STUDENT NUMBER: ____________________

HAND IN

Answers recorded on exam paper.

DEPARTMENT OF MATHEMATICS AND STATISTICS

QUEEN'S UNIVERSITY AT KINGSTON

DEC 2015

APSC 171

A. Ableson, P. Li, C. Rozins

INSTRUCTIONS:

- Answer **all questions**, writing clearly in the space provided. If you need more room, continue your answer on the **blank pages at the back of the exam**, providing clear directions to the marker.
- For full marks, you must show all your work and explain how you arrived at your answers, unless explicitly told to do otherwise.
- Only **CASIO FX-991**, **Gold Sticker** or **Blue Sticker** calculators are permitted.
- Write your student number **clearly** at the top of each page.
- You have three hours to complete the examination.
- Wherever appropriate, **include units in your answers**.
- When drawing graphs, **add labels and scales on all axes**.

PLEASE NOTE: Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer questions as written.

I	II	III	IV	V	VI	VII	VIII	IX	X	Total
9	6	6	7	7	6	8	6	6	9	70

STUDENT NUMBER:

I -Paths

1. Would two particles on the paths $\vec{u}(t) = \langle e^{t+1}, t\rangle$ and $\vec{v}(t) = \langle e^{2t}, t^2\rangle$ collide? If so, indicate the time and location of the collision. *[/3]*

collide: same location (x, y) at same time.

x: $e^{t+1} = e^{2t}$ (1)

y: $t = t^2$

$0 = t^2 - t = t(t-1) \Rightarrow t=0$ or $t=1$

Sub into (1): $t=0$: $e^1 = e^0$ ✗ not a solution for x

$t=1$: $e^2 = e^2$ ✓ t=1 is a solution for x and y

$\Rightarrow$ collide at $t=1$, at position $\langle e^2, 1\rangle$

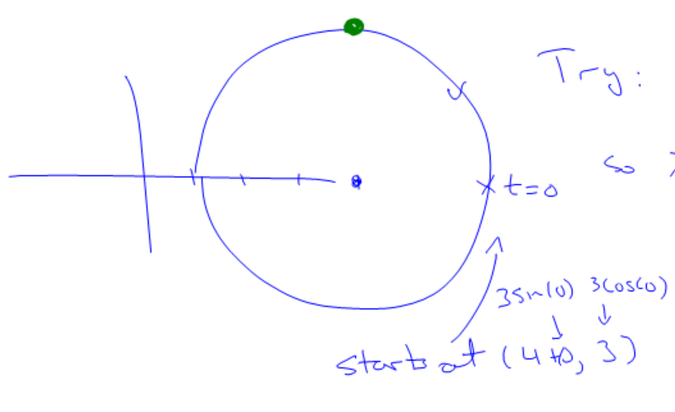
2. Eliminate the t parameter in $\vec{w}(t) = \langle e^t - 1, e^{3t}\rangle$ to find a Cartesian equation for the graph of $\vec{w}(t)$. Put your answer in the form $y = \ldots$, and simplify your answer using any appropriate log/exponential/power rules. *[/3]*

$x = e^t - 1 \Rightarrow x + 1 = e^t \Rightarrow t = \ln(x+1)$ (1)

(2) $y = e^{3t} \xrightarrow{(1)} y = e^{3\ln(x+1)} = \left(e^{\ln(x+1)}\right)^3 = (x+1)^3$

so $y = (x+1)^3$

3. Define a position function $\vec{r}(t)$ (position in meters, t in seconds) that traces the circle in the xy plane with radius 3 m and center at (4, 0), with constant speed 5 m/s. The particle should move in a **clockwise** direction, and at $t = 0$ be at the point (4, 3). *[/3]*



Try: $\vec{r}(t) = \langle 4 + 3\sin(bt), 3\cos(bt)\rangle$

so $\vec{r}\,'(t) = \langle 3b\cos(bt), 3b\sin(bt)\rangle$

so speed $= 3b = 5$ m/s

set so $\boxed{b = 5/3}$

$\vec{r}(t) = \langle 4 + 3\sin(\frac{5}{3}t), 3\cos(\frac{5}{3}t)\rangle$

STUDENT NUMBER:

II - Implicit Derivatives. The relation

$$x^2 - 2xy + y^2 + 6x - 10y + 29 = 0 \quad (1)$$

defines a parabola.

