ECON4020

Advanced Microeconomic Theory

Answers to Problem Set Two

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Note: There are two parts to this document. In the first part (pages ii – vii) are the answers to exercises 7.4, 8.2, 8.8, and the two additional exercises. In the second part are selected pages from Varian's own answers to the exercises.

7.4 Consider the indirect utility function given by

$$v(p_1, p_2, m) = \frac{m}{p_1 + p_2}.$$

a) What are the demand functions?

Use Roy's identity $x_i = -\frac{\partial v(p,m)/\partial p_i}{\partial v(p,m)/\partial m}$ for $i = 1, \ldots, k$.

$$\frac{\partial v(p,m)}{\partial p_1} = \frac{\partial v(p,m)}{\partial p_2} = -\frac{m}{(p_1 + p_2)^2}$$

$$\frac{\partial v(p,m)}{\partial m} = \frac{1}{p_1 + p_2}$$

$$x_1 = x_2 = - \; -\frac{m}{(p_1 + p_2)^2} \times \frac{p_1 + p_2}{1} = \frac{m}{p_1 + p_2}$$

b) What is the expenditure function?

Use the identity $v(\boldsymbol{p}, e(\boldsymbol{p}, u)) \equiv u$.

$u = \frac{m}{p_1 + p_2}$

⇨ $m = (p_1 + p_2)u$

⇨ $e(p, u) = (p_1 + p_2)u$

c) What is the direct utility function?

By observation, at the optimum, $u^* = x_1{}^* = x_2{}^*$, which means consuming either good 1 or good 2 gives the optimum level of utility for the agent. Since consumer preference is assumed to be convex; that indicates these two goods are perfect complements. Therefore, the direct utility function will be $u = \min\{x_1, x_2\}$.

8.2 Calculate the substitution matrix for the Cobb- Douglas demand system with two goods. Verify that the diagonal terms are negative and the cross-price effects are symmetric.

By the definition of the substitution matrix,

$$S = \begin{bmatrix} \frac{\partial x_1(p,m)}{\partial p_1} + \frac{\partial x_1(p,m)}{\partial m} x_1 & \frac{\partial x_1(p,m)}{\partial p_2} + \frac{\partial x_1(p,m)}{\partial m} x_2 \\ \frac{\partial x_2(p,m)}{\partial p_1} + \frac{\partial x_2(p,m)}{\partial m} x_1 & \frac{\partial x_2(p,m)}{\partial p_2} + \frac{\partial x_2(p,m)}{\partial m} x_2 \end{bmatrix}$$

As shown on page 111 of *Varian*, the Marshallian demand functions associated with the Cobb-Douglas utility function $= x_1{}^a x_2{}^b$ (where $a + b = 1, a > 0, b > 0$) are:

$$x_1(p_1, p_2, m) = \frac{am}{p_1}, \quad x_2(p_1, p_2, m) = \frac{bm}{p_2}.$$

Then,

$$\frac{\partial x_1(p,m)}{\partial p_1} + \frac{\partial x_1(p,m)}{\partial m} x_1 = -\frac{am}{(p_1)^2} + \frac{ax_1}{p_1} = -\frac{am}{(p_1)^2} + \frac{a}{p_1}\frac{am}{p_1} = \frac{am}{(p_1)^2}(a-1) = -\frac{abm}{(p_1)^2}$$

$$\frac{\partial x_1(p,m)}{\partial p_2} + \frac{\partial x_1(p,m)}{\partial m} x_2 = \frac{ax_2}{p_1} = \frac{a}{p_1}\frac{bm}{p_2}$$

$$\frac{\partial x_2(p,m)}{\partial p_1} + \frac{\partial x_2(p,m)}{\partial m} x_1 = \frac{bx_1}{p_2} = \frac{b}{p_2}\frac{am}{p_1}$$

$$\frac{\partial x_2(p,m)}{\partial p_2} + \frac{\partial x_2(p,m)}{\partial m} x_2 = -\frac{bm}{(p_2)^2} + \frac{bx_2}{p_2} = -\frac{bm}{(p_2)^2} + \frac{b}{p_2}\frac{bm}{p_2} = (b-1)\frac{bm}{(p_2)^2} = -\frac{abm}{(p_2)^2}$$

$$\Rightarrow S = \begin{bmatrix} -\frac{abm}{(p_1)^2} & \frac{a}{p_1}\frac{bm}{p_2} \\ \frac{b}{p_2}\frac{am}{p_1} & -\frac{abm}{(p_2)^2} \end{bmatrix}$$

Since $a > 0$ and $b > 0$, we have $-\frac{abm}{(p_1)^2} < 0$ and $-\frac{abm}{(p_2)^2} < 0$, i.e., the diagonal terms are negative. Moreover,

$$|S| = \left[-\frac{abm}{(p_1)^2}\right]\left[-\frac{abm}{(p_2)^2}\right] - \frac{abm}{(p_1)^2}\frac{abm}{(p_2)^2} = 0.$$

Hence, S is negative semi-definite.

