ECON4020

Advanced Microeconomic Theory

Answers to Problem Set One

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Note: There are two parts to this document. In the first part (pages ii – x) are the answers to exercises 1.2, 1.3, 2.3, 2.7, 3.1, 3.3, 3.4, 5.1, 5.8, 5.9, and the two additional exercises. In the second part are selected pages from Varian's own answers to the exercises. However, some of these answers by Varian lack specifics. That is why I have added the first part.

Chapter 1

1.2 The CES production function: $\boldsymbol{y = (a_1 x_1^{\rho} + a_2 x_2^{\rho})^{\frac{1}{\rho}}}$

$$\frac{dy}{dx_1} = \frac{1}{\rho}(a_1 x_1^{\rho} + a_2 x_2^{\rho})^{\frac{1}{\rho}-1}\rho a_1 x_1^{\rho-1} = (a_1 x_1^{\rho} + a_2 x_2^{\rho})^{\frac{1}{\rho}-1} a_1 x_1^{\rho-1} \quad (1)$$

$$\frac{dy}{dx_2} = \frac{1}{\rho}(a_1 x_1^{\rho} + a_2 x_2^{\rho})^{\frac{1}{\rho}-1}\rho a_2 x_2^{\rho-1} = (a_1 x_1^{\rho} + a_2 x_2^{\rho})^{\frac{1}{\rho}-1} a_2 x_2^{\rho-1} \quad (2)$$

Using (1) and (2) to obtain:

$$TRS = -\frac{\partial y/\partial x_1}{\partial y/\partial x_2} = -\frac{a_1 x_1^{\rho-1}}{a_2 x_2^{\rho-1}} = -\frac{a_1}{a_2}\cdot\left(\frac{x_1}{x_2}\right)^{\rho-1} \quad (3)$$

$$\Rightarrow \frac{x_1}{x_2} = -\left(\frac{a_2}{a_1}\right)^{\frac{1}{\rho-1}}\cdot(TRS)^{\frac{1}{\rho-1}}$$

$$\Rightarrow \frac{x_2}{x_1} = -\left(\frac{a_2}{a_1}\right)^{\frac{1}{1-\rho}}\cdot(TRS)^{\frac{1}{1-\rho}} \quad (4)$$

$$\Rightarrow \frac{d(x_2/x_1)}{d\,TRS} = -\left(\frac{a_2}{a_1}\right)^{\frac{1}{1-\rho}}\cdot\frac{1}{1-\rho}(TRS)^{\frac{1}{1-\rho}-1} \quad (5)$$

Substitute (3) (4) (5) into the elasticity of substitution formula:

$$\Rightarrow \sigma = \frac{TRS}{x_2/x_1}\cdot\frac{dx_2/x_1}{d\,TRS} = -\frac{TRS}{x_2/x_1}\cdot\frac{1}{1-\rho}\left(\frac{a_2}{a_1}\right)^{\frac{1}{1-\rho}}(TRS)^{\frac{1}{1-\rho}-1} = -\frac{(TRS)^{\frac{1}{1-\rho}}}{-\left(\frac{a_2}{a_1}\right)^{\frac{1}{1-\rho}}\cdot(TRS)^{\frac{1}{1-\rho}}}\cdot\frac{1}{1-\rho}\left(\frac{a_2}{a_1}\right)^{\frac{1}{1-\rho}} = \frac{1}{1-\rho}$$

1.3 The output elasticity of a factor 1 and 2:

$$\epsilon_1(\mathbf{x}) = \frac{\partial f(x_1, x_2)}{\partial x_1}\frac{x_1}{f(x_1, x_2)} = a x_1^{a-1} x_2^{b}\frac{x_1}{x_1^{a} x_2^{b}} = a$$

$$\epsilon_2(\mathbf{x}) = \frac{\partial f(x_1, x_2)}{\partial x_2}\frac{x_2}{f(x_1, x_2)} = a x_1^{a} x_2^{b-1}\frac{x_2}{x_1^{a} x_2^{b}} = b$$

Chapter 2

2.3 The production function is $\boldsymbol{y = x^{a}, 0 < a < 1}$.

a) The profit-maximization problem: $\pi = max_x\, px^{a} - wx$.

F.O.C. $pa\,x^{\,a-1} = w \implies x(p,w) = \left(\frac{w}{ap}\right)^{\frac{1}{a-1}}$

S.O.C.: $pa\,(a-1)\,x^{\,a-2} \le 0$ is satisfied because $0 < a < 1$.

Hence, the input demand and output supply functions are:

$$x(p,w) = \left(\frac{w}{ap}\right)^{\frac{1}{a-1}} \quad \text{and} \quad y = x^{a} = \left(\frac{w}{ap}\right)^{\frac{a}{a-1}}$$

The profit function: $\pi = py - wx = px^a - wx = p\left(\frac{w}{ap}\right)^{\frac{a}{a-1}} - w\left(\frac{w}{ap}\right)^{\frac{1}{a-1}}$

We have $\pi = p\left(\frac{w}{ap}\right)^{\frac{a}{a-1}} - w\left(\frac{w}{ap}\right)^{\frac{1}{a-1}} = p^{\frac{1}{1-a}} w^{\frac{a}{a-1}}\left(a^{\frac{a}{1-a}} - a^{\frac{1}{1-a}}\right)$

To prove that the profit function is homogenous, note that

$\pi(tp, tw) = tw\left(\frac{1-a}{a}\right)\left(\frac{tw}{atp}\right)^{\frac{1}{a-1}} = t\pi(p, w)$, which indicates that $\pi(p, w)$ is homogenous of degree 1.

The profit function π is convex in (p, w) if and only if the Hessian Matrix

$$D^2\pi(p, w) = \begin{pmatrix} \frac{\partial^2\pi}{\partial p^2} & \frac{\partial^2\pi}{\partial p\partial w} \\ \frac{\partial^2\pi}{\partial w\partial p} & \frac{\partial^2\pi}{\partial w^2} \end{pmatrix}$$

is positive semidefinite.