(a) Find the point(s) where the curve has a horizontal tangent line. *[/3]*

Take Implicit Deriv: $\frac{d}{dx}\left(x^2 - 2xy + y^2 + 6x - 10y + 29\right) = \frac{d}{dx}(0)$

(2) $2x - 2y - 2x\frac{dy}{dx} + 2y\frac{dy}{dx} + 6 - 10\frac{dy}{dx} = 0$

Want tgt line horizontal $\Rightarrow \frac{dy}{dx} = 0$ $\quad 2x - 2y + 6 = 0$

$\boxed{x = y - 3}$

Sub back into (1) $(y-3)^2 - 2(y-3)y + y^2 + 6(y-3) - 10y + 29 = 0$

$y^2 - 6y + 9 - 2y^2 + 6y + y^2 + 6y - 18 - 10y + 29 = 0$

$-4y + 20 = 0 \quad \boxed{y = 5}$

and $\boxed{x = y - 3 = 2}$ at $(2, 5)$, the tangent line is horizontal.

(b) Find the formula for the tangent line to the parabola at the point (5, 6). *[/2]*

at $(5,6)$, (2) $\Rightarrow 2(5) - 2(6) - 2(5)y' + 2(6)y' - 4 = 0$

$-2 + 2y' - 4 = 0 \quad y' = 3$

so tgt line is $y = 3(x-5) + 6$

(c) Using your answer to part (b), estimate a value of y on the graph at $x = 4.95$. *[/1]*

at $x = 4.95$, $y = 3(4.95 - 5) + 6$

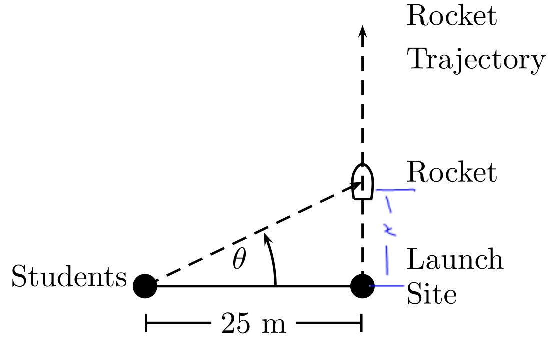
$y = -0.15 + 6$

$\boxed{y = 5.85}$

STUDENT NUMBER:

III - Rocket Launch. A student team is launching a model rocket. The rocket goes straight up from its launch site, and is tracked by the team, which is standing 25 m away from the launch site.
When the rocket reaches the point where the angle between the rocket appears to be at an angle of $\frac{\pi}{3}$ radians above the horizon, the students measure that the angle is increasing at a rate of 0.25 rad/s. At what rate, in m/s, is the rocket rising at this moment?
Note: you must define any new variables you introduce, either in words on on the diagram.

[/6]



have $\frac{d\theta}{dt} = 0.25$ rad/s

want $\frac{dx}{dt}$.

$\tan(\theta) = \frac{x}{25}$

Take $\frac{d}{dt}$ of both sides

$\frac{d}{dt}\tan(\theta) = \frac{d}{dt}\frac{1}{25}x$

$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{25}\frac{dx}{dt}$

sub in $\theta = \pi/3$, $\frac{d\theta}{dt} = 0.25$

$\frac{1}{\cos^2(\pi/3)} 0.25 = \frac{1}{25}\frac{dx}{dt}$

$\frac{1}{\cos^2(\pi/3)} = 1/(1/2)^2 = 2^2$

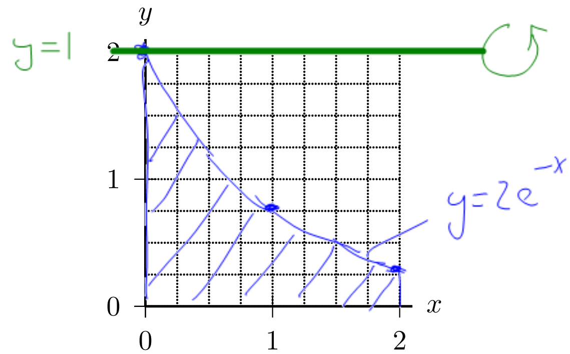
$25 = \frac{dx}{dt}$

The rocket is rising at 25 m/s

STUDENT NUMBER:

IV - Volumes and Work

1. Consider the region R bounded by the y axis and x axis, the graph of $y = 2e^{-x}$ and $x = 2$.
(a) Sketch the region on the axes to the right. *[/1]*