The cross-price effects are both equal to $\frac{a}{p_1}\frac{bm}{p_2}$, i.e., they are symmetric.

8.6 Use utility function $u(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{3}}$ and the budget constraint $m = p_1 x_1 + p_2 x_2$ to calculate $x(p,m), v(p,m), h(p,u)$ and $e(p,u)$.

8.8 Repeat the previous exercise using $u^*(x_1, x_2) = \frac{1}{2}\, lnx_1 + \frac{1}{3} lnx_2$ and show that all the previous formula hold provided u is replaced by e^{u^*} (Here, the "previous exercise" refers to 8.6).

$max_{x_1,x_2}\ u = \frac{1}{2}\, lnx_1 + \frac{1}{3} lnx_2$ subject to $p_1x_1 + p_2x_2 = m$

$$\mathcal{L}(\lambda, x) = \frac{1}{2}\, lnx_1 + \frac{1}{3} lnx_2 + \lambda(m - p_1x_1 - p_2x_2)$$

F.O.C.s:

$$\frac{\partial \mathcal{L}(\lambda, x)}{\partial x_1} = \frac{1}{2x_1} - \lambda p_1 = 0 \Rightarrow \frac{1}{2x_1} = \lambda p_1 \qquad (1)$$

$$\frac{\partial \mathcal{L}(\lambda, x)}{\partial x_2} = \frac{1}{3x_2} - \lambda p_2 = 0 \Rightarrow \frac{1}{3x_2} = \lambda p_2 \qquad (2)$$

$$\frac{\partial \mathcal{L}(\lambda, x)}{\partial \lambda} = m - p_1 x_1 - p_2 x_2 = 0$$

(1) ÷ (2):

$$\frac{3x_2}{2x_1} = \frac{p_1}{p_2}$$

$$\Longrightarrow x_2 = \frac{2}{3}\frac{p_1}{p_2}x_1 \qquad (3)$$

Substitute (3) into the budget constraint:

$$p_1 x_1 + p_2 \frac{2}{3}\frac{p_1}{p_2}x_1 = m \Longrightarrow \frac{5}{3}p_1 x_1 = m$$

$$\Longrightarrow {x_1}^* = \frac{3}{5}\frac{m}{p_1} \qquad (4)$$

$$\Longrightarrow {x_2}^* = \frac{2}{3}\frac{p_1}{p_2}\frac{3}{5}\frac{m}{p_1} = \frac{2}{5}\frac{m}{p_2} \qquad (5)$$

Hence, the indirect utility function associated with this utility function is

$$v^*(\mathbf{p}, m) = \frac{1}{2}\ln\left(\frac{3}{5}\frac{m}{p_1}\right) + \frac{1}{3}\ln\left(\frac{2}{5}\frac{m}{p_2}\right) \qquad (6)$$

The indirect utility function derived from Exercise 8.6 is

$$v(\mathbf{p}, m) = \left(\frac{3}{5}\frac{m}{p_1}\right)^{1/2}\left(\frac{2}{5}\frac{m}{p_2}\right)^{1/3}$$

Since $x = e^{lnx}$, we have

$$e^{v^*(\mathbf{p},m)} = e^{\frac{1}{2}\ln\left(\frac{3m}{5p_1}\right)+\frac{1}{3}\ln\left(\frac{2m}{5p_2}\right)} = e^{ln\left[\left(\frac{3m}{5p_1}\right)^{1/2}\left(\frac{2m}{5p_2}\right)^{1/3}\right]} = \left(\frac{3}{5}\frac{m}{p_1}\right)^{1/2}\left(\frac{2}{5}\frac{m}{p_2}\right)^{1/3} = v(\mathbf{p}, m).$$

