MATH1119
Linear Algebra with Applications to Business and Economics

Caroline El-Chaar

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Chapter 1

Systems of Linear Equations

1.1 Introduction to Linear Systems

Definition (Linear Equation)**.** *A **linear equation** is an equation of the form*

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

*where $a_1, a_2, \ldots, a_n$ are known constants called **coefficients**, b is a known constant, and $x_1, x_2, \ldots, x_n$ are **variables** that are unknown.*

Example 1. *Which of the following equations is/are linear?*

1. $6x + 3y = 2$
2. $2xy - 3z^3 = 8$
3. $\sqrt{3}x - 4y + z = -1$
4. $2x - \sqrt{y} = 3z$

Note. *Variables of linear equations occur only to the first power and do not appear as arguments of, for example, trigonometric, logarithmic and exponential functions.*

Definition (System of Linear Equations)**.** *A **system of linear equations** (or a **linear system**) is a collection of linear equations involving the same set of variables (or unknowns). A general system of m equations in n variables $x_1, x_2, \ldots, x_n$, often referred to as an $m \times n$ system, can be written as*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \quad \vdots &= \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

Note. $a_{i\,j}$ $m \times n$

Example 2. *The following is a* 2×4 *linear system:*

$$\begin{aligned} 3x_1 + x_2 + x_4 &= 1 \\ 2x_3 - x_4 &= 4 \end{aligned}$$

Definition (Systems Terminology)**.**

- *An* $m \times n$ *system is said to be* ***homogeneous*** *if* $b_1 = b_2 = \ldots = b_m = 0$*. Otherwise, the system is said to be* ***non-homogenenous****.*
- *A* ***solution*** *of a linear system is an assignment of numbers to the variables such that all the equations are simultaneously satisfied.*
  - *A solution of an* $m \times n$ *linear system is a list of numbers* $(x_1, x_2, \ldots, x_n)$ *that satisfies all* m *equations simultaneously.*
- *The set of* <u>all</u> *possible solutions is called the* ***solution set*** *of the linear system.*
- *Two linear systems are said to be* ***equivalent*** *if they have the same solution set.*
- *A linear system is said to be* ***consistent*** *if it has at least one solution and* ***inconsistent*** *if it has no solutions.*

Example 3. *Consider once again the system*

$$\begin{aligned} 3x_1 + x_2 + x_4 &= 1, \\ 2x_3 - x_4 &= 4. \end{aligned}$$

(a) *Is this a homogeneous system?*

(b) *Determine if each of the following is a solution to the system.*

   (a) $(0, 0, 0, 1)$

   (b) $(1, 0, 1, -2)$

Example 4. *Consider the system*

$$\begin{aligned} 3x_1 + x_2 + x_4 &= 0, \\ 2x_3 - x_4 &= 0. \end{aligned}$$

(a) Is this a homogeneous system?

(b) Determine if each of the following is a solution to the system.

(a) $(0, 0, 0, 0)$

(b) $(1, 0, 1, -2)$

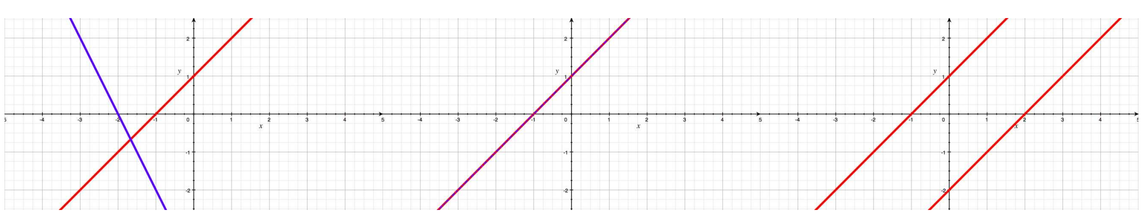
Note. $[0, 0, \ldots, 0] \in \mathbb{R}^n$ *is ALWAYS a solution of a homogeneous* $m \times n$ *linear system. Therefore,* ***a homogeneous linear system is ALWAYS*** ________________ *.*

1.1.1 2×2 Linear Systems

The (algebraic) solution of a linear system with two equations and two variables is the (geometric) intersection of two lines in two-space ($\mathbb{R}^2$ or xy-plane).

