CONCORDIA UNIVERSITY
Department of Mathematics & Statistics

Course	**Number**	**Sections**
Mathematics	201	All
Examination	**Date**	**Pages**
Final	April 2015	2
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Special Instructions:	Only approved calculators are allowed. **Show all your work for full marks.**	

MARKS

[11] **1.** (a) Write the equation of the line passing through the points $(3,-1)$ and $(-3,3)$.

(b) Write the equation of the line orthogonal to the line $y = 3x + 2$ and passing through the point $(-2,4)$.

(c) Find the equation of the circle with the centre at the point $(6,4)$ and passing through the point $(2,5)$

(d) Find the domain of the function $f(x) = \sqrt{x^2-1} + \sqrt{3x+6}$.

[24] **2.** Find the solutions of the following equations:

(a) $e^{2x} - 5 \cdot e^{x} - 6 = 0$

(b) $\log_3(2x) + \log_3(\frac{x-2}{2}) = 1$

(c) $8^{x-1} - 2^{-x} = 0$

(d) $\ln(x-2) + \ln(x+2) - \ln(x^2) = -\ln(3)$

[12] **3.** Consider the quadratic function $f(x) = 2x^2 + 6x$.

(a) Express $f(x)$ in standard form.

(b) Find the coordinates of the vertex and indicate whether it corresponds to the maximum or the minimum of f.

(c) Find the $x-$intercepts and the $y-$intercept.

(d) Sketch the graph of $f(x)$ using the information above.

[8] **4.** (a) Consider $f(x) = e^{2x} - 3$. Find the inverse function $f^{-1}(x)$.

(b) Determine the domain and the range of $f(x)$ and $f^{-1}(x)$.

[7] **5.** (a) The area of the sector with the central angle $\alpha = \frac{2\pi}{3}$ radians is equal to 12 square meters. Find the radius of the circle.

(b) Find the length of the arc of a circle intercepted by the central angle with the measure $\alpha = 149°$ given the radius of the circle $r = 15$ cm.

[8] **6.** Verify the identities

(a) $\frac{1+\cos x}{1-\cos x} - \frac{1-\cos x}{1+\cos x} = 4 \csc x \cot x$

(b) $(\tan x - \sin x)(\tan x + \sin x) = (\sin x \tan x)^2$

[18] **7.** Solve the triangle ABC (i.e. find the missing sides and angles).

(a) $\angle A = 53°$, $\angle C = 30°$, $b = 120$ cm

(b) $\angle A = 120°$, $b = 15$ cm, $c = 10$ cm

(c) $a = 15$ cm, $b = 18$ cm, $c = 30$ cm

[6] **8.** (a) Find the amplitude, the period, and the phase shift of $y = 5\sin[\pi(x - \frac{1}{4})]$.

(b) Find $f \circ g$ and $g \circ f$ if $f(x) = \sqrt{x-1}$ and $g(x) = e^{x^2+1}$.

[6] **9.** Find all horizontal and all vertical asymptotes of the function

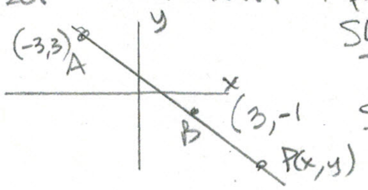
$$f(x) = \frac{x\sqrt{4x^2+2x+1} + 2x^2}{x^2 - 25}.$$

[5] **10.** **Bonus Question**

Express $\cos(\theta)$ as a function of the parameter A if it is known that $\theta = \arctan(A)$, and $0 \leq \theta < \frac{\pi}{2}$.

Reminder: *arctan* is the inverse of *tan*, i.e. $\arctan(A) \equiv \tan^{-1}(A)$.