Set $u^* = v^*(\mathbf{p}, m)$ in (6) and solve for m,

$$u^* = \frac{1}{2}\ln\left(\frac{3}{5}\frac{m}{p_1}\right) + \frac{1}{3}\ln\left(\frac{2}{5}\frac{m}{p_2}\right) = \ln\left[\left(\frac{3}{p_1}\right)^{1/2}\left(\frac{2}{p_2}\right)^{1/3}\left(\frac{m}{5}\right)^{5/6}\right] \Longrightarrow e^{u^*} = \left[\left(\frac{3}{p_1}\right)^{1/2}\left(\frac{2}{p_2}\right)^{1/3}\left(\frac{m}{5}\right)^{5/6}\right]$$

$$\Longrightarrow e(\mathbf{p}, u^*) = m = 5\left[e^{u^*}\left(\frac{p_1}{3}\right)^{1/2}\left(\frac{p_2}{2}\right)^{1/3}\right]^{\frac{6}{5}} = 5\left(e^{u^*}\right)^{6/5}\left(\frac{p_1}{3}\right)^{3/5}\left(\frac{p_2}{2}\right)^{2/5} \qquad (7)$$

In (7), if we replace e^{u^*} by u, we obtain the same expenditure function as in Exercise 8.6.

$$h_1(\mathbf{p}, u^*) = \frac{\partial e(\mathbf{p}, u)}{\partial p_1} = \left(e^{u^*}\right)^{6/5} \left(\frac{p_1}{3}\right)^{-2/5} \left(\frac{p_2}{2}\right)^{2/5} \qquad (8)$$

$$h_2(\mathbf{p}, u^*) = \frac{\partial e(\mathbf{p}, u)}{\partial p_2} = \left(e^{u^*}\right)^{6/5} \left(\frac{p_1}{3}\right)^{3/5} \left(\frac{p_2}{2}\right)^{-3/5} \qquad (9)$$

In (8) and (9), if we replace e^{u^*} by u, we obtain the same Hicksian demand functions as in Exercise 8.6.

Additional Exercise 1

1. The indirect utility function of a consumer is given by $v(p_1, p_2, m) = \frac{Am}{p_1^a p_2^{1-a}}$ where $a \in (0,1)$ and $A>0$.

(a) Use the Roy's identity to find the Marshallian demand function for goods 1 and 2.

Use the Roy's Identity

$$x_i = -\frac{\partial v/\partial p_i}{\partial v/\partial m}.$$

Since by $v(p_1, p_2, m) = \frac{Am}{P_1^a P_2^{1-a}} = AmP_1^{-a}P_2^{a-1}$

$$\frac{\partial v}{\partial p_1} = -aAmP_1^{-a-1}P_2^{a-1}$$

$$\frac{\partial v}{\partial p_2} = (a-1)AmP_1^{-a}P_2^{a-2}$$

$$\frac{\partial v}{\partial m} = AP_1^{-a}P_2^{a-1}$$

Therefore, the Marshallian demand function for good 1 and good 2 are:

$$x_1 = -\frac{\partial v/\partial p_1}{\partial v/\partial m} = -\frac{-aAmp_1^{-a-1}p_2^{a-1}}{AP_1^{-a}P_2^{a-1}} = \frac{am}{p_1}$$

$$x_2 = -\frac{\partial v/\partial p_2}{\partial v/\partial m} = -\frac{(a-1)Amp_1^{-a}p_2^{a-2}}{AP_1^{-a}P_2^{a-1}} = \frac{(1-a)m}{p_2}$$

(b) Find the expenditure function.

Solve $v(p_1, p_2, m) = \frac{Am}{P_1^a P_2^{1-a}}$ for m and set $e(\mathbf{p}, u) = m$, we obtain the expenditure function

$$e(\mathbf{p}, u) = \frac{u(p_1^a p_2^{1-a})}{A}.$$

(c) Use the Shephard's lemma to find the Hicksian demand function for goods 1 and 2.

The Shephard's lemma $h_i(p, u) = \frac{\partial e(p,u)}{\partial p_i}$.

Then the Hicksian demand functions are:

$$h_1(\mathbf{p}, u) = \frac{\partial e(\mathbf{p}, u)}{\partial p_1} = \frac{au(p_1^{a-1}p_2^{1-a})}{A} = \frac{au}{A}\left(\frac{p_2}{p_1}\right)^{1-a}$$

$$h_2(\mathbf{p}, u) = \frac{\partial e(\mathbf{p}, u)}{\partial p_2} = \frac{(1-a)u(p_1^a p_2^{-a})}{A} = \frac{(1-a)u}{A}\left(\frac{p_1}{p_2}\right)^{a}$$

(d) Verify the Slutsky equation for $\partial x_1/\partial p_1$

The Slutsky equation gives

$$\frac{\partial x_1(p,m)}{\partial p_1} = \frac{\partial h_1}{\partial p_1} - \frac{\partial x_1(p,m)}{\partial m} \cdot x_1$$