Given $0 < a < 1$, so $a^{\frac{a}{1-a}} - a^{\frac{1}{1-a}} > 0$

$$\frac{\partial\pi}{\partial p} = \frac{1}{1-a} p^{\frac{a}{1-a}} w^{\frac{a}{a-1}}\left(a^{\frac{a}{1-a}} - a^{\frac{1}{1-a}}\right)$$

$$\frac{\partial^2\pi}{\partial p^2} = \frac{a}{(1-a)^2} p^{\frac{2a-1}{1-a}} w^{\frac{a}{a-1}}\left(a^{\frac{a}{1-a}} - a^{\frac{1}{1-a}}\right) > 0$$

$$\frac{\partial^2\pi}{\partial p\partial w} = \frac{-a}{(1-a)^2} p^{\frac{a}{1-a}} w^{\frac{1}{a-1}}\left(a^{\frac{a}{1-a}} - a^{\frac{1}{1-a}}\right)$$

$$\frac{\partial\pi}{\partial w} = \frac{a}{a-1} p^{\frac{1}{1-a}} w^{\frac{1}{a-1}}\left(a^{\frac{a}{1-a}} - a^{\frac{1}{1-a}}\right)$$

$$\frac{\partial^2\pi}{\partial w^2} = \frac{a}{(a-1)^2} p^{\frac{1}{1-a}} w^{\frac{2-a}{a-1}}\left(a^{\frac{a}{1-a}} - a^{\frac{1}{1-a}}\right)$$

=> $|D_1| = \left|\frac{\partial^2\pi}{\partial p^2}\right| > 0$

=> $|D_2| = \begin{vmatrix} \frac{\partial^2\pi}{\partial p^2} & \frac{\partial^2\pi}{\partial p\partial w} \\ \frac{\partial^2\pi}{\partial w\partial p} & \frac{\partial^2\pi}{\partial w^2} \end{vmatrix}$

$$= \begin{vmatrix} \frac{a}{(a-1)^2} p^{\frac{2a-1}{1-a}} w^{\frac{a}{a-1}}\left(a^{\frac{a}{1-a}} - a^{\frac{1}{1-a}}\right) & \frac{-a}{(a-1)^2} p^{\frac{a}{1-a}} w^{\frac{1}{a-1}}\left(a^{\frac{a}{1-a}} - a^{\frac{1}{1-a}}\right) \\ \frac{-a}{(a-1)^2} p^{\frac{a}{1-a}} w^{\frac{1}{a-1}}\left(a^{\frac{a}{1-a}} - a^{\frac{1}{1-a}}\right) & \frac{a}{(a-1)^2} p^{\frac{1}{1-a}} w^{\frac{2-a}{a-1}}\left(a^{\frac{a}{1-a}} - a^{\frac{1}{1-a}}\right)\frac{\partial^2\pi}{\partial w^2} \end{vmatrix} = 0$$

$\therefore D^2\pi(p, w)$ is positive semidefinite

2.7 The production function is $\boldsymbol{f(x) = 20x - x^2}$ and the price of output is normalized to 1.
a) The profit maximization problem:

$$\pi = max_x \ (20x - x^2 - wx)$$

The first-order condition for an interior solution: $\frac{d\pi}{dx} = 20 - 2x - w = 0$

=> $x^* = 10 - \frac{w}{2}$

Note that $x^* > 0$ if and only if $w < 20$.

b) For what values of w will the optimal x be zero?

Note that $\frac{d\pi}{dx} \leq 0$ when $w \geq 20$. Hence, $x^* = 0$ if $w \geq 20$.

c) For what values of w will the optimal x be 10?

Setting $x^* = 10 - \frac{w}{2} = 0$ to obtain $w = 0$.

d) The factor demand function is:

$x = 10 - \frac{w}{2}$.

e) The profit function:

$$\pi = (20 - w - x)x = \left(20 - w - 10 + \frac{w}{2}\right)\left(10 - \frac{w}{2}\right) = \left(10 - \frac{w}{2}\right)^2$$

f) The derivative of the profit function w.r.t. w:

$$\frac{d\pi(w)}{dw} = 2\left(10 - \frac{w}{2}\right)\left(-\frac{1}{2}\right) = \frac{w}{2} - 10$$

Chapter 3

3.1 The profit function is $\boldsymbol{\pi(\mathbf{w}) = \Phi_1(w_1) + \Phi_1(w_2)}$, with $p = 1$.
a) The profit function is non-increasing in w_i. Hence, $\frac{\partial\pi(\mathbf{w})}{\partial w_1} = \Phi_1{}'(w_1) \leq 0$, and $\frac{\partial\pi(\mathbf{w})}{\partial w_2} = \Phi_2{}'(w_2) \leq 0$.

Let $\pi_{ij} \equiv \frac{\partial^2\pi}{\partial w_i \partial w_j}$. Because π is convex in prices, the Hessian Matrix is positive semidefinite. Each element along the principal diagonal of a positive semidefinite matrix must be non-negative. Hence

$\pi_{11} = \Phi_1{}''(w_1) \geq 0$, and $\pi_{22} = \Phi_2{}''(w_2) \geq 0$. Note also that $\pi_{12} = \pi_{21} = 0$.

b) By Hotelling's lemma,

$x_1 = -\frac{\partial \pi(\mathbf{w})}{\partial w_1} = -\Phi_1'(w_1)$, and $x_2 = -\frac{\partial \pi(\mathbf{w})}{\partial w_2} = -\Phi_2'(w_2)$.

Then $\frac{\partial x_i}{\partial w_j} = -\frac{\partial^2 \pi}{\partial w_i \partial w_j} = 0$.

c) Note that $x_i = -\frac{\partial \pi}{\partial w_i} = -\Phi_i'(w_i)$. The demand for factor *i* is only a function of the *i*th price. This indicated that the production function must be in the form that the marginal product of factor i depends only on x_i. It follows that the production function takes the form $f(x_1, x_2) = g(x_1) + g(x_2)$. To see this, consider the profit maximization problem (recalling that *p* = 1):

$$max_{x_1,x_2}\ g(x_1) + g(x_2) - wx_1 - w\,x_2$$

From the first-order conditions, we have $g'(x_1) = w_1$ and $g'(x_2) = w_2$. This confirms that the demand for factor *i* is a function of w_i only.

3.3 Solve the profit maximization problem:

$$\pi = max_{\ x_1,x_2}\, pf(x_1,x_2) - x_1w_1 - x_2w_2 = max_{\ x_1,x_2}\, p(a_1 lnx_1 + a_2 lnx_2) - x_1w_1 - x_2w_2$$

F.O.C.s:

$\frac{\partial \pi}{\partial x_1} = p\frac{a_1}{x_1} - w_1 = 0$ => $p\frac{a_1}{x_1} = w_1$ => Demand for factor 1: $x_1^* = \frac{pa_1}{w_1}$.

$\frac{\partial \pi}{\partial x_2} = p\frac{a_2}{x_2} - w_2 = 0$ => $p\frac{a_2}{x_2} = w_2$ => Demand for factor 2: $x_2^* = \frac{pa_2}{w_2}$.

Output supply function: $y = f(x_1^*, x_2^*) = (a_1 ln\frac{pa_1}{w_1} + a_2 ln\frac{pa_2}{w_2})$.

The profit function: $\pi = p(a_1 ln\frac{pa_1}{w_1} + a_2 ln\frac{pa_2}{w_2}) - pa_1 - pa_2$.