Math 201 Final Exam April 2015 UNEDITED

1a) 

Step 1 Slope $AB = \frac{y_2-y_1}{x_2-x_1} = \frac{-1-3}{3-[-3]} = \frac{-4}{6} = -\frac{2}{3}$

Step 2 let $P(x,y)$ be any pt on line

Slope $AP = \frac{y_2-y_1}{x_2-x_1} =$

$-\frac{2}{3} = \frac{y-3}{x-[-3]}$

$3(y-3) = -2(x+3)$

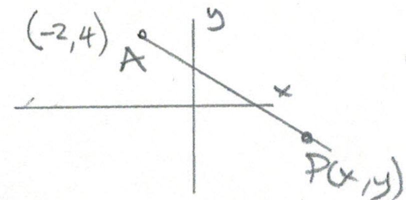
$3y-9 = -2x-6$

$3y = -2x-6+9$

$y = -\frac{2}{3}x+1$

b) Step 1 Slope of given line is coefficient of $x \Rightarrow$ slope $= \frac{3}{1}$

$\Rightarrow$ slope of line $\perp$ (orthogonal) $= -\frac{1}{3}$



Step 2 let $P(x,y)$ be any pt on line

Slope $AP = \frac{y_2-y_1}{x_2-x_1}$

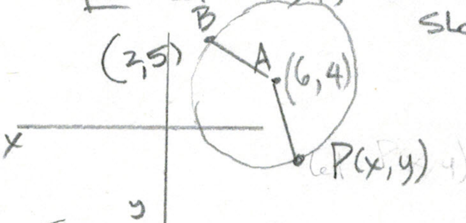
$-\frac{1}{3} = \frac{y-4}{x-[-2]}$

$3(y-4) = -1(x+2)$

$3y-12 = -x-2$

$3y = -x-2+12$

$y = -\frac{1}{3}x+\frac{10}{3}$

c) 

Step 1 Find Radius $AB = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$= \sqrt{(6-2)^2+(4-5)^2}$

$= \sqrt{4^2+(-1)^2}$

$= \sqrt{17}$

Step 2 let $P(x,y)$ be any pt on circle

$AP = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$\sqrt{17} = \sqrt{(x-6)^2+(y-4)^2}$

$x^2-12x+36+y^2-8y+16 = 17$

$x^2+y^2-12x-8y+35=0$

d) Domain:

① $x^2-1 \geq 0$

$x^2 \geq 1$

$x \geq 1$ or $x \leq -1$

② $3x+6 \geq 0$

$3x \geq -6$

$x \geq -2$

Final Domain $x \in R \mid -2 \leq x \leq -1$ OR $x \geq 1$ OR write $[-2,-1] \cup [1,\infty)$

2. a) $(e^x)^2-5(e^x)-6=0$

$(e^x-6)(e^x+1)=0$

$e^x-6=0$	$e^x+1=0$
$e^x=6$	$e^x=-1$
	No x here (why?)

$\Rightarrow \log_e 6 = x$

$x = \ln 6$

looks like:

$A^2-5A-6=0$ Except we have e^x instead of A

$(A-6)(A+1)=0$

$A-6=0$	$A+1=0$
$A=6$	$A=-1$

b) $\log_3 2x + \log_3 \frac{x-2}{2} = 1$

$\log_3 (\cancel{2}x)\left(\frac{x-2}{\cancel{2}}\right) = 1$

$3^1 = x(x-2)$

$x^2-2x-3=0$

$(x-3)(x+1)=0$

$x-3=0$	$x+1=0$
$x=3$	$x=-1$
	DISCARD, $x=-1$ Not in Domain of these logs.

c) $8^{x-1} = 2^{-x}$

$(2^3)^{x-1} = 2^{-x}$

$2^{3x-3} = 2^{-x}$

$\Rightarrow 3x-3 = -x$

$3x+x = 3$

$4x = 3$

$x = \frac{3}{4}$

d) $\ln(x-2)+\ln(x+2)-\ln x^2 = -\ln 3$

$\ln (x-2)(x+2) - \ln x^2 = (-1)\ln 3$

$\ln \frac{(x-2)(x+2)}{x^2} = \ln 3^{-1}$

$\frac{(x-2)(x+2)}{x^2} = \frac{1}{3}$

$3(x^2-4) = 1(x^2)$

$3x^2-12-x^2=0$

$2x^2 = 12$

$x^2 = 6$

$x = \pm\sqrt{6}$

DISCARD $x=-\sqrt{6}$ Not in domain of 1st 2 logs.