Using the Marshallian demand derived in (a) , we find

$$\frac{\partial x_1(p,m)}{\partial p_1} = -\frac{am}{(p_1)^2}$$
$$\frac{\partial x_1(p,m)}{\partial m} \cdot x_1 = \frac{aam}{p_1 p_1} = \frac{a^2 m}{p_1^2}$$

Using the Hicksian demand derived in (b), we find

$$\frac{\partial h_1}{\partial p_1} = (a-1)\frac{au(p_1^{a-2}p_2^{1-a})}{A} = (a-1)\frac{Am}{p_1^a p_2^{1-a}}\frac{a(p_1^{a-2}p_2^{1-a})}{A} = \frac{a(a-1)m}{p_1^2}$$

Hence

$$\frac{\partial h_1}{\partial p_1} - \frac{\partial x_1(p,m)}{\partial m} \cdot x_2 = \frac{a(a-1)m}{p_1^2} - \frac{a^2 m}{p_1^2} = -\frac{am}{(p_1)^2} = \frac{\partial x_1(p,m)}{\partial p_1}$$

∴ The Slutsky equation for $\partial x_1/\partial p_1$is verified.

Additional Exercise 2

2. From Exercise 8.6, we find the following indirect utility function and the expenditure function:

$$v(p_1,p_2,m) = [\frac{m}{5}]^{5/6}[\frac{3}{p_1}]^{1/2}[\frac{2}{p_2}]^{1/3}$$
$$e(p_1,p_2,u) = 5u^{6/5}[\frac{p_1}{3}]^{3/5}[\frac{p_2}{2}]^{2/5}$$

(a) Verify that the indirect utility function is increasing m and decreasing in p_i.

$$\frac{\partial v(p_1,p_2,m)}{\partial p_1} = -(\frac{1}{2})[\frac{m}{5}]^{5/6}[3]^{1/2}[p_1]^{-3/2}[\frac{2}{p_2}]^{1/3} < 0$$
$$\frac{\partial v(p_1,p_2,m)}{\partial p_2} = -(\frac{1}{3})[\frac{m}{5}]^{5/6}[\frac{3}{p_1}]^{1/2}[2]^{1/3}[p_2]^{-4/3} < 0$$
$$\frac{\partial v(p_1,p_2,m)}{\partial m} = \left(\frac{5}{6}\right)[5]^{-5/6}[m]^{-1/6}\left[\frac{3}{p_1}\right]^{1/2}\left[\frac{2}{p_2}\right]^{1/3} > 0$$

Therefore, the indirect utility function is increasing m and decreasing in p_i.

(b) Verify that the indirect utility function is homogeneous of degree 0 in (p_1,p_2,m).

For any $t > 1$,

$$v(tp_1,tp_2,tm) = [\frac{tm}{5}]^{5/6}[\frac{3}{tp_1}]^{1/2}[\frac{2}{tp_2}]^{1/3} = t^{5/6-1/2-1/3}[\frac{m}{5}]^{5/6}[\frac{3}{p_1}]^{1/2}[\frac{2}{p_2}]^{1/3}$$
$$= t^0[\frac{m}{5}]^{5/6}[\frac{3}{p_1}]^{1/2}[\frac{2}{p_2}]^{1/3} = [\frac{m}{5}]^{5/6}[\frac{3}{p_1}]^{1/2}[\frac{2}{p_2}]^{1/3} = v(p_1,p_2,m)$$

Therefore, the indirect utility function is homogeneous of degree 0 in (p_1,p_2,m).

(c) Verify that the expenditure function is increasing in u and p_i.

$$\frac{\partial e(p_1,p_2,u)}{\partial u} = 6u^{1/5}[\frac{p_1}{3}]^{3/5}[\frac{p_2}{2}]^{2/5} > 0$$
$$\frac{\partial e(p_1,p_2,u)}{\partial p_1} = 5\frac{3}{5}\frac{1}{3}u^{5/6}[\frac{p_1}{3}]^{-2/5}[\frac{p_2}{2}]^{2/5} = u^{6/5}[\frac{p_1}{3}]^{-2/5}[\frac{p_2}{2}]^{2/5} > 0$$
$$\frac{\partial e(p_1,p_2,u)}{\partial p_2} = 5\frac{2}{5}\frac{1}{2}u^{5/6}[\frac{p_1}{3}]^{3/5}[\frac{p_2}{2}]^{-3/5} = u^{6/5}[\frac{p_1}{3}]^{3/5}[\frac{p_2}{2}]^{-3/5} > 0$$

Therefore, the expenditure function is increasing in u and p_i.