3.4 Solve for profit maximization problem:

$$\pi = max_{\ x_1,x_2}\, pf(x_1,x_2) - x_1w_1 - x_2w_2 = max_{\ x_1,x_2}\, p(x_1^{a_1}x_2^{a_2}) - x_1w_1 - x_2w_2$$

F.O.C.s:

$\frac{\partial \pi}{\partial x_1} = pa_1x_1^{a_1-1}x_2^{a_2} - w_1 = 0$ => $pa_1x_1^{a_1-1}x_2^{a_2} = w_1$ (1)

$\frac{\partial \pi}{\partial x_2} = pa_2x_2^{a_2-1}x_1^{a_1} - w_2 = 0$ => $pa_2x_2^{a_2-1}x_1^{a_1} = w_2$ (2)

(1)÷(2):

$\frac{a_1x_2}{a_2x_1} = \frac{w_1}{w_2}$ => $x_2 = \frac{w_1}{w_2}\frac{a_2}{a_1}x_1$ (3)

Substitute (3) into (1):

$$pa_1 x_1^{a_1-1}(\frac{w_1}{w_2}\frac{a_2}{a_1}x_1)^{a_2} = w_1 \quad => \quad x_1^{a_1-1+a_2} = \frac{w_1^{1-a_2}\, w_2^{a_2}}{pa_1^{1-a_2}\, a_2^{a_2}}$$

$$=> \quad x_1^* = p^{\frac{1}{1-a_1-a_2}}(\frac{w_1}{a_1})^{\frac{1-a_2}{a_1+a_2-1}}\,(\frac{w_2}{a_2})^{\frac{a_2}{a_1+a_2-1}} \quad (4)$$

Substitute (4) into (3):

$$x_2^* = \frac{w_1}{w_2}\frac{a_2}{a_1}p^{\frac{1}{1-a_1-a_2}}\left(\frac{w_1}{a_1}\right)^{\frac{1-a_2}{a_1+a_2-1}}\left(\frac{w_2}{a_2}\right)^{\frac{a_2}{a_1+a_2-1}}$$

$$= p^{\frac{1}{1-a_1-a_2}}\, w_1^{\frac{a_1}{a_1+a_2-1}} a_1^{\frac{-a_1}{a_1+a_2-1}} w_2^{\frac{1-a_1}{a_1+a_2-1}}\, a_2^{\frac{-(1-a_1)}{a_1+a_2-1}}$$

$$= p^{\frac{1}{1-a_1-a_2}}(\frac{w_1}{a_1})^{\frac{a_1}{a_1+a_2-1}}\,(\frac{w_2}{a_2})^{\frac{1-a_1}{a_1+a_2-1}}$$

In order to satisfy the second-order conditions of the profit maximization problem, the technology must exhibit decreasing returns to scale; so $a_1 + a_2 < 1$.

Chapter 5

5.1 The cost functions of the two plants:

$$c_1(y_1) = y_1^2$$

$$c_2(y_2) = y_2^2$$

The firm chooses the output levels at the two plants to minimize its total cost of production:

$$c(y) = \min_{y_1,y_2}\ \{c_1(y_1) + c_2(y_2)\} = \min\{y_1^2 + y_2^2\} \quad s.t.\, y_1 + y_2 = y$$

F.O.Cs:

$$2y_1 - \lambda = 0$$

$$2y_2 - \lambda = 0$$

So we must have $y_1^* = y_2^* = \frac{y}{2}$. Substituting into the objective function yields

$$c(y) = (\frac{y}{2})^2 + (\frac{y}{2})^2 = \frac{y^2}{2}$$

5.8 Using the cost function, we write the firm's profit as $\boldsymbol{\pi = py - y^2 - 1}$ for $\boldsymbol{y > 0}$. To maximize its profit, the firm sets

$\frac{d\pi}{dy} = p - 2y = 0 \qquad => y^* = {}^{p}/_{2}$

Note that if the price of the output is too low, , the firm would earn a negative profit by producing a positive quantity. For example,

at $p = 1$ => $y^* = \frac{1}{2}, \pi = \frac{1}{2} - \frac{1}{4} - 1 = -\frac{3}{4} < 0.$

In such a situation, the firm would rather produce no output and earn zero profit, since $c(y) = 0$ at $y = 0$. Therefore, the firm's profit function is

$$\pi(p) = \max\ \{\frac{p^2}{2} - \frac{p^2}{4} - 1, 0\} = \max\ \{\frac{p^2}{4} - 1, 0\}.$$

At $p = 2, \frac{p^2}{4} - 1 = 0$. Hence, the firm will produce according to $y^* = {}^{p}/_{2} = 1.$

5.9 Since fraction 1- $\boldsymbol{\alpha}$ of the firm's output are defective and cannot be sold, its revenue from producing y units of output is $\boldsymbol{\alpha y}$.

a) The firm's profit function: $\pi = max_y\ p\alpha y - c(y)$.
   Using the envelope theorem, we obtain: $\frac{d\pi}{d\alpha} = py > 0$.
   The sign of the above derivative is positive because $p > 0\ and\ y > 0$.
b) To find the output y, we take derivative of π w.r.t. y

$\frac{d\pi}{dy} = \alpha p - c'(y) = 0$ => $\alpha p = c'(y)$ (1)

Note that the solution of y depends on α. Differentiating (1) w.r.t. α, we have

$p = c''(y) \cdot \frac{dy}{d\alpha}$ => $\frac{dy}{d\alpha} = \frac{p}{c''(y)}$ (2)

Since $c''(y) > 0$ (rising marginal cost), we have $\frac{dy}{d\alpha} > 0$.

c) With n identical chip producers, the total supply of (functional) chips are $n\alpha y$. The equilibrium price is determined by the market clearing condition:

$D(p) = n\alpha y$ (3)

=> $\frac{D(p)}{n\alpha} = y$ (4)

Sub (4) into (1), we obtain

$\alpha p = c'(\frac{D(p)}{n\alpha})$ (5)

Equation (5) determines the equilibrium price p as a function of α. Differentiate (5) w.r.t. α:

=> $p + \alpha p'(\alpha) = c''(\frac{D(p)}{n\alpha}) \cdot \frac{d(\frac{D(p)}{n\alpha})}{d\alpha}$ (6)

And $\frac{d(\frac{D(p)}{n\alpha})}{d\alpha} = \frac{1}{n}\frac{D'(p)p'(\alpha)\alpha - D(p)}{(\alpha)^2} = \frac{1}{n\alpha}\,p'(\alpha)D'(p) - \frac{D(p)}{n\alpha^2}$

=> $p + \alpha p'(\alpha) = c''\left(\frac{D(p)}{n\alpha}\right) \cdot \left(\frac{1}{n\alpha}\,p'(\alpha)D'(p) - \frac{D(p)}{n\alpha^2}\right) = \frac{1}{n\alpha}p'(\alpha)\,c''(y)D'(p) - c''(y)\frac{D(p)}{n\alpha^2}$

=> $p + c''(y)\frac{D(p)}{n\alpha^2} = p'(\alpha)[\frac{c''(y)D'}{n\alpha} - \alpha]$

=> $p'(\alpha) = \frac{p + c''(y)\frac{D(p)}{n\alpha^2}}{\frac{c''(y)D'}{n\alpha} - \alpha}$ and since $D(p) = n\alpha y$,

$$\Rightarrow p'(\alpha) = \frac{p + c''(y)\frac{y}{\alpha}}{\frac{c''(y)D'}{n\alpha} - \alpha} = \frac{n\alpha p + c''(y)ny}{c''(y)D' - n\alpha^2} = \frac{n\alpha\frac{p}{c''} + ny}{D' - \frac{n\alpha^2}{c''}} = \frac{n(\alpha\frac{p}{c''} + y)}{D' - \frac{n\alpha^2}{c''}}$$

Note that $p'(\alpha) < 0$ because $D' < 0$ and $c'' > 0$. The sign of $(\frac{dp}{d\alpha})/p$ is the same as that of $p'(\alpha)$.