Geometrically, there are three possibilities for the intersection of two lines in two-space:

- Intersect at a point
- Coincident
- Parallel



Example 5. *1. The system of equations*

$$\begin{aligned} x - y &= -1 \\ 2x + y &= -4 \end{aligned}$$

represents two lines that are not parallel. Since we are in $\mathbb{R}^2$*, they must therefore intersect in a single point. It turns out that the system has unique solution* $x = -\frac{5}{3}$ *and* $y = -\frac{2}{3}$*. We can also write the solution as* $(x, y) = \left(-\frac{5}{3}, -\frac{2}{3}\right)$*. This is the point of intersection!*

2. When considering the system of equations

$$\begin{aligned} x - y &= -1 \\ 2x - 2y &= -2, \end{aligned}$$

we note that the second equation is simply the first multiplied by 2. Therefore, geometrically, these are coinciding lines! The system has infinitely many solutions where for all values of $x = t$*, we let* $y = t + 1$*. So, all points of the form* $(t, t + 1)$ *are solutions for the system.*

3. When considering the system of equations

$$\begin{aligned} x - y &= -1 \\ x - y &= 2, \end{aligned}$$

we note that the only difference are the known constants. Geometrically, these are parallel (distinct) lines. In other words, the lines do not share any point and so the system has no solution. There is no value of x *and* y *satisfying both equations.*

We conclude that, algebraically, 2×2 linear systems have three possibilities for their solutions:

- Unique solution
- Infinitely many solutions
- No solution.

It turns out that this is true for **ALL linear systems**!

Conclusion: A linear system of equation has

1. no solution, or
2. a unique solution, or
3. infinitely many solutions.

1.1.2 Matrix Notation

The current method we have to solve a linear system is to manipulate the equations algebraically in a way that does not alter the solution and that produces a simpler system until we can tell if the system is consistent and, if so, then find the solutions.

If we perform one of the following three operations on a linear system, we would obtain an equivalent system (i.e. with the same solution set).

- **Scaling:** Multiplying an equation by a non-zero constant.
- **Interchanging** two equations.
- **Replacing** an equation by its sum with a multiple of another equation in the system.

Note that, as our systems get bigger, solving them gets increasingly difficult. We will therefore work towards introducing a method that will be a bit more systematic using the concept of matrices.

Definition (Matrix)**.** *A **matrix** is a rectangular array of numbers. A matrix with m rows and n equations is called an $m \times n$ **matrix**. (Note that the plural of matrix is **matrices**.)*

Example 6. *The following are examples of matrices:*

$$\begin{bmatrix} 0 & 1 & 1 \\ 2 & 4 & 6 \\ 1 & 2 & 4 \end{bmatrix} \qquad \begin{bmatrix} 5 \\ 8 \\ 5 \end{bmatrix} \qquad \begin{bmatrix} 3 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 1 \end{bmatrix}$$

A matrix can be used to summarize the essential information present in a linear system.

Example 7. *The 3×3 linear system*

$$\begin{array}{rcrcrcl} & & x_2 & + & x_3 & = & 5 \\ 2x_1 & + & 4x_2 & + & 6x_3 & = & 8 \\ x_1 & + & 2x_2 & + & 4x_3 & = & 5 \end{array}$$

can be represented in a matrix consisting of the coefficients and the constants as follows:

*The matrix is called the **augmented matrix** of the system. The array of coefficients $[A]$ is called the **coefficient matrix** (or **matrix of coefficients**) of the system and $[B]$ is called the **constant matrix** of the system (can also be seen as a vector in $\mathbb{R}^3$ in this example).*

Q: How can we use matrix notation to solve linear systems?

First, we need some terminology...

1.2 Row Reduction and Echelon Forms

1.2.1 Echelon Forms

Definition. *A **leading entry** of a non-zero row refers to the left-most non-zero entry.*

Definition (Row-Echelon Form (REF))**.** *A matrix is said to be in **row-echelon form** (and will be called a **row-echelon matrix** if it satisfies the following properties:*

1. *All zero rows (i.e. consisting entirely of zeros) are at the bottom of the matrix.*
2. *Each leading entry is in a column to the left of any leading entries below it. (Note that this implies that all entries below leading entries are zeros.)*

Example 8. $\begin{bmatrix} 3 & 1 & 1 & -1 \\ 0 & 0 & -5 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ *is an example of a matrix in REF.*

Exercise 1. *Determine whether or not each of the following matrices are in REF.*

