integers coefficients (denoted here  $\mathcal{A}$ ) is

$$\mathcal{A} = \bigcup_{p \in P_{\mathbb{Z}}} \mathcal{R}(p)$$

This is the countable union of finite sets, hence countable by the same theorem used above.

**Solution of Problem 7.** By definition of convergent

$$\exists L \in \mathbb{R} : \forall \frac{\epsilon}{2} > 0 \quad \exists N_0 \in \mathbb{N} : \forall n \geq N_0 \quad |x_n - L| < \frac{\epsilon}{2}$$

Let  $\epsilon > 0$  be now arbitrary and let  $L, N_0$  as indicated here above. Then  $\forall n, m \geq N_0$  we have, by the triangle inequality ( $|A + B| \leq |A| + |B|$ )

$$|x_n - x_m| = |x_n - L + L - x_m| \leq |x_n - L| + |L - x_m| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$