Additional Exercises

A1. Derive the cost functions associated with the following production technologies:

(a) Cobb-Douglas technology: $\boldsymbol{q = Ax_1{}^a x_2{}^b}$

$$\min_{x_1,x_2}\{w_1x_1 + w_2x_2\}\ s.t.\ \mathrm{A}x_1{}^a x_2{}^b = q$$

$$\mathcal{L}(\lambda, \mathbf{x}) = w_1x_1 + w_2x_2 - \lambda\left(\mathrm{A}x_1{}^a x_2{}^b - q\right)$$

F.O.C.s:

$\frac{\partial\mathcal{L}}{\partial x_1} = w_1 - A\lambda a x_1{}^{a-1}x_2{}^b = 0$ => $\quad w_1 = A\lambda a x_1{}^{a-1}x_2{}^b \qquad (1)$

$\frac{\partial\mathcal{L}}{\partial x_2} = w_2 - A\lambda a x_1{}^a x_2{}^{b-1} = 0$ => $\quad w_2 = A\lambda b x_1{}^a x_2{}^{b-1} \qquad (2)$

$\frac{\partial\mathcal{L}}{\partial\lambda} = \mathrm{A}x_1{}^a x_2{}^b - q = 0$

(1) ÷ (2):

$$\frac{w_1}{w_2} = \frac{A\lambda a x_1{}^{a-1}x_2{}^b}{A\lambda b x_1{}^a x_2{}^{b-1}} = \frac{a}{b}\frac{x_2}{x_1}$$

$$\Rightarrow x_2 = \frac{w_1}{w_2}\frac{b}{a}x_1 \qquad (3)$$

Sub (3) back into the production function:

$\mathrm{A}x_1{}^a(\frac{w_1}{w_2}\frac{b}{a}x_1)^b = q$

$$=> \ x_1^{a+b} = \frac{q}{A}\left[\frac{w_2}{w_1}\frac{a}{b}\right]^b \qquad => \qquad x_1^* = \left(\frac{q}{A}\right)^{\frac{1}{a+b}}\left[\frac{w_2}{w_1}\frac{a}{b}\right]^{\frac{b}{a+b}}$$

$$=> x_2^* = \frac{w_1}{w_2}\frac{b}{a}\,x_1 = \frac{w_1}{w_2}\frac{b}{a}\left(\frac{q}{A}\right)^{\frac{1}{a+b}}\left[\frac{w_2}{w_1}\frac{a}{b}\right]^{\frac{b}{a+b}} = \left(\frac{q}{A}\right)^{\frac{1}{a+b}}\left(\frac{w_2}{w_1}\frac{a}{b}\right)^{\frac{b}{a+b}-1} = \left(\frac{q}{A}\right)^{\frac{1}{a+b}}\left(\frac{w_2}{w_1}\frac{a}{b}\right)^{\frac{-a}{a+b}}$$

$$c(q) = w_1x_1^* + w_2x_2^* = w_1\left(\frac{q}{A}\right)^{\frac{1}{a+b}}\left[\frac{w_2}{w_1}\frac{a}{b}\right]^{\frac{b}{a+b}} + w_2\left(\frac{q}{A}\right)^{\frac{1}{a+b}}\left(\frac{w_2}{w_1}\frac{a}{b}\right)^{\frac{-a}{a+b}}$$

$$= \left(\frac{q}{A}\right)^{\frac{1}{a+b}}\left(\left(\frac{a}{b}\right)^{\frac{b}{a+b}} w_1^{a/(a+b)}w_2^{b/(a+b)} + \left(\frac{a}{b}\right)^{\frac{-a}{a+b}} w_1^{a/(a+b)}w_2^{b/(a+b)}\right)$$

$$= \left(\frac{q}{A}\right)^{\frac{1}{a+b}}\left[\left(\frac{a}{b}\right)^{\frac{b}{a+b}} + \left(\frac{a}{b}\right)^{\frac{-a}{a+b}}\right] w_1^{a/(a+b)}w_2^{b/(a+b)}$$

(b) Leontief technology: $q = \min\{ax_1,\ bx_2\}$

If $ax_1 > bx_2$, the output level is given by $q = bx_2$. In this case, the firm can reduce the amount of input 1 used in the production to the point $ax_1 = bx_2$ and still produce $q = bx_2$ units of output. Similarly, if $ax_1 < bx_2$, the output level is given by $q = ax_1$, in which case the firm can reduce the amount of input 2 to the point $ax_1 = bx_2$ and still produces $q = ax_1$ units of output. Therefore, to minimize costs the firm should choose x_1 and x_2 in such a way that $q = ax_1 = bx_2$. Then, we have

$$x_1^* = \frac{q}{a}, x_2^* = \frac{q}{b}.$$

Therefore, the cost function is $c(w_1, w_2, q) = w_1\frac{q}{a} + w_2\frac{q}{b} = q\left(\frac{w_1}{a} + \frac{w_2}{b}\right)$

(c) Linear technology: $q = ax_1 + bx_2$.

In this case, inputs 1 and 2 are perfect substitutes. Specifically, 1 unit of input 1 alone can produce a units of output, and 1 unit of input 2 alone can produce *b* units of output. This implies that the firm can use either $1/a$ units of input 1 or $1/b$ units of input 2 to produce 1 unit of output. The cost of producing 1 unit output is w_1/a if the firm uses only input 1 and w_2/b if the firm uses only input 2. Since the two inputs are perfect substitutes, the firm will use whichever is cheaper. Accordingly, the firm uses only input 1 (i.e., $x_1 = q/a, x_2 = 0$) if $w_1/a < w_2/b$; and it uses only input 2 only if (i.e., $x_1 = 0, x_2 = q/b$) if $w_1/a > w_2/b$. Hence, the cost function is $c(w_1, w_2, q) = q \cdot \min\{\frac{w_1}{a}, \frac{w_2}{b}\}$.

A2 (a) Set $x_2 = k$ in the output constraint: $q = a\ln x_1 + b\ln k$. Solve this equation to obtain the amount of input 1 needed to produce q units of output:

$$x_1 = e^{(q-b\ln k)/a}.$$

The firm's short-run total cost function:

$$STC = w_1x_1 + w_2k = w_1e^{(q-b\ln k)/a} + w_2k.$$

(b) The short-run marginal cost function:

$$SMC = \frac{dSTC}{dq} = \frac{w_1}{a}e^{(q-b\ln k)/a}.$$

Since

$$\frac{dSMC}{dk} = -\frac{b}{ak}\frac{w_1}{a}e^{(q-b\ln k)/a} < 0,$$

SMC falls if the firm has a larger *k*.

