

Solutions for Midterm 1

ECON 4020B

February 16, 2016

Q1: (15 pts) Input requirement set

Let (x_1, x_2, x_3) be the input vector and y be the output. Define the *input requirement set*, and explain what it means. Write down the input requirement set associated with the production technology $y = x_1^a x_2^b x_3^c$.

(5 pts) For the input vector (x_1, x_2, x_3) and output y , the input requirement set is defined as:

$$V(y) = \{(x_1, x_2, x_3) \in \mathbb{R}_+^3 \mid (y, -x_1, -x_2, -x_3) \in Y\}$$

(5 pts) It is the set of all input bundles that produce at least y output.

[Note: Students lose two marks if they do not specialize the arbitrary-dimensional textbook definition to the three-dimensional input of this question. Specializing a definition to a particular question shows that the definition is understood, not merely memorized.]

(5 pts) With production technology $y = x_1^a x_2^b x_3^c$, the input requirement set is:

$$V(y) = \{(x_1, x_2, x_3) \in \mathbb{R}_+^3 \mid x_1^a x_2^b x_3^c \geq y\}$$

Q2: (20 pts) Leontief production technology

Consider the Leontief production technology represented by:

$$q = f(x_1, x_2) = \min\{ax_1, bx_2\}$$

(a) (8 pts) Returns to scale

Does this technology exhibit (global) constant returns to scale, increasing returns to scale, or decreasing returns to scale? Be sure to demonstrate your answer.

We scale all inputs by a common factor t , and see what happens to production:

$$\begin{aligned} f(tx_1, tx_2) &= \min\{a(tx_1), b(tx_2)\} \\ &= \min\{atx_1, btx_2\} \\ &= t \cdot \min\{ax_1, bx_2\} \\ &= t \cdot f(x_1, x_2) \end{aligned}$$

Therefore, the production technology exhibits **constant** returns to scale.

(b) (12 pts) Cost minimization, homogeneity

Set up the cost-minimization problem, and derive the cost function associated with this production technology. Verify that this cost function is homogeneous of degree 1 in (w_1, w_2) .

(8 pts) The cost minimization problem is:

$$\begin{aligned} &\underset{x_1, x_2}{\text{minimize}} && w_1x_1 + w_2x_2 \\ &\text{such that} && q = \min\{ax_1, bx_2\} \end{aligned}$$

[Note: the Lagrangian multiplier method will not be helpful here because our production function is not differentiable when $ax_1 = bx_2$]

We begin with the piecewise definition of ‘min’ in our production function:

$$q = \min\{ax_1, bx_2\} = \begin{cases} ax_1 & \text{if } ax_1 > bx_2 \\ bx_2 & \text{if } ax_1 < bx_2 \end{cases}$$

This means that when ax_1 exceeds bx_2 , the firm can reduce its use of the first input without affecting the production level q . Similarly, if bx_2 exceeds ax_1 , the firm can reduce x_2 without production loss. The firm incurs a smaller cost when it uses fewer inputs to produce the same output. Cost is minimized thus:

$$ax_1^* = bx_2^* = q$$

Isolating, we get the optimal input levels as functions of output:

$$x_1^*(q) = \frac{q}{a} \quad x_2^*(q) = \frac{q}{b}$$

Now, we can express the cost function:

$$\begin{aligned} c(w_1, w_2, q) &= w_1 x_1^*(q) + w_2 x_2^*(q) \\ &= q \left[\frac{w_1}{a} + \frac{w_2}{b} \right] \end{aligned}$$

(4 pts) To verify first-degree homogeneity of cost in the input prices, we scale both input prices by a common factor t :

$$\begin{aligned} c(tw_1, tw_2, q) &= q \left[\frac{tw_1}{a} + \frac{tw_2}{b} \right] \\ &= t \cdot q \left[\frac{w_1}{a} + \frac{w_2}{b} \right] \\ &= t \cdot c(w_1, w_2, q) \end{aligned}$$

The cost function is indeed first-degree homogeneous in (w_1, w_2) .

Q3: (30 pts) Profit maximization

Suppose a firm's production function is given by:

$$q = 3 \ln x$$

Let p denote the price of output, and w the price of input. Suppose $p > w$.

(a) (18 pts) Input demand and output supply

Derive the input demand function and output supply function.