1. $\begin{bmatrix} 3 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 1 \end{bmatrix}$ 2. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ *3.* $\begin{bmatrix} 0 & 4 & 0 & 1 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ *4.* $\begin{bmatrix} 3 & 1 & 1 & 0 \\ 0 & 0 & 10 & -10 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

Definition (Reduced Row-Echelon Form (RREF))**.** *A matrix is said to be in **reduced row-echelon form** and will be called a **reduced row-echelon matrix**) if it satisfies the following properties:*

1. *It is in row-echelon form.*
2. *The leading entry in each nonzero row is a 1 (called the **leading 1**).*
3. *Each column containing a leading 1 has zeros everywhere else.*

Example 9. $\begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ *is an example of a matrix in RREF.*

Exercise 2. *Determine whether or not each of the following matrices are in RREF.*

1. $\begin{bmatrix} 3 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 1 \end{bmatrix}$ 2. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ *3.* $\begin{bmatrix} 0 & 1 & 9 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ *4.* $\begin{bmatrix} 1 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Definition. *Here are some terminology that we will use:*

- ***Pivot position:*** *position of the leading entry in an echelon form of a matrix.*
- ***Pivot:*** *a non-zero number that is in a pivot position.*
- ***Pivot column:*** *a column that contains a pivot position.*
- ***Basic/Leading variable:*** *a variable that corresponds to a pivot column.*
- ***Free variable:*** *a variable that is not a basic variable, i.e. a variable that corresponds to a non-pivot column.*
- ***Rank of a matrix*** A***:*** *Number of pivot positions of* A*, denoted rank(*A*).*

Example 10. $A = \begin{bmatrix} -1 & 5 & -1 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

Note. *If we can get the coefficient matrix of a linear system in RREF, finding the solution is pretty simple.*

Example 11. *Determine a solution (if any) of the systems with the following augmented matrices:*

1. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right]$

2. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 9 \end{array}\right]$

3. $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 8 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Exercise 3. *Determine the general solution of the following linear system.*

$$\begin{aligned} x_1 + 5x_2 + 2x_4 &= 4 \\ x_3 - 5x_4 &= 6 \end{aligned}$$

Note that the augmented matrices of most systems aren't always so simple. How can we get the to the REF or RREF?

1.2.2 Row Reduction

Since the rows of an augmented matrix correspond to the equations in the associated system, the operations that can be performed on a linear system that won't affect its solution set correspond to allowable operations on rows of its augmented matrix.

Definition (Elementary Row Operations)**.** *The following **elementary row operations** can be performed on a matrix.*

- ***Scaling:*** *Multiplying a row by a non-zero constant.*
- ***Interchanging*** *two rows.*
- ***Replacing*** *a row by its sum with a multiple of another row.*

Definition (Row Equivalent Matrices)**.** *Let A and B be two matrices of the same size. We say that A and B are **row equivalent** if there is a sequence of elementary row operations that converts A to B. Notation: $A \sim B$*

Next, we will explore a step-by-step **row reduction** procedure that can be used to reduce any augmented matrix into row-echelon form (REF) and then into reduced row-echelon form (RREF).

This procedure consists of 2 parts:

- a **forward phase** in which we will proceed from left to right and obtain a REF, and then
- a **backward phase** in which we will proceed from right to left and obtain the RREF.

The steps of the procedure will be illustrated through the reduction of the following matrix:

$$A = \begin{bmatrix} 0 & 0 & 2 & 1 & 9 \\ -2 & -6 & 2 & 0 & 2 \\ 2 & 6 & -2 & 2 & 0 \\ 3 & 9 & 2 & 2 & 19 \end{bmatrix}$$

STEP 1: Locate the leftmost nonzero column and place a leading 1 at the top of the column.

If the top entry of the column is already a 1, proceed to step 2. Otherwise:

- If the top entry of the column is a zero, interchange rows to place a nonzero entry in that position.

$$A \quad \sim \quad \begin{bmatrix} -2 & -6 & 2 & 0 & 2 \\ 0 & 0 & 2 & 1 & 9 \\ 2 & 6 & -2 & 2 & 0 \\ 3 & 9 & 2 & 2 & 19 \end{bmatrix}$$

- If the top entry is not zero, make it a 1 by either multiplying the top row by a suitable constant or by interchanging rows.

$$\begin{bmatrix} -2 & -6 & 2 & 0 & 2 \\ 0 & 0 & 2 & 1 & 9 \\ 2 & 6 & -2 & 2 & 0 \\ 3 & 9 & 2 & 2 & 19 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 & 0 & -1 \\ 0 & 0 & 2 & 1 & 9 \\ 2 & 6 & -2 & 2 & 0 \\ 3 & 9 & 2 & 2 & 19 \end{bmatrix}$$