(b) Consider the solid generated by rotating the region R around the line $y = 2$. Write down a definite integral whose value is the volume of this solid. You do **not** need to evaluate this integral. *[/3]*

Disks r_{in}, $r_{out} = 2$

$$\int_{x=0}^{2} \left(\pi r_{out}^2 - \pi r_{in}^2\right) dx = \int_0^2 \left[\pi 2^2 - \pi\left(2 - 2e^{-x}\right)^2\right] dx$$

OR Shells $r = 2-y$, $h = x$ on $y = 2e^{-x}$, $h = -\ln(y/2)$

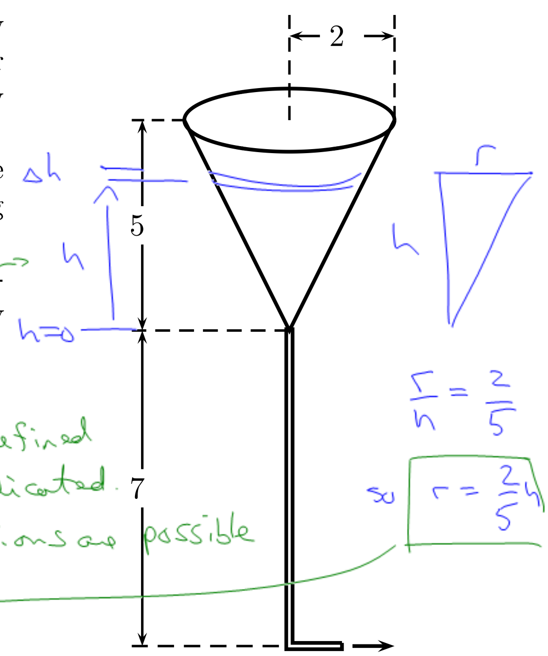
Harder as there are two boundaries

$h = 2$

$$\int_{y=0}^{y=2e^{-2}} 2\pi(2-y)\cdot 2 \cdot dy + \int_{y=2e^{-2}}^{y=2} 2\pi(2-y)\left(-\ln(y/2)\right) dy$$

($2\pi r$, h, Thickness)

2. A conical reservoir 5 m tall has radius 2 m at the top, and an outflow pipe that terminates 7 m below the bottom of the cone. The reservoir starts completely full of water. (The pipe under the reservoir is narrow enough that its water contents are negligible.)
Write an integral that represents the amount of work that could be done (in J) by letting the water in the top 3 m of the tank drain out (leaving water in the bottom 2 m). You do **not** need to evaluate this integral. Clearly define any variables you introduce, either in words or on the diagram. The density of water is $\rho = 1000$ kg/m^3; acceleration due to gravity is $g = 9.8$ m/s^2.



(Not to scale.) *[/4]*

$\frac{r}{h} = \frac{2}{5}$ so $r = \frac{2}{5}h$

work done by one slice = F · (fall dist)
$= mg(h+7)$ with h defined as indicated. Other options are possible
$= V \cdot \rho \cdot g(h+7)$
$= \pi r^2 \Delta h \cdot \rho \cdot g \cdot (h+7)$
$= \pi\left(\frac{2}{5}h\right)^2 \Delta h \, \rho g (h+7)$

$$\text{Total work} = \int_{h=2}^{h=5} \pi\left(\tfrac{2}{5}h\right)^2 \rho g (h+7)\, dh$$

STUDENT NUMBER:

V - Integration 1

(a) Evaluate the integral $\int \frac{x^2-4}{x^2(3-x)}\,dx$ [/4]

Partial fractions: $\frac{x^2-4}{x^2(3-x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{3-x}$

or $x^2-4 = Ax(3-x) + B(3-x) + C(x^2)$

$x=0: \quad -4 = B(3) \qquad B = -4/3$

$x=3: \quad 9-4 = C(9) \qquad C = 5/9$

x^2 coeffs: $\quad 1 = -A + C \qquad A = C - 1 = 5/9 - 1 = -4/9$

so $\int \frac{x^2-4}{x^2(3-x)}\,dx = \int \left(\frac{-4}{9}\frac{1}{x} - \frac{4}{3}\frac{1}{x^2} + \frac{5}{9}\frac{1}{(3-x)}\right)dx$

$= -\frac{4}{9}\ln|x| + \frac{4}{3}x^{-1} - \frac{5}{9}\ln|3-x| + C$

note sign change, from $(-x)$ in denom./chain rule

(b) Evaluate the integral $\int \frac{3x}{(1+x^2)^4}\,dx$. [/3]