(d) Verify that the expenditure function is concave in (p_1, p_2).

Use the Hessian matrix of $e(p_1, p_2, u)$.

$$\frac{\partial^2 e(p_1,p_2,u)}{\partial p_1^2} = -\frac{2}{15}u^{6/5}\left[\frac{p_1}{3}\right]^{-7/5}\left[\frac{p_2}{2}\right]^{2/5} < 0$$

$$\frac{\partial^2 e(p_1,p_2,u)}{\partial p_2^2} = -\frac{3}{10}u^{6/5}\left[\frac{p_1}{3}\right]^{3/5}\left[\frac{p_2}{2}\right]^{-8/5} < 0$$
$$\frac{\partial^2 e(p_1,p_2,u)}{\partial p_1\,\partial p_2} = \frac{\partial^2 e(p_1,p_2,u)}{\partial p_2\,\partial p_1} = \frac{1}{5}u^{6/5}[\frac{p_1}{3}]^{-2/5}[\frac{p_2}{2}]^{-3/5}$$

The determinant of the Hessian matrix,

$$|H| = \frac{\partial^2 e(p_1,p_2,u)}{\partial p_1^2}\frac{\partial^2 e(p_1,p_2,u)}{\partial p_2^2} - \frac{\partial^2 e(p_1,p_2,u)}{\partial p_1\,\partial p_2}\frac{\partial^2 e(p_1,p_2,u)}{\partial p_2\,\partial p_1}$$
$$= -\frac{2}{15}u^{6/5}\left[\frac{p_1}{3}\right]^{-7/5}\left[\frac{p_2}{2}\right]^{2/5}\left[-\frac{3}{10}\right]u^{6/5}\left[\frac{p_1}{3}\right]^{3/5}\left[\frac{p_2}{2}\right]^{-8/5} - \left\{\frac{1}{5}u^{6/5}\left[\frac{p_1}{3}\right]^{-2/5}\left[\frac{p_2}{2}\right]^{-3/5}\right\}^2 = 0.$$

The signs of the leading principal minors of the Hessian matrix confirm that it is negative semi-definite. Hence, this function is concave in (p_1, p_2).

(e) Find the money metric utility function and the money metric indirect utility function.

We need to use the direct utility function gave in exercise 8.6: $u(x_1, x_2) = {x_1}^{1/2}{x_2}^{1/3}$

The money metric utility function is

$$m(\mathbf{p}, x) \equiv e\big(\mathbf{p}, u(x)\big) = 5({x_1}^{\frac{1}{2}}{x_2}^{\frac{1}{3}})^{6/5}\left[\frac{p_1}{3}\right]^{3/5}\left[\frac{p_2}{2}\right]^{2/5} = 5{x_1}^{3/5}{x_2}^{2/5}\left[\frac{p_1}{3}\right]^{3/5}\left[\frac{p_2}{2}\right]^{2/5}$$

And the money metric indirect utility function is

$$\mu(\mathbf{p}; \mathbf{q}, m) \equiv e(\mathbf{p}, v(\mathbf{q}, m)) = 5\left\{\left[\frac{m}{5}\right]^{5/6}\left[\frac{3}{q_1}\right]^{1/2}\left[\frac{2}{q_2}\right]^{1/3}\right\}^{6/5}\left[\frac{p_1}{3}\right]^{3/5}\left[\frac{p_2}{2}\right]^{2/5}$$
$$= m\left[\frac{3}{q_1}\right]^{3/5}\left[\frac{2}{q_2}\right]^{2/5}\left[\frac{p_1}{3}\right]^{3/5}\left[\frac{p_2}{2}\right]^{2/5} = m\left[\frac{p_1}{q_1}\right]^{3/5}\left[\frac{p_2}{q_2}\right]^{2/5}.$$

Answers to Exercises

Microeconomic Analysis

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Chapter 7. Utility Maximization

7.1 The preferences exhibit local nonsatiation, *except* at $(0,0)$. The consumer will choose this consumption point when faced with positive prices.

7.2 The demand function is

$$x_1 = \begin{cases} m/p_1 & \text{if } p_1 < p_2 \\ \text{any } x_1 \text{ and } x_2 \text{ such that } p_1 x_2 + p_2 x_2 = m & \text{if } p_1 = p_2 \\ 0 & \text{if } p_1 > p_2 \end{cases}$$

The indirect utility function is $v(p_1, p_2, m) = \max\{m/p_1, m/p_2\}$, and the expenditure function is $e(p_1, p_2, u) = u \min\{p_1, p_2\}$.