STEP 2: By adding or subtracting suitable multiples of the row containing the leading 1, make each entry below it zero.

$$\begin{bmatrix} 1 & 3 & -1 & 0 & -1 \\ 0 & 0 & 2 & 1 & 9 \\ 2 & 6 & -2 & 2 & 0 \\ 3 & 9 & 2 & 2 & 19 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 & 0 & -1 \\ 0 & 0 & 2 & 1 & 9 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 5 & 2 & 22 \end{bmatrix}$$

If the matrix is now in REF, then you have performed **Gaussian elimination**. You can either stop here or continue to step 3 in order to obtain the RREF.

If the matrix is not yet in REF, then ignore the first row and apply steps 1-2 to the sub-matrix that remains.

In our example, we are not yet in the REF, therefore we must repeat steps 1 and 2.

$$\begin{bmatrix} 1 & 3 & -1 & 0 & -1 \\ 0 & 0 & 2 & 1 & 9 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 5 & 2 & 22 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{9}{2} \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 5 & 2 & 22 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{9}{2} \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Row 2 is now complete but we are still not in REF, therefore ignore the first 2 rows and once again repeat steps 1-2 to the remaining sub-matrix.

$$\begin{bmatrix} 1 & 3 & -1 & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{9}{2} \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{9}{2} \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{9}{2} \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We are now in REF! Now on to the backward phase...

STEP 3: Beginning with the rightmost leading 1, create zeros above it by adding appropriate multiples of the row containing the leading 1.

$$\begin{bmatrix} 1 & 3 & -1 & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{9}{2} \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

If the matrix is now in RREF, then you have performed **Gauss-Jordan elimination** and you're done!

If the matrix is not yet in RREF, then move to the left towards the next leading 1 and repeat step 3.

In our example, we are not yet in the RREF, therefore we must repeat step 3.

$$\begin{bmatrix} 1 & 3 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in RREF! :-)

Theorem 1.2.1 (Uniqueness of RREF)**.** *Each matrix is row equivalent to one and only one reduced echelon matrix.*

Note. *A matrix can be equivalent to multiple matrices in REF; i.e. REFs are not unique. But all of the REFs will have the same number of zero rows and the leading entries always occur in the same positions (the pivot positions).*

Example 12. *Solve the following system:*

$$\begin{aligned} 3x + 2y &= 4, \\ -x + 3y + 3z &= -2, \\ y + z &= 0. \end{aligned}$$

Solution: *The augmented matrix of the system is:*

We start by row reducing to REF:

*We can stop here and proceed by **back substitution** as follows:*

We could have also continued to find the RREF of the augmented matrix and used it to find the solution:

Theorem 1.2.2 (Existence and Uniqueness Theorem). *A linear system is consistent if and only if the rightmost column of the augmented matrix is **not** a pivot column – that is, if and only if an echelon form of the augmented matrix has no row of the form*

$$\left[\begin{array}{ccc|c} 0 & \cdots & 0 & b \end{array}\right] \qquad \text{with } b \text{ non-zero}.$$

If a linear system is consistent, then the solution set contains either

(i) a unique solution - when there are no free variables, or

(ii) infinitely many solutions - when there is at least one free variable.

Example 13. *Consider the following matrices in REF and determine how many solutions the corresponding systems has.*

1. $\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & -3 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

2. $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

3. $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

4. $\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 4 & 5 & 3 \\ 0 & 1 & 3 & 2 & -2 & 4 \\ 0 & 0 & 0 & 4 & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Example 14. *Solve the following linear system:*

$$\begin{aligned} x + 2y + z &= 3 \\ x - y + z &= 1 \\ -2x - 4y - 2z &= 4. \end{aligned}$$

Example 15. *Solve the following linear system:*

$$\begin{aligned} x + 2y - 3z &= 3 \\ -2x - 5y + 4z &= 5 \\ -5x - 13y + 9z &= 18. \end{aligned}$$

Example 16. *Find the value of the constant k such that the following system has (i) no solution, (ii) infinitely many solutions, (iii) a unique solution.*

$$\begin{aligned} x + 2y - z &= 1 \\ -2x - 3y + 2z &= -1 \\ -5x - 8y + 5z &= k. \end{aligned}$$