Substitution: let $u = 1+x^2$

so $\frac{du}{dx} = 2x \rightarrow \frac{1}{2x}\,du = dx$

Rewriting integral $\int \frac{3x}{(1+x^2)^4}\,dx = \int 3x \frac{1}{u^4}\left(\frac{1}{2x}\,du\right) = \frac{3}{2}\int \frac{1}{u^4}\,du$

$= 3/2 \, \frac{u^{-3}}{-3} + C$

$= \frac{-1}{2}\frac{1}{(1+x^2)^3} + C$

STUDENT NUMBER:

VI - Miscellaneous

(a) Evaluate the integral $\int_0^5 \frac{1}{(x-1)^3}\,dx$. [/3]

Careful: $\frac{1}{(x-1)^3}$ undefined at $x=1$, so this integral is improper.

Break integral at discontinuity, $x=1$: $= \int_0^1 (x-1)^{-3}\,dx + \int_1^5 (x-1)^{-3}\,dx$

Look at either integral first. We'll start w/ $\int_0^1 (x-1)^{-3}\,dx$

defn for improper integral: $\lim_{b\to 1^-} \int_0^b (x-1)^{-3}\,dx = \lim_{b\to 1^-} \left.\frac{(x-1)^{-2}}{-2}\right|_0^b = \lim_{b\to 1^-} -\frac{1}{2}\left(\frac{1}{(b-1)^2} - \frac{1}{(0-1)^2}\right)$ ($\frac{1}{(b-1)^2} \to \infty$ as $b\to 1$)

This limit is undefined so:

$\int_0^1 (x-1)^{-3}\,dx$ diverges, so the original $\int_0^5 (x-1)^{-3}\,dx$ also diverges

(b) Evaluate the limit $\lim_{x\to\infty} \frac{x}{\ln(x)}$. [/3]

$= \frac{\infty}{\infty}$ indeterminate

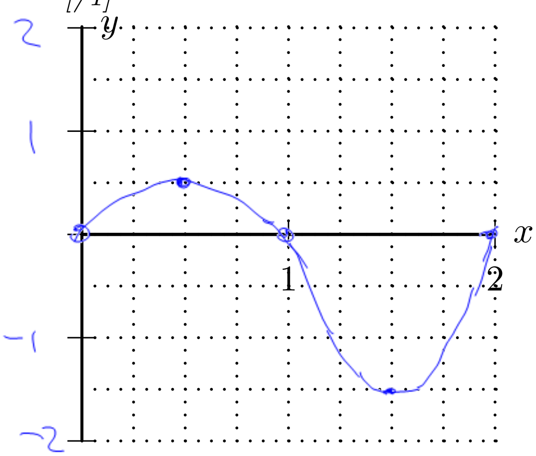
Try l'Hôpital's rule: $= \lim_{x\to\infty} \frac{1}{1/x} = \lim_{x\to\infty} x \to \infty$

The limit does not exist.

STUDENT NUMBER:

VII - Integration 2

(a) Sketch the graph of the function $f(x) = x\sin(\pi x)$. *[/1]*



x	$x\sin(\pi x)$
0	0
0.5	0.5
1	0
1.5	−1.5
2	0

(b) Based only on your sketch, what is the sign of $\int_0^2 x\sin(\pi x)\,dx$? *[/1]*

Negative: the region below the x-axis is larger than the region above

(c) Estimate the value of the integral $\int_0^2 x\sin(\pi x)\,dx$ using Simpson's rule, and 4 intervals. *[/3]*

Using the table of values from the sketch:

w/ $\Delta x = \frac{2}{4} = \frac{1}{2}$

$$\int_0^2 x\sin(\pi x)\,dx \cong \frac{(1/2)}{3}\cdot\left(1\cdot 0 + 4\cdot 0.5 + 2\cdot 0 + 4(-1.5) + 1\cdot 0\right)$$

$\frac{\Delta x}{3}$

$$= \frac{1}{6}\left(2 + (-6)\right) = \frac{-2}{3} \cong 0.6667$$

(d) Find the exact value of the integral $\int_0^2 x\sin(\pi x)\,dx$ using the Fundamental Theorem of Calculus. *[/3]*