7.3 The expenditure function is $e(p_1, p_2, u) = u \min\{p_1, p_2\}$. The utility function is $u(x_1, x_2) = x_1 + x_2$ (or any monotonic transformation), and the demand function is

$$x_1 = \begin{cases} m/p_1 & \text{if } p_1 < p_2 \\ \text{any } x_1 \text{ and } x_2 \text{ such that } p_1 x_1 + p_2 x_2 = m & \text{if } p_1 = p_2 \\ 0 & \text{if } p_1 > p_2 \end{cases}$$

7.4.a Demand functions are $x_1 = m/(p_1 + p_2)$, $x_2 = m/(p_1 + p_2)$.

7.4.b $e(p_1, p_2, u) = (p_1 + p + 2)u$

7.4.c $u(x_1, x_2) = \min\{x_1, x_2\}$

7.5.a Quasilinear preferences.

7.5.b Less than $u(1)$.

7.5.c $v(p_1, p_2, m) = \max\{u(1) - p_1 + m, m\}$

7.6.a Homothetic.

7.6.b $e(\mathbf{p}, u) = u/A(p)$

7.6.c $\mu(\mathbf{p}; \mathbf{q}, m) = mA(\mathbf{q})/A(\mathbf{p})$

7.6.d It will be the same, since this is just a monotonic transformation.

Chapter 8. Choice

8.1 We know that

$$x_j(\mathbf{p}, m) \equiv h_j(\mathbf{p}, v(\mathbf{p}, m)) \equiv \partial e(\mathbf{p}, v(\mathbf{p}, m))/\partial p_j. \tag{0.1}$$

8.5 Write the Lagrangian

$$\mathcal{L}(\mathbf{x}, \lambda) = \frac{3}{2} \ln x_1 + \ln x_2 - \lambda(3x_1 + 4x_2 - 100).$$

(Be sure you understand why we can transform u this way.) Now, equating the derivatives with respect to x_1, x_2, and λ to zero, we get three equations in three unknowns

$$\frac{3}{2x_1} = 3\lambda,$$
$$\frac{1}{x_2} = 4\lambda,$$
$$3x_1 + 4x_2 = 100.$$

Solving, we get

$$x_1(3, 4, 100) = 20, \text{ and } x_2(3, 4, 100) = 10.$$

Note that if you are going to interpret the Lagrange multiplier as the marginal utility of income, you must be explicit as to which utility function you are referring to. Thus, the marginal utility of income can be measured in original *'utils'* or in '$\ln$ *utils*'. Let $u^* = \ln u$ and, correspondingly, $v^* = \ln v$; then

$$\lambda = \frac{\partial v^*(\mathbf{p}, m)}{\partial m} = \frac{\frac{\partial v(\mathbf{p}, m)}{\partial m}}{v(\mathbf{p}, m)} = \frac{\mu}{v(\mathbf{p}, m)},$$

where μ denotes the Lagrange multiplier in the Lagrangian

$$L(\mathbf{x}, \mu) = x_1^{\frac{3}{2}} x_2 - \mu(3x_1 + 4x_2 - 100).$$

Check that in this problem we'd get $\mu = \frac{20^{\frac{3}{2}}}{4}$, $\lambda = \frac{1}{40}$, and $v(3, 4, 100) = 20^{\frac{3}{2}} 10$.

8.6 The Lagrangian for the utility maximization problem is

$$\mathcal{L}(\mathbf{x}, \lambda) = x_1^{\frac{1}{2}} x_2^{\frac{1}{3}} - \lambda(p_1 x_1 + p_2 x_2 - m),$$

taking derivatives,

$$\frac{1}{2} x_1^{-\frac{1}{2}} x_2^{\frac{1}{3}} = \lambda p_1,$$
$$\frac{1}{3} x_1^{\frac{1}{2}} x_2^{-\frac{2}{3}} = \lambda p_2,$$
$$p_1 x_1 + p_2 x_2 = m.$$

Solving, we get

$$x_1(\mathbf{p}, m) = \frac{3}{5} \frac{m}{p_1}, \; x_2(\mathbf{p}, m) = \frac{2}{5} \frac{m}{p_2}.$$

Plugging these demands into the utility function, we get the indirect utility function

$$v(\mathbf{p}, m) = U(\mathbf{x}(\mathbf{p}, m)) = \left(\frac{3}{5}\frac{m}{p_1}\right)^{\frac{1}{2}} \left(\frac{2}{5}\frac{m}{p_2}\right)^{\frac{1}{3}} = \left(\frac{m}{5}\right)^{\frac{5}{6}} \left(\frac{3}{p_1}\right)^{\frac{1}{2}} \left(\frac{2}{p_2}\right)^{\frac{1}{3}}.$$