Answers to Exercises

Microeconomic Analysis

Third Edition

Hal R. Varian

University of California at Berkeley

W. W. Norton & Company • New York • London

ANSWERS

Chapter 1. Technology

1.1 False. There are many counterexamples. Consider the technology generated by a production function $f(x) = x^2$. The production set is $Y = \{(y, -x) : y \leq x^2\}$ which is certainly not convex, but the input requirement set is $V(y) = \{x : x \geq \sqrt{y}\}$ which is a convex set.

1.2 It doesn't change.

1.3 $\epsilon_1 = a$ and $\epsilon_2 = b$.

1.4 Let $y(t) = f(t\mathbf{x})$. Then

$$\frac{dy}{dt} = \sum_{i=1}^{n} \frac{\partial f(\mathbf{x})}{\partial x_i} x_i,$$

so that

$$\frac{1}{y}\frac{dy}{dt} = \frac{1}{f(\mathbf{x})} \sum_{i=1}^{n} \frac{\partial f(\mathbf{x})}{\partial x_i} x_i.$$

1.5 Substitute tx_i for $i = 1, 2$ to get

$$f(tx_1, tx_2) = [(tx_1)^\rho + (tx_2)^\rho]^{\frac{1}{\rho}} = t[x_1^\rho + x_2^\rho]^{\frac{1}{\rho}} = tf(x_1, x_2).$$

This implies that the CES function exhibits constant returns to scale and hence has an elasticity of scale of 1.

1.6 This is half true: if $g'(x) > 0$, then the function must be strictly increasing, but the converse is not true. Consider, for example, the function $g(x) = x^3$. This is strictly increasing, but $g'(0) = 0$.

1.7 Let $f(\mathbf{x}) = g(h(\mathbf{x}))$ and suppose that $g(h(\mathbf{x})) = g(h(\mathbf{x}'))$. Since g is monotonic, it follows that $h(\mathbf{x}) = h(\mathbf{x}')$. Now $g(h(t\mathbf{x})) = g(th(\mathbf{x}))$ and $g(h(t\mathbf{x}')) = g(th(\mathbf{x}'))$ which gives us the required result.

1.8 A homothetic function can be written as $g(h(\mathbf{x}))$ where $h(\mathbf{x})$ is homogeneous of degree 1. Hence the TRS of a homothetic function has the

2.1 For profit maximization, the Kuhn-Tucker theorem requires the following three inequalities to hold

$$\left(p\frac{\partial f(\mathbf{x}^*)}{\partial x_j} - \mathbf{w}_j\right) x_j^* = 0,$$
$$p\frac{\partial f(\mathbf{x}^*)}{\partial x_j} - \mathbf{w}_j \leq 0,$$
$$x_j^* \geq 0.$$

Note that if $x_j^* > 0$, then we must have $w_j/p = \partial f(\mathbf{x}^*)/\partial x_j$.

2.2 Suppose that $\mathbf{x}'$ is a profit-maximizing bundle with positive profits $\pi(\mathbf{x}') > 0$. Since

$$f(t\mathbf{x}') > tf(\mathbf{x}'),$$

for $t > 1$, we have

$$\pi(t\mathbf{x}') = pf(t\mathbf{x}') - t\mathbf{w}\mathbf{x}' > t(pf(\mathbf{x}') - \mathbf{w}\mathbf{x}') > t\pi(\mathbf{x}') > \pi(\mathbf{x}').$$

Therefore, $\mathbf{x}'$ could not possibly be a profit-maximizing bundle.

2.3 In the text the supply function and the factor demands were computed for this technology. Using those results, the profit function is given by

$$\pi(p,w) = p\left(\frac{w}{ap}\right)^{\frac{a}{a-1}} - w\left(\frac{w}{ap}\right)^{\frac{1}{a-1}}.$$

To prove homogeneity, note that

$$\pi(tp,tw) = tp\left(\frac{w}{ap}\right)^{\frac{a}{a-1}} - tw\left(\frac{w}{ap}\right)^{\frac{1}{a-1}} = t\pi(p,w),$$

which implies that $\pi(p,w)$ is a homogeneous function of degree 1.

Before computing the Hessian matrix, factor the profit function in the following way:

$$\pi(p,w) = p^{\frac{1}{1-a}} w^{\frac{a}{a-1}} \left(a^{\frac{a}{1-a}} - a^{\frac{1}{1-a}}\right) = p^{\frac{1}{1-a}} w^{\frac{a}{a-1}} \phi(a),$$

where $\phi(a)$ is strictly positive for $0 < a < 1$.

The Hessian matrix can now be written as

$$D^2\pi(p,\omega) = \begin{pmatrix} \frac{\partial^2\pi(p,w)}{\partial p^2} & \frac{\partial^2\pi(p,w)}{\partial p\partial w} \\ \frac{\partial^2\pi(p,w)}{\partial w\partial p} & \frac{\partial^2\pi(p,w)}{\partial w^2} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{a}{(1-a)^2} p^{\frac{2a-1}{1-a}} w^{\frac{a}{a-1}} & -\frac{a}{(1-a)^2} p^{\frac{a}{1-a}} w^{\frac{1}{a-1}} \\ & \\ -\frac{a}{(1-a)^2} p^{\frac{a}{1-a}} w^{\frac{1}{a-1}} & \frac{a}{(1-a)^2} p^{\frac{1}{1-a}} w^{\frac{2-a}{a-1}} \end{pmatrix} \phi(a).$$

The principal minors of this matrix are

$$\frac{a}{(1-a)^2} p^{\frac{2a-1}{1-a}} w^{\frac{a}{a-1}} \phi(a) > 0$$

and 0. Therefore, the Hessian is a positive semidefinite matrix, which implies that $\pi(p, w)$ is convex in (p, w).

2.4 By profit maximization, we have

$$|TRS| = \frac{\frac{\partial f}{\partial x_1}}{\frac{\partial f}{\partial x_2}} = \frac{w_1}{w_2}.$$

Now, note that

$$\ln(w_2 x_2 / w_1 x_1) = -(\ln(w_1/w_2) + \ln(x_1/x_2)).$$

Therefore,

$$\frac{d \ln(w_2 x_2 / w_1 x_1)}{d \ln(x_1/x_2)} = \frac{d \ln(w_1/w_2)}{d \ln(x_2/x_1)} - 1 = \frac{d \ln |TRS|}{d \ln(x_2/x_1)} - 1 = 1/\sigma - 1.$$

2.5 From the previous exercise, we know that

$$\ln(w_2 x_2 / w_1 x_1) = \ln(w_2/w_1) + \ln(x_2/x_1),$$

Differentiating, we get

$$\frac{d \ln(w_2 x_2 / w_1 x_1)}{d \ln(w_2/w_1)} = 1 - \frac{d \ln(x_2/x_1)}{d \ln |TRS|} = 1 - \sigma.$$

2.6 We know from the text that $YO \supset Y \supset YI$. Hence for any $\mathbf{p}$, the maximum of $\mathbf{py}$ over YO must be larger than the maximum over Y, and this in turn must be larger than the maximum over YI.