3. a) $f(x) = 2x^2 + 6x$

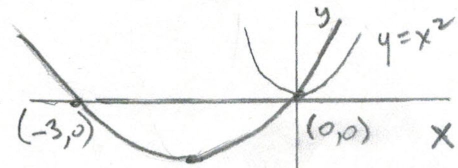
$y = 2\left(x^2 + 3x + \frac{9}{4}\right) - 2\left(\frac{9}{4}\right)$

$y = 2\left(x + \frac{3}{2}\right)\left(x + \frac{3}{2}\right) - \frac{9}{2}$

$y = 2\left(x + \frac{3}{2}\right)^2 - \frac{9}{2}$

OR

$\frac{1}{2}\left(y + \frac{9}{2}\right) = \left(x + \frac{3}{2}\right)^2$



b) $\longrightarrow$ ① Vertex $\longrightarrow$ $\left(-\frac{3}{2}, -\frac{9}{2}\right)$ $y = x^2$ shifts: $3/2$ horiz to left, $9/2$ vert. down

② because coeff of y term is positive $\Rightarrow$ opens up $\Rightarrow$ Vertex is a Minimum y for f

c)

x int	y int
Let $y = 0$	let $x = 0$
$0 = 2x^2 + 6x$	$y = 2(0)^2 + 6(0)$
$x(2x + 6)$	$y = 0$

$x = 0$ | $2x + 6 = 0$, $x = -3$

d) See graph above.

4 a) $f(x) = e^{2x} - 3$

$y = e^{2x} - 3$

$x = e^{2y} - 3$

$e^{2y} = x + 3$

$\log_e (x+3) = 2y$

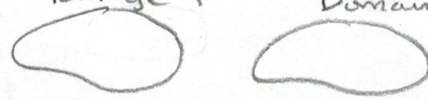
$\frac{1}{2} \ln(x+3) = y$

$f^{-1}(x) = \frac{1}{2} \ln(x+3)$

b) Domain of f is all Real numbers or $(-\infty, \infty)$

Range of f : $y + 3 = e^x$. Since $e^x > 0 \Rightarrow y + 3 > 0 \Rightarrow y > -3$ or $(-3, \infty)$

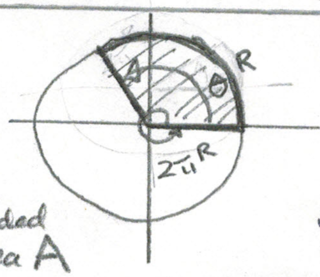
Domain f, Range f^{-1} — Range f, Domain f^{-1}



$\Rightarrow$ Domain f^{-1} $x > -3$

Range f^{-1} All Reals

5a)



Find Shaded Area A

Area of Complete circle $= \pi r^2$ (r is Radius)

$= \frac{1}{2}(2\pi r^2)$ Note 2π is N° Radians at centre for complete circle.

Area of part of circle called a Sector

$A = \frac{1}{2}(\theta r^2)$ where θ is angle at centre in Radians

$12 = \frac{1}{2}\left(\frac{2\pi}{3}\right) r^2$

$(12)(6) = 2\pi r^2$

$\frac{(12)(6)}{2\pi} = r^2$

$r = \sqrt{\frac{(12)(3)}{\pi}} = 3.39\text{ m}$

b) Step 1 Chang degrees to Radians:

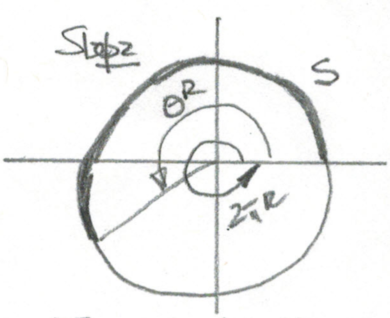
$\pi^R = 180°$

$x^R = 149°$

$\frac{x}{149} = \frac{\pi}{180}$

$x = 149 * \frac{\pi}{180}^R$

Step 2



Find arc length S

Circumference of Complete circle $= 2\pi r$ (r is Radius)

2π is angle at Centre

part of circumference Called an arc

$S = \theta r$

$S = \left(149 * \frac{\pi}{180}\right) 15$

$S = 39\text{ cm}.$

6. a) $\frac{1+\cos x}{1-\cos x} - \frac{1-\cos x}{1+\cos x}$

$= \frac{(1+\cos x)(1+\cos x)-(1-\cos x)(1-\cos x)}{(1-\cos x)(1+\cos x)}$

$= \frac{\cancel{1}+2\cos x+\cancel{\cos^2 x}-\cancel{1}+2\cos x-\cancel{\cos^2 x}}{1-\cos^2 x}$