Rewrite the above expression replacing $v(\mathbf{p}, m)$ by u and m by $e(\mathbf{p}, u)$. Then solve it for $e(\cdot)$ to get

$$e(\mathbf{p}, u) = 5\left(\frac{p_1}{3}\right)^{\frac{3}{5}} \left(\frac{p_2}{2}\right)^{\frac{2}{5}} u^{\frac{6}{5}}.$$

Finally, since $h_i = \partial e/\partial p_i$, the Hicksian demands are

$$h_1(\mathbf{p}, u) = \left(\frac{p_1}{3}\right)^{-\frac{2}{5}} \left(\frac{p_2}{2}\right)^{\frac{2}{5}} u^{\frac{6}{5}},$$

and

$$h_2(\mathbf{p}, u) = \left(\frac{p_1}{3}\right)^{\frac{3}{5}} \left(\frac{p_2}{2}\right)^{-\frac{3}{5}} u^{\frac{6}{5}}.$$

8.7 Instead of starting from the utility maximization problem, let's now start from the expenditure minimization problem. The Lagrangian is

$$\mathcal{L}(\mathbf{x}, \mu) = p_1 x_1 + p_2 x_2 - \mu((x_1 - \alpha_1)^{\beta_1}(x_2 - \alpha_2)^{\beta_2} - u);$$

the first-order conditions are

$$\begin{aligned} p_1 &= \mu\beta_1(x_1 - \alpha_1)^{\beta_1 - 1}(x_2 - \alpha_2)^{\beta_2}, \\ p_2 &= \mu\beta_2(x_1 - \alpha_1)^{\beta_1}(x_2 - \alpha_2)^{\beta_2 - 1}, \\ (x_1 - \alpha_1)^{\beta_1}(x_2 - \alpha_2)^{\beta_2} &= u. \end{aligned}$$

Divide the first equation by the second

$$\frac{p_1\beta_2}{p_2\beta_1} = \frac{x_2 - \alpha_2}{x_1 - \alpha_1},$$

using the last equation

$$x_2 - \alpha_2 = \left((x_1 - \alpha_1)^{-\beta_1} u\right)^{\frac{1}{\beta_2}};$$

substituting and solving,

$$h_1(\mathbf{p}, u) = \alpha_1 + \left(\frac{p_2\beta_1}{p_1\beta_2} u^{\frac{1}{\beta_2}}\right)^{\frac{\beta_2}{\beta_1 + \beta_2}},$$

and

$$h_2(\mathbf{p}, u) = \alpha_2 + \left(\frac{p_1\beta_2}{p_2\beta_1}u^{\frac{1}{\beta_1}}\right)^{\frac{\beta_1}{\beta_1+\beta_2}}.$$

Verify that

$$\frac{\partial h_1(\mathbf{p}, m)}{\partial p_2} = \left(\frac{u}{\beta_1+\beta_2}\left(\frac{\beta_1}{p_1}\right)^{\beta_2}\left(\frac{\beta_2}{p_2}\right)^{\beta_1}\right)^{\frac{1}{\beta_1+\beta_2}} = \frac{\partial h_2(\mathbf{p}, m)}{\partial p_1}.$$

The expenditure function is

$$e(\mathbf{p}, u) = p_1\left(\alpha_1 + \left(\frac{p_2\beta_1}{p_1\beta_2}u^{\frac{1}{\beta_2}}\right)^{\frac{\beta_2}{\beta_1+\beta_2}}\right) + p_2\left(\alpha_2 + \left(\frac{p_1\beta_2}{p_2\beta_1}u^{\frac{1}{\beta_1}}\right)^{\frac{\beta_1}{\beta_1+\beta_2}}\right).$$

Solving for u, we get the indirect utility function

$$v(\mathbf{p}, m) = \left(\frac{\beta_1}{\beta_1+\beta_2}\left(\frac{m-\alpha_2 p_2}{p_1} - \alpha_1\right)\right)^{\beta_1}\left(\frac{\beta_2}{\beta_1+\beta_2}\left(\frac{m-\alpha_1 p_1}{p_2} - \alpha_2\right)\right)^{\beta_2}.$$

By Roy's law we get the Marshallian demands

$$x_1(\mathbf{p}, m) = \frac{1}{\beta_1+\beta_2}\left(\beta_1\alpha_2 + \beta_2\frac{m-\alpha_1 p_1}{p_2}\right),$$

and

$$x_2(\mathbf{p}, m) = \frac{1}{\beta_1+\beta_2}\left(\beta_2\alpha_1 + \beta_1\frac{m-\alpha_2 p_2}{p_1}\right).$$