2.7.a We want to maximize $20x - x^2 - wx$. The first-order condition is $20 - 2x - w = 0$.

2.7.b For the optimal x to be zero, the derivative of profit with respect to x must be nonpositive at $x = 0$: $20 - 2x - w < 0$ when $x = 0$, or $w \geq 20$.

2.7.c The optimal x will be 10 when $w = 0$.

2.7.d The factor demand function is $x = 10 - w/2$, or, to be more precise, $x = \max\{10 - w/2, 0\}$.

2.7.e Profits as a function of output are

$$20x - x^2 - wx = [20 - w - x]x.$$

Substitute $x = 10 - w/2$ to find

$$\pi(w) = \left[10 - \frac{w}{2}\right]^2.$$

2.7.f The derivative of profit with respect to w is $-(10 - w/2)$, which is, of course, the negative of the factor demand.

Chapter 3. Profit Function

3.1.a Since the profit function is convex and a decreasing function of the factor prices, we know that $\phi_i'(w_i) \leq 0$ and $\phi_i''(w_i) \geq 0$.

3.1.b It is zero.

3.1.c The demand for factor i is only a function of the i^{th} price. Therefore the marginal product of factor i can only depend on the amount of factor i. It follows that $f(x_1, x_2) = g_1(x_1) + g_2(x_2)$.

3.2 The first-order conditions are $p/x = w$, which gives us the demand function $x = p/w$ and the supply function $y = \ln(p/w)$. The profits from operating at this point are $p \ln(p/w) - p$. Since the firm can always choose $x = 0$ and make zero profits, the profit function becomes $\pi(p, w) = \max\{p \ln(p/w) - p, 0\}$.

3.3 The first-order conditions are

$$a_1 \frac{p}{x_1} - w_1 = 0$$
$$a_2 \frac{p}{x_2} - w_2 = 0,$$

which can easily be solved for the factor demand functions. Substituting into the objective function gives us the profit function.

3.4 The first-order conditions are

$$pa_1 x_1^{a_1 - 1} x_2^{a_2} - w_1 = 0$$
$$pa_2 x_2^{a_2 - 1} x_1^{a_1} - w_2 = 0,$$

which can easily be solved for the factor demands. Substituting into the objective function gives us the profit function for this technology. In order for this to be meaningful, the technology must exhibit decreasing returns to scale, so $a_1 + a_2 < 1$.

3.5 If w_i is strictly positive, the firm will never use more of factor i than it needs to, which implies $x_1 = x_2$. Hence the profit maximization problem can be written as

$$\max \; p x_1^a - w_1 x_1 - w_2 x_2.$$

The first-order condition is

$$p a x_1^{a-1} - (w_1 + w_2) = 0.$$

The factor demand function and the profit function are the same as if the production function were $f(x) = x^a$, but the factor price is $w_1 + w_2$ rather than w. In order for a maximum to exist, $a < 1$.

Chapter 4. Cost Minimization

4.1 Let $\mathbf{x}^*$ be a profit-maximizing input vector for prices $(p, \mathbf{w})$. This means that $\mathbf{x}^*$ must satisfy $pf(\mathbf{x}^*) - \mathbf{w}\mathbf{x}^* \geq pf(\mathbf{x}) - \mathbf{w}\mathbf{x}$ for all permissible $\mathbf{x}$. Assume that $\mathbf{x}^*$ does not minimize cost for the output $f(\mathbf{x}^*)$; i.e., there exists a vector $\mathbf{x}^{**}$ such that $f(\mathbf{x}^{**}) \geq f(\mathbf{x}^*)$ and $\mathbf{w}(\mathbf{x}^{**} - \mathbf{x}^*) < 0$. But then the profits achieved with $\mathbf{x}^{**}$ must be greater than those achieved with $\mathbf{x}^*$:

$$\begin{aligned} pf(\mathbf{x}^{**}) - \mathbf{w}\mathbf{x}^{**} &\geq pf(\mathbf{x}^*) - \mathbf{w}\mathbf{x}^{**} \\ &> pf(\mathbf{x}^*) - \mathbf{w}\mathbf{x}^*, \end{aligned}$$

which contradicts the assumption that $\mathbf{x}^*$ was profit-maximizing.

4.2 The complete set of conditions turns out to be

$$\begin{aligned} \left(t \frac{\partial f(\mathbf{x}^*)}{\partial x_j} - \mathbf{w}_j \right) x_j^* &= 0, \\ t \frac{\partial f(\mathbf{x}^*)}{\partial x_j} - \mathbf{w}_j &\leq 0, \\ x_j^* &\geq 0, \\ (y - f(\mathbf{x}^*))\, t &= 0, \\ y - f(\mathbf{x}^*) &\leq 0, \\ t &\geq 0. \end{aligned}$$

If, for instance, we have $x_i^* > 0$ and $x_j^* = 0$, the above conditions imply

$$\frac{\dfrac{\partial f(\mathbf{x}^*)}{\partial x_i}}{\dfrac{\partial f(\mathbf{x}^*)}{\partial x_j}} \geq \frac{\mathbf{w}_i}{\mathbf{w}_j}.$$

doesn't make sense—you are producing more output in the plant with the higher costs!

It turns out that this corresponds to a constrained *maximum* and not to the desired minimum. Check the second-order conditions to verify this.

Since the cost function is concave, rather than convex, the optimal solution will always occur at a boundary. That is, you will produce all output at the cheaper plant so $c(y) = 2\sqrt{y}$.

4.7 No, the data violate WACM. It costs 40 to produce 100 units of output, but at the same prices it would only cost 38 to produce 110 units of output.

4.8 Set up the minimization problem

$$\min x_1 + x_2$$
$$x_1 x_2 = y.$$

Substitute to get the unconstrained minimization problem

$$\min \; x_1 + y/x_1.$$

The first-order condition is

$$1 - y/x_1^2,$$

which implies $x_1 = \sqrt{y}$. By symmetry, $x_2 = \sqrt{y}$. We are given that $2\sqrt{y} = 4$, so $\sqrt{y} = 2$, from which it follows that $y = 4$.

Chapter 5. Cost Function

5.1 The firm wants to minimize the cost of producing a given level of output:

$$c(y) = \min_{y_1, y_2} y_1^2 + y_2^2$$
$$\text{such that } y_1 + y_2 = y.$$

The solution has $y_1 = y_2 = y/2$. Substituting into the objective function yields

$$c(y) = (y/2)^2 + (y/2)^2 = y^2/2.$$

5.2 The first-order conditions are $6y_1 = 2y_2$, or $y_2 = 3y_1$. We also require $y_1 + y_2 = y$. Solving these two equations in two unknowns yields $y_1 = y/4$ and $y_2 = 3y/4$. The cost function is

$$c(y) = 3\left[\frac{y}{4}\right]^2 + \left[\frac{3y}{4}\right]^2 = \frac{3y^2}{4}.$$

5.8 If $p = 2$, the firm will produce 1 unit of output. If $p = 1$, the first-order condition suggests $y = 1/2$, but this yields negative profits. The firm can get zero profits by choosing $y = 0$. The profit function is $\pi(p) = \max\{p^2/4 - 1, 0\}$.