$= \frac{4\cos x}{\sin^2 x}$

$= 4\ \frac{1}{\sin x} * \frac{\cos x}{\sin x}$

$= 4\csc x\ \cot x$

b) $(\tan x - \sin x)(\tan x + \sin x)$

$= \tan^2 x + \cancel{\tan x\sin x} - \cancel{\tan x \sin x} - \sin^2 x$

$= \tan^2 x - \sin^2 x$

$= \frac{\sin^2 x}{\cos^2 x} - \frac{\sin^2 x}{1}$

$= \frac{\sin^2 x - \sin^2 x\cos^2 x}{\cos^2 x}$

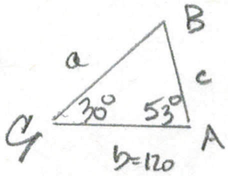
$= \frac{\sin^2 x(1-\cos^2 x)}{\cos^2 x}$

$= \frac{\sin^2 x}{1} * \frac{\sin^2 x}{\cos^2 x}$

$= \sin^2 x * \tan^2 x$

$= (\sin x \tan x)^2$

7 a)



Step 1 $\angle B$: $\angle A + \angle B + \angle C = 180^\circ$

$\angle B = 180^\circ - 30^\circ - 53^\circ$

$\angle B = 97^\circ$

Step 2 a: $\frac{\sin A}{a} = \frac{\sin B}{b}$

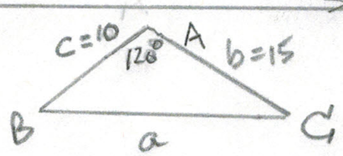
$\frac{\sin 53}{a} = \frac{\sin 97}{120}$

$a = \frac{(\sin 53)120}{\sin 97} = 96.56\text{cm}$

Step 3 c: $\frac{\sin C}{c} = \frac{\sin B}{b}$

$c = \frac{(\sin 30)120}{\sin 97} = 60.45\text{cm}$

b)



Step 1 a: $a^2 = b^2 + c^2 - 2bc\cos A$

$a = \sqrt{15^2 + 10^2 - 2(15)(10)\cos 120^\circ}$

$a = 21.79\text{cm}$

Step 2 C: $\frac{\sin C}{c} = \frac{\sin A}{a}$

$\sin C = \frac{10\sin 120^\circ}{a}$

$C = 23.41^\circ$ or $C = 180^\circ - 23.41 = 156.9^\circ$

But this is discarded (why?)

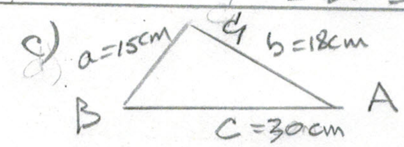
Step 3 B:

$\angle A + \angle B + \angle C = 180^\circ$

$\angle B = 180 - 23.41 - 120$

$\angle B = 36.59^\circ$

c)



Step 1 C: $c^2 = a^2 + b^2 - 2ab\cos C$

$30^2 - 15^2 - 18^2 = -2(15)(18)\cos C$

(Note MINUS Sign)

$\frac{30^2 - 15^2 - 18^2}{-2(15)(18)} = \cos C$

$C = 130.54^\circ$

Step 2 $\angle A$ $\frac{\sin A}{a} = \frac{\sin C}{c}$

$\frac{\sin A}{15} = \frac{\sin 130.54^\circ}{30}$

$\sin A = \frac{15(\sin 130.54)}{30}$

$A = 22.33^\circ$ OR $A = 180 - 22.33 = 157.67^\circ$

DISCARD, why?