8.8 Easy—a monotonic transformation of utility doesn't change anything about observed behavior.

8.9 By definition, the Marshallian demands $\mathbf{x}(\mathbf{p}, m)$ maximize $\phi(\mathbf{x})$ subject to $\mathbf{px} = m$. We claim that they also maximize $\psi(\phi(\mathbf{x}))$ subject to the same budget constraint. Suppose not. Then, there would exist some other choice $\mathbf{x}'$ such that $\psi(\phi(\mathbf{x}')) > \psi(\phi(\mathbf{x}(\mathbf{p}, m)))$ and $\mathbf{px}' = m$. But since applying the transformation $\psi^{-1}(\cdot)$ to both sides of the inequality will preserve it, we would have $\phi(\mathbf{x}') > \phi(\mathbf{x}(\mathbf{p}, m))$ and $\mathbf{px}' = m$, which contradicts our initial assumption that $\mathbf{x}(\mathbf{p}, m)$ maximized $\phi(\mathbf{x})$ subject to $\mathbf{px} = m$. Therefore $\mathbf{x}(\mathbf{p}, m) = \mathbf{x}^*(\mathbf{p}, m)$. (Check that the reverse proposition also holds—i.e., the choice that maximizes u^* also maximizes u when the the same budget constraint has to be verified in both cases.)

$$v^*(\mathbf{p}, m) = \psi(\phi(\mathbf{x}^*(\mathbf{p}, m))) = \psi(\phi(\mathbf{x}(\mathbf{p}, m)) = \psi(v(\mathbf{p}, m)),$$

8.10.c The rate of return—also known as "own rate of interest"—on good x is $(p_1/p_2) - 1$

8.11 No, because his demand behavior violates GARP. When prices are $(2, 4)$ he spends 10. At these prices he could afford the bundle $(2, 1)$, but rejects it; therefore, $(1, 2) \succ (2, 1)$. When prices are $(6, 3)$ he spends 15. At these prices he could afford the bundle $(1, 2)$ but rejects it; therefore, $(2, 1) \succ (1, 2)$.

8.12 Inverting, we have $e(p, u) = u/f(p)$. Substituting, we have

$$\mu(p; q, y) = v(q, y)/f(p) = f(q)y/f(p).$$

8.13.a Draw the lines $x_2 + 2x_1 = 20$ and $x_1 + 2x_2 = 20$. The indifference curve is the northeast boundary of this X.

8.13.b The slope of a budget line is $-p_1/p_2$. If the budget line is steeper than 2, $x_1 = 0$. Hence the condition is $p_1/p_2 > 2$.

8.13.c Similarly, if the budget line is flatter than 1/2, x_2 will equal 0, so the condition is $p_1/p_2 < 1/2$.

8.13.d If the optimum is unique, it must occur where $x_2 - 2x_1 = x_1 - 2x_2$. This implies that $x_1 = x_2$, so that $x_1/x_2 = 1$.

8.14.a This is an ordinary Cobb-Douglas demand: $S_1 = \frac{\alpha}{\alpha+\beta+\gamma}Y$ and $S_2 = \frac{\beta}{\alpha+\beta+\gamma}Y$.

8.14.b In this case the utility function becomes $U(C, S_1, L) = S_1^\alpha L^\beta C^\gamma$. The L term is just a constant, so applying the standard Cobb-Douglas formula $S_1 = \frac{\alpha}{\alpha+\gamma}Y$.

8.15 Use Slutsky's equation to write: $\dfrac{\partial L}{\partial w} = \dfrac{\partial L^s}{\partial w} + (\overline{L} - L)\dfrac{\partial L}{\partial m}$. Note that the substitution effect is always negative, $(\overline{L} - L)$ is always positive, and hence if leisure is inferior, $\dfrac{\partial L}{\partial w}$ is necessarily negative. Thus the slope of the labor supply curve is positive.

8.16.a True. With the grant, the consumer will maximize $u(x_1, x_2)$ subject to $x_1 + x_2 \leq m + g_1$ and $x_1 \geq g_1$. We know that when he maximizes his utility subject to $x_1 + x_2 \leq m$, he chooses $x_1^* \geq g_1$. Since x_1 is a normal good, the amount of good 1 that he will choose if given an unconstrained grant of g_1 is some number $x_1' > x_1^* \geq g_1$. Since this choice satisfies the constraint $x_1' \geq g_1$, it is also the choice he would make when forced to spend g_1 on good 1.