By parts: let $u = x$, $dv = \sin(\pi x)\,dx$

so $du = dx$, $v = \frac{-\cos(\pi x)}{\pi}$

$$= x\left(\frac{-\cos(\pi x)}{\pi}\right)\Bigg|_0^2 - \frac{-1}{\pi}\int \cos(\pi x)\,dx$$

$$= -\frac{x}{\pi}\cos(\pi x) + \frac{1}{\pi}\left(\frac{1}{\pi}\sin(\pi x)\right)\Bigg|_0^2$$

$$= \left(\frac{-2}{\pi}\underbrace{\cos(2\pi)}_{=1} + \frac{1}{\pi^2}\underbrace{\sin(2\pi)}_{=0}\right) - \left(0 + \frac{1}{\pi^2}\underbrace{\sin(0)}_{=0}\right) = \frac{-2}{\pi} \cong -0.6366$$

STUDENT NUMBER:

VIII - Differential Equations. Consider the differential equation $\dfrac{dy}{dt} = 0.5(3y - 200)$.

(a) Find the general solution to this differential equation. *[/4]*

Separate and Integrate $\displaystyle\int \frac{1}{3y-200}\,dy = \int 0.5\,dt$

$\frac{1}{3}\ln|3y-200| = 0.5t + C$

Solving for y: exponentiate $e^{\ln|3y-200|} = e^{(1.5t+3C)}$

$|3y-200| = e^{1.5t}\cdot e^{3C}$ Let $A = \pm e^{3C}$

$3y - 200 = Ae^{1.5t}$

$3y = Ae^{1.5t} + 200$

$$y = \frac{A}{3}e^{1.5t} + \frac{200}{3}$$

(b) Verify whether the proposed solution you found in part (a) is in fact a solution to the original DE. *[/1]*

$LHS = y' = \left(\frac{A}{3}e^{1.5t} + \frac{200}{3}\right)' = \frac{A}{3}(1.5)e^{1.5t} = \frac{A}{2}e^{1.5t}$

$RHS = 0.5(3y-200) = 0.5\left(3\left(\frac{A}{3}e^{1.5t} + \frac{200}{3}\right) - 200\right) = 0.5Ae^{1.5t}$

LHS = RHS when $y = A/3\, e^{1.5t} + \frac{200}{3}$ so the solution is correct

(c) Find any constant solution(s) to the differential equation. *[/1]*

constant solution $\Rightarrow y = k \Rightarrow y' = 0$

Sub into DE: $0 = 0.5(3k - 200)$

$0 = 3k - 200$

$200 = 3k$

$k = \frac{200}{3}$ so $y = \frac{200}{3}$ is a constant solution

STUDENT NUMBER:

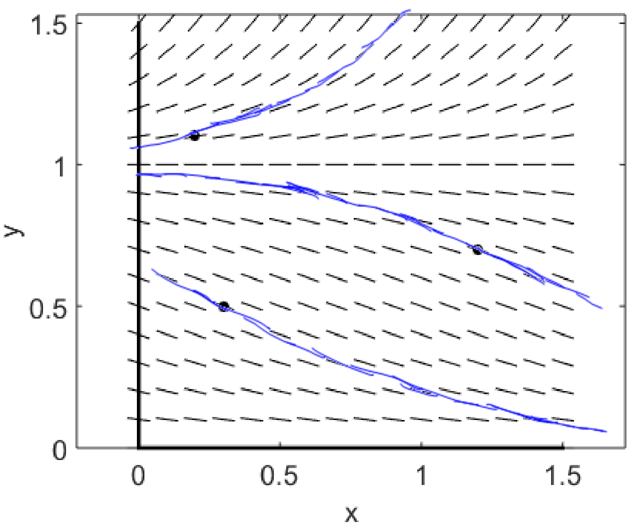
IX - Differential Equations 2.

1. (a) Circle the equation below that is associated with the slope field shown to the right. The constant k is positive.

$y' = ky(y-1)$ $\qquad$ $y' = ky^2$

y=0, y=1 ⇒ y'=0 / horiz'l slopes
and
y > 1 ⇒ y' pos ⇒ upwards slopes

$y' = ky(1-y)$ $\qquad$ $y' = -ky^2$ $\qquad$ *[/1]*



(b) On the slope field, sketch the solutions to the DE that pass through the three points indicated by dots. Your solutions should be drawn on the entire x range shown. *[/2]*

2. Consider a frictionless spring/mass system, which starts at rest from an initial displacement of 1 cm from equilibrium, and then is released to oscillate.
If you double the mass at the end of the same spring, specify whether each of the following properties of the oscillation will change or not, and if so, by how much. Support your answers with a brief argument or calculation. *[/3]*

(a) The period.