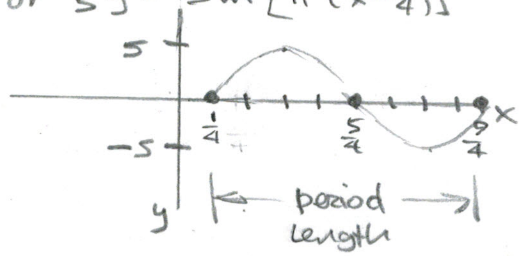
Step 3 $\angle B$: $\angle A + \angle B + \angle C = 180^\circ$

$\angle B = 180^\circ - 22.33 - 130.54$

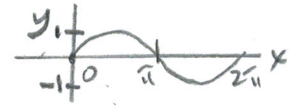
$\angle B = 27.13^\circ$

8 a) $y = 5\sin\left[\pi\left(x-\frac{1}{4}\right)\right]$

or $\frac{1}{5}y = \sin\left[\pi\left(x-\frac{1}{4}\right)\right]$



BASIC Sine graph has x intercepts when Angle :



$\pi(x-\frac{1}{4}) = 0 \Rightarrow \pi x - \frac{\pi}{4} \Rightarrow \pi x = \frac{\pi}{4} \Rightarrow x = \frac{1}{4}$

$\pi(x-\frac{1}{4}) = \pi \Rightarrow x - \frac{1}{4} = 1 \Rightarrow x = 1 + \frac{1}{4} = \frac{5}{4}$

$\pi(x-\frac{1}{4}) = 2\pi \rightarrow x - \frac{1}{4} = 2 \Rightarrow x = 2 + \frac{1}{4} = \frac{9}{4}$

Amplitude = 5

period : $\frac{9}{4} - \frac{1}{4} = \frac{8}{4} = 2$

phase shift : $\frac{1}{4}$

b) $f(x) = \sqrt{x-1}$

$f(g(x)) = \sqrt{g(x)-1}$

$(f \circ g)(x) = \sqrt{e^{x^2+1} - 1}$

$g(x) = e^{x^2+1}$

$g(f(x)) = e^{[f(x)]^2+1}$

$(g \circ f)(x) = e^{(\sqrt{x-1})^2+1}$

$(g \circ f)(x) = e^{x-1+1}$

$(g \circ f)(x) = e^{x}$

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9. VA : let $x^2 - 25 = 0$

$x^2 = 25$

$x = \pm 5$

Check Numerator when

$x = 5$	$x = -5$
$5\sqrt{4(5^2)+2(5)+1} + 2x^2$	$(-5)\sqrt{4(-5)^2+2(-5)+1} + 2(-5)^2$
$\neq 0$	$\neq 0$
$\Rightarrow x = 5$ is a VA	$\Rightarrow x = -5$ is VA

HA :

$\lim_{x \to \pm\infty} \frac{x\sqrt{4x^2+2x+1} + 2x^2}{x^2-25} = \frac{\infty}{\infty}$

$= \lim_{x \to \pm\infty} \frac{\frac{x}{x}\frac{\sqrt{x^2+2x+1}}{x} + \frac{2x^2}{x^2}}{\frac{x^2}{x^2} - \frac{25}{x^2}}$

$= \lim_{x \to \pm\infty} \frac{1\sqrt{\frac{x^2}{x^2} + \frac{2x}{x^2} + \frac{1}{x^2}} + 2}{1 - \frac{25}{x^2}}$

$= \lim_{x \to \pm\infty} \frac{\sqrt{1 + \frac{2}{x} + \frac{1}{x^2}} + 2}{1 - \frac{25}{x^2}}$

$= \frac{3}{1}$

$\Rightarrow$ HA : $y = 3$

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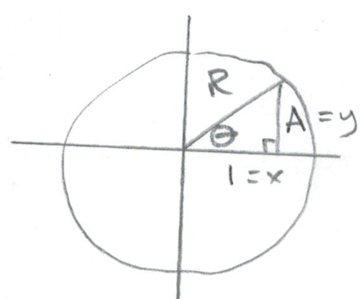
10. Bonus: $\arctan A = \theta$

Step 1 $\Rightarrow$ $A = \tan\theta$

OR $\tan\theta = \frac{A}{1} = \frac{y}{x}$



Step 2 $\cos\theta = \frac{x}{R}$

$\cos\theta = \frac{1}{\sqrt{A^2+1}}$

$R^2 = x^2 + y^2$

$R = \sqrt{1^2 + A^2}$

$R = \sqrt{A^2+1}$