5.9.a $d\pi/d\alpha = py > 0$.

5.9.b $dy/d\alpha = p/c''(y) > 0$.

5.9.c $p'(\alpha) = n[y + \alpha p/c'']/[D'(p) - n\alpha/c''] < 0$.

5.10 Let $y(p, \mathbf{w})$ be the supply function. Totally differentiating, we have

$$dy = \sum_{i=1}^{n} \frac{\partial y(p, \mathbf{w})}{\partial w_i} dw_i = -\sum_{i=1}^{n} \frac{\partial x_i(p, \mathbf{w})}{\partial p} dw_i = -\sum_{i=1}^{n} \frac{\partial x_i(\mathbf{w}, y)}{\partial y} \frac{\partial y(p, \mathbf{w})}{\partial p} dw_i.$$

The first equality is a definition; the second uses the symmetry of the substitution matrix; the third uses the chain rule and the fact that the unconditional factor demand, $x_i(p, \mathbf{w})$, and the conditional factor demand, $x_i(\mathbf{w}, y)$, satisfy the identity $x_i(\mathbf{w}, y(p, \mathbf{w})) = x_i(p, \mathbf{w})$. The last expression on the right shows that if there are no inferior factors then the output of the firm must increase.

5.11.a $\mathbf{x} = (1, 1, 0, 0)$.

5.11.b $\min\{w_1 + w_2, w_3 + w_4\}y$.

5.11.c Constant returns to scale.

5.11.d $\mathbf{x} = (1, 0, 1, 0)$.

5.11.e $c(w, y) = [\min\{w_1, w_2\} + \min\{w_3, w_4\}]y$.

5.11.f Constant.

5.12.a The diagram is the same as the diagram for an inferior good in consumer theory.

5.12.b If the technology is CRS, then conditional factor demands take the form $x_i(\mathbf{w}, 1)y$. Hence the derivative of a factor demand function with respect to output is $x_i(\mathbf{w}) \geq 0$.

5.12.c The hypothesis can be written as

$$\partial c(\mathbf{w}, y)^2/\partial y \partial w_i < 0.$$

But

$$\partial c(\mathbf{w}, y)^2/\partial y \partial w_i = \partial c(\mathbf{w}, y)^2/\partial w_i \partial y = \partial x_i(\mathbf{w}, y)/\partial y.$$

5.13.a Factor demand curves slope downward, so the demand for unskilled workers must decrease when their wage increases.

5.13.b We are given that $\partial l/\partial p < 0$. But by duality, $\partial l/\partial p = -\partial^2\pi/\partial p\partial w = -\partial^2\pi/\partial w\partial p = -\partial y/\partial w$. It follows that $\partial y/\partial w > 0$.

5.14 Take a total derivative of the cost function to get:

$$dc = \sum_{i=1}^{n} \frac{\partial c}{\partial w_i} dw_i + \frac{\partial c}{\partial y} dy.$$

It follows that

$$\frac{\partial c}{\partial y} = \frac{dc - \sum_{i=1}^{n} \frac{\partial c}{\partial w_i} dw_i}{dy}.$$

Now substitute the first differences for the dy, dc, dw_i terms and you're done.

5.15 By the linearity of the function, we know we will use either x_1, or a combination of x_2 and x_3 to produce y. By the properties of the Leontief function, we know that if we use x_2 and x_3 to produce y, we must use 3 units of both x_2 and x_3 to produce one unit of y. Thus, if the cost of using one unit of x_1 is less than the cost of using one unit of both x_2 and x_3, then we will use only x_1, and conversely. The conditional factor demands can be written as:

$$x_1 = \begin{cases} 3y & \text{if } w_1 < w_2 + w_3 \\ 0 & \text{if } w_1 > w_2 + w_3 \end{cases}$$

$$x_2 = \begin{cases} 0 & \text{if } w_1 < w_2 + w_3 \\ 3y & \text{if } w_1 > w_2 + w_3 \end{cases}$$

$$x_3 = \begin{cases} 0 & \text{if } w_1 < w_2 + w_3 \\ 3y & \text{if } w_1 > w_2 + w_3 \end{cases}$$

if $w_1 = w_2 + w_3$, then any bundle (x_1, x_2, x_3) with $x_2 = x_3$ and $x_1 + x_2 = 3y$ (or $x_1 + x_3 = 3y$) minimizes cost.

The cost function is

$$c(w, y) = 3y \min(w_1, w_2 + w_3).$$

5.16.a *Homogeneous*:

$$\begin{aligned} c(t\mathbf{w}, y) &= y^{1/2}(tw_1 tw_2)^{3/4} \\ &= t^{3/2}(y^{1/2}(w_1 w_2)^{3/4}) \\ &= t^{3/2}c(\mathbf{w}, y) \quad \text{No.} \end{aligned}$$

Monotone:

$$\frac{\partial c}{\partial w_1} = \frac{3}{4} y^{1/2} w_1^{-1/4} w_2^{3/4} > 0 \qquad \frac{\partial c}{\partial w_2} = \frac{3}{4} y^{1/2} w_1^{3/4} w_2^{-1/4} > 0 \quad \text{Yes.}$$

Concave:

$$\text{Hessian } = \begin{bmatrix} -\frac{3}{16} y^{1/2} w_1^{-5/4} w_2^{3/4} & \frac{9}{16} y^{1/2} w_1^{-1/4} w_2^{-1/4} \\ \frac{9}{16} y^{1/2} w_1^{-1/4} w_2^{-1/4} & -\frac{3}{16} y^{1/2} w_1^{3/4} w_2^{-5/4} \end{bmatrix}$$

$$|H_1| < 0$$

$$\begin{aligned} |H_2| &= \frac{9}{256} y w_1^{-1/2} w_2^{-1/2} - \frac{81}{256} y w_1^{-1/2} w_2^{-1/2} \\ &= -\frac{72}{256} \frac{y}{\sqrt{w_1 w_2}} < 0 \quad \text{No} \end{aligned}$$

Continuous: Yes

5.16.b *Homogeneous*:

$$\begin{aligned} c(t\mathbf{w}, y) &= y(tw_1 + \sqrt{tw_1 tw_2} + tw_2) \\ &= ty(w_1 + \sqrt{w_1 w_2} + w_2) \\ &= tc(y, \overline{w}) \quad \text{Yes} \end{aligned}$$

Monotone:

$$\frac{\partial c}{\partial w_1} = y\left(1 + \frac{1}{2}\sqrt{\frac{w_2}{w_1}}\right) > 0 \quad \frac{\partial c}{\partial w_2} = y\left(1 + \frac{1}{2}\sqrt{\frac{w_1}{w_2}}\right) > 0 \quad \text{Yes}$$