Will change. Since $\omega = \sqrt{\frac{K}{m}}$,
and period $= \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{K}}$, doubling m will make period increase by a factor of $\sqrt{2}$.

(b) The frequency.

Will change. Since $\omega = \sqrt{\frac{K}{m}}$, doubling m will decrease freq
is a frequency (rad/s)
Multiplier will be $\frac{1}{\sqrt{2}}$

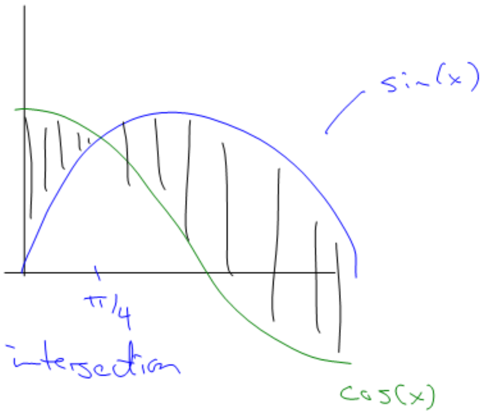
(c) The amplitude.

Will not change. If the initial displacement is the same 1cm, the oscillations will remain at 1cm of amplitude.

STUDENT NUMBER:

X - Areas and Averages.

1. Find the area enclosed between the graphs of $y = \cos(x)$ and $y = \sin(x)$ on the interval $x \in [0, \pi]$. *[/4]*



$$\text{Area enclosed} = \int_0^{\pi/4} \cos(x) - \sin(x)\, dx + \int_{\pi/4}^{\pi} \sin(x) - \cos(x)\, dx$$

$$= \sin(x) - (-\cos(x)) \Big|_0^{\pi/4} + (-\cos(x) - \sin(x)) \Big|_{\pi/4}^{\pi}$$

$$= \left(\sin(\tfrac{\pi}{4}) + \cos(\tfrac{\pi}{4})\right) - \left(\sin(0) + \cos(0)\right)$$

$$+ \left(-\cos(\pi) - \sin(\pi)\right) - \left(-\cos(\tfrac{\pi}{4}) - \sin(\tfrac{\pi}{4})\right)$$

$$= \left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}} - 1\right) + \left(1 + \tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}\right) = \tfrac{4}{\sqrt{2}} \approx 2.828$$

2. The number of hours, H, of daylight in Madrid as a function of date is approximated by the formula

$$H = 12 + 2.4\sin(0.0172(t - 80)),$$

where t is the number of days since the start of the year. (We can think of $t = 0$ as the stroke of midnight on Dec. 31/Jan 1; thus, January falls between $t = 0$ and $t = 31$, February falls between $t = 31$ and $t = 59$, etc.).

(a) Find the average number of hours of daylight in Madrid in October ($t = 273$ to 304). *[/3]*

$$\text{avg daylight} = \frac{1}{304 - 273} \int_{273}^{304} 12 + 2.4 \sin(0.0172(t - 80))\, dt$$

$$= \frac{1}{31}\left(12t + \frac{2.4}{0.0172}\left(-\cos(0.0172(t-80))\right)\right)\Bigg|_{273}^{304} = \frac{1}{31}\left(12(304) - \frac{2.4}{0.0172}\cos(0.0172(304-80)) - \left(12(273) - \frac{2.4}{0.0172}\cos(0.0172(273-80))\right)\right)$$

$$= 10.4799 \text{ hours on average}$$

(b) Find the average number of hours of daylight in Madrid over the entire year ($t = 0$ to 365). *[/2]*

$$= \frac{1}{365}\int_0^{365} 12 + 2.4 \sin(0.0172(t - 80))\, dt$$

$$= \frac{1}{365}\left[\left(12(365) - \frac{2.4}{0.0172}\cos(0.0172(365-80))\right) - 12 \cdot 0 - \frac{2.4}{0.0172}\cos(0.0172(0-80))\right]$$

$$= 12 + (-0.0019) \simeq 12 \text{ hours on average.}$$

STUDENT NUMBER:

Space for additional work. **Indicate clearly which question you are continuing if you use this space.**

STUDENT NUMBER: ______

Space for additional work. **Indicate clearly which question you are continuing if you use this space.**

STUDENT NUMBER:

Space for additional work. **Indicate clearly which question you are continuing if you use this space.**