Concave:

$$H = \begin{bmatrix} -\frac{1}{4} y w_2^{1/2} w_1^{-3/2} & \frac{1}{4} y w_2^{-1/2} w_1^{-1/2} \\ \frac{1}{4} y w_2^{-1/2} w_1^{-1/2} & -\frac{1}{4} y w_2^{-3/2} w_1^{1/2} \end{bmatrix}$$

$$|H_1| < 0$$

$$|H_2| = \frac{1}{16} y w_2^{-1} w_1^{-1} - \frac{1}{16} y w_2^{-1} w_1^{-1} = 0 \quad \text{Yes}$$

Continuous: Yes

Production Function:

$$x_1(\mathbf{w}, y) = y\left(1 + \frac{1}{2}\sqrt{\frac{w_2}{w_1}}\right) \tag{1}$$

$$x_2(\mathbf{w}, y) = y\left(1 + \frac{1}{2}\sqrt{\frac{w_1}{w_2}}\right) \tag{2}$$

Rearranging these equations:

$$x_1 - y = \frac{y}{2}\sqrt{\frac{w_2}{w_1}} \tag{1$'$}$$

$$x_2 - y = \frac{y}{2}\sqrt{\frac{w_1}{w_2}} \tag{2$'$}$$

Multiply (1′) and (2′): $(x_1 - y)(x_2 - y) = \frac{y^2}{4}$. This is a quadratic equation which gives $y = \frac{2}{3}(x_2 + x_1) \pm \frac{2}{3}\sqrt{x_1^2 + x_2^2 + 2 - x_1 x_2}$.

5.16.c *Homogeneous*:

$$\begin{aligned} c(t\mathbf{w}, y) &= y(tw_1 e^{-tw_1} + tw_2) \\ &= ty(w_1 e^{-tw_1} + w_2) \\ &\neq tc(\mathbf{w}, y) \quad \text{No} \end{aligned}$$

Monotone:

$$\frac{\partial c}{\partial w_1} = y(-w_1 e^{-w_1} + e^{-w_1}) = ye^{-w_1}(1 - w_1)$$

This is positive only if $w_1 < 1$.

$$\frac{\partial c}{\partial w_2} = y > 0 \quad \text{No}$$

Concave:

$$H = \begin{bmatrix} y(w_1 - 2)e^{-w_1} & 0 \\ 0 & 0 \end{bmatrix}$$

$$|H_1| = y(w_1 - 2)e^{-w_1}$$

This is less than zero only if $w_1 < 2$.

$$|H_2| = 0 \qquad \text{No}$$

Continuous: Yes

5.16.d *Homogeneous:*

$$\begin{aligned} c(t\mathbf{w}, y) &= y(tw_1 - \sqrt{tw_1 tw_2} + tw_2 \\ &= ty(w_1 - \sqrt{w_1 w_2} + w_2) \\ &= tc(\mathbf{w}, y) \qquad \text{Yes} \end{aligned}$$

Monotone:

$$\frac{\partial c}{\partial w_1} = y(1 - \frac{1}{2}\sqrt{\frac{w_2}{w_1}})$$

This is greater than 0 only if $1 > \frac{1}{2}\sqrt{\frac{w_2}{w_1}}$

$$\frac{\partial c}{\partial w_2} = y(1 - \frac{1}{2}\sqrt{\frac{w_1}{w_2}})$$

This is greater than 0 only if $2 > \sqrt{\frac{w_2}{w_1}}$

$$w_2 > \frac{1}{4}w_1 \quad \text{(by symmetry)} \quad 2\sqrt{w_1} > \sqrt{w_2}$$

or

$$w_1 < 4w_2 \qquad w_1 > \frac{1}{4}w_2$$

Monotone only if $\frac{1}{4}w_2 < w_1 < 4w_2$. No.

Concave:

$$H = \begin{bmatrix} \frac{1}{4}yw_1^{-3/2}w_2^{1/2} & -\frac{1}{4}yw_1^{-1/2}w_2^{-1/2} \\ -\frac{1}{4}yw_1^{-1/2}w_2^{-1/2} & \frac{1}{4}yw_1^{1/2}w_2^{-1/2} \end{bmatrix}$$

$$\begin{aligned} |H_1| &= \frac{1}{4}yw_1^{-3/2}w_2^{1/2} > 0 \\ |H_2| &= 0 \qquad \text{No (it is convex)} \end{aligned}$$

Continuous: Yes

5.16.e

Homogeneous:

$$\begin{aligned} c(t\mathbf{w}, y) &= (y + \frac{1}{y}\sqrt{tw_1 tw_2}) \\ &= tc(y, \overline{w}) \qquad \text{Yes} \end{aligned}$$

Monotone in w:

$$\frac{\partial c}{\partial w_1} = \frac{1}{2}(y + \frac{1}{y})\sqrt{\frac{w_2}{w_1}} > 0 \qquad \frac{\partial c}{\partial w_2} = \frac{1}{2}(y + \frac{1}{2})\sqrt{\frac{w_1}{w_2}} > 0 \qquad \text{Yes}$$

Concave:

$$H = \begin{bmatrix} -\frac{1}{4}(y + \frac{1}{y}w_1^{-3/2}w_2^{1/2} & \frac{1}{4}(y + \frac{1}{y})w_1^{-1/2}w_2^{-1/2} \\ \frac{1}{4}(y, \frac{1}{y})w_1^{-1/2}w_2^{-1/2} & -\frac{1}{4}(y + \frac{1}{y})w_1^{1/2}w_2^{-3/2} \end{bmatrix} \quad \text{But not in } y!$$

$$\begin{matrix} |H_1| < 0 \\ |H_2| = 0 \end{matrix} \qquad \text{Yes}$$

Continuous: Not for $y = 0$.

5.17.a $y = \sqrt{ax_1 + bx_2}$

5.17.b Note that this function is exactly like a linear function, except that the linear combination of x_1 and x_2 will produce y^2, rather than just y. So, we know that if x_1 is relatively cheaper, we will use all x_1 and no x_2, and conversely.

5.17.c The cost function is $c(w, y) = y^2 \min(\frac{w_1}{a}, \frac{w_2}{b})$.

Chapter 6. Duality

6.1 The production function is $f(x_1, x_2) = x_1 + x_2$. The conditional factor demands have the form

$$x_i = \begin{cases} y & \text{if } w_i < w_j \\ 0 & \text{if } w_i > w_j \\ \text{any amount between 0 and } y & \text{if } w_i = w_j. \end{cases}$$

6.2 The conditional factor demands can be found by differentiating. They are $x_1(w_1, w_2, y) = x_2(w_1, w_2, y) = y$. The production function is

$$f(x_1, x_2) = \min\{x_1, x_2\}.$$

6.3 The cost function must be increasing in both prices, so a and b are both nonnegative. The cost function must be concave in both prices, so a and b are both less than 1. Finally, the cost function must be homogeneous of degree 1, so $a = 1 - b$.