The profit-maximization problem is:

$$\begin{aligned} & \underset{x}{\text{maximize}} && pq - wx \\ & \text{such that} && q = 3 \ln x \end{aligned}$$

Substitution yields an equivalent unconstrained problem:

$$\underset{x}{\text{maximize}} \quad p(3 \ln x) - wx$$

(6pts) The first-order condition for maximum is:

$$0 = \frac{\partial}{\partial x} (p(3 \ln x) - wx) = \frac{3p}{x^*} - w$$

(6 pts) Isolating x , we get the input demand function:

$$x(p, w) := x^* = \frac{3p}{w}$$

(6 pts) Substitution into the production function yields the output supply:

$$q(p, w) := 3 \ln x^* = 3 \ln \left(\frac{3p}{w} \right) = 3[\ln p - \ln w + \ln 3]$$

(b) (12 pts) Convexity

(3 pts) The profit function is:

$$\begin{aligned}\pi(p, w) &:= pq(p, w) - wx(p, w) \\ &= 3p[\ln p - \ln w + \ln 3] - w \frac{3p}{w} \\ &= 3p[\ln p - \ln w + \ln 3 - 1]\end{aligned}$$

(9 pts) For the profit function $\pi(p, w)$ to be convex in its arguments, its Hessian matrix H must be positive-semidefinite. The Hessian matrix for π is defined as the matrix of all of its second-order partial derivatives:

$$H(p, w) := \begin{bmatrix} \frac{\partial^2 \pi}{\partial p^2}(p, w) & \frac{\partial^2 \pi}{\partial p \partial w}(p, w) \\ \frac{\partial^2 \pi}{\partial p \partial w}(p, w) & \frac{\partial^2 \pi}{\partial w^2}(p, w) \end{bmatrix}$$

We begin with the first-order derivatives of the profit function.

$$\begin{aligned}\frac{\partial \pi}{\partial p}(p, w) &= 3[\ln p - \ln w + \ln 3 - 1] + \frac{3p}{p} \\ &= 3[\ln p - \ln w + \ln 3] \\ \frac{\partial \pi}{\partial w}(p, w) &= -\frac{3p}{w}\end{aligned}$$

The second-order derivatives of the profit function are:

$$\begin{aligned}\frac{\partial}{\partial p} \frac{\partial \pi}{\partial p}(p, w) &= \frac{3}{p} \\ \frac{\partial}{\partial w} \frac{\partial \pi}{\partial p}(p, w) &= -\frac{3}{w} \\ \frac{\partial}{\partial w} \frac{\partial \pi}{\partial w}(p, w) &= \frac{3p}{w^2}\end{aligned}$$

Next, we check that the principal minors of the Hessian matrix are non-negative:

$$\begin{aligned}\frac{\partial^2 \pi}{\partial p^2} &= \frac{3}{p} \geq 0 \\ \frac{\partial^2 \pi}{\partial w^2} &= \frac{3p}{w^2} \geq 0 \\ \det H &= \frac{\partial^2 \pi}{\partial p^2} \cdot \frac{\partial^2 \pi}{\partial w^2} - \left[\frac{\partial^2 \pi}{\partial p \partial w} \right]^2 \\ &= \frac{3}{p} \cdot \frac{3p}{w^2} - \left[-\frac{3}{w} \right]^2 = 0\end{aligned}$$

Therefore, the profit function is convex in (p, w) , because its Hessian matrix is positive-semidefinite.

Q4: (35 pts) Elasticity

A firm's production technology is represented by:

$$q = f(x_1, x_2) = \frac{kx_1x_2}{2 + x_1x_2}$$

where k is a positive constant and $q < k$.

(a) (15 pts) Substitution

Calculate the elasticity of substitution between the two inputs.

(10 pts) We start with the first derivatives of the production function.

$$\begin{aligned} f'_1 &= \frac{kx_2}{2 + x_1x_2} - \frac{kx_1x_2 \cdot x_2}{[2 + x_1x_2]^2} \\ &= \frac{kx_2[2 + x_1x_2 - x_1x_2]}{[2 + x_1x_2]^2} \\ &= \frac{2kx_2}{[2 + x_1x_2]^2} \\ f'_2 &= \frac{kx_2}{2 + x_1x_2} - \frac{kx_1x_2 \cdot x_1}{[2 + x_1x_2]^2} \\ &= \frac{kx_1[2 + x_1x_2 - x_1x_2]}{[2 + x_1x_2]^2} \\ &= \frac{2kx_1}{[2 + x_1x_2]^2} \end{aligned}$$

(5 pts) Next, we find the ratio of the partial derivatives:

$$\frac{f'_1}{f'_2} = \frac{x_2 \cdot 2k[2 + x_1x_2]^{-2}}{x_1 \cdot 2k[2 + x_1x_2]^{-2}} = \frac{x_2}{x_1}$$

The elasticity of substitution between the two inputs is:

$$\sigma = \frac{\partial \ln(f'_1/f'_2)}{\partial \ln(x_2/x_1)} = \frac{\partial \ln(x_2/x_1)}{\partial \ln(x_2/x_1)} = 1$$

(b) (12 pts) Scale

Calculate the elasticity of scale.

(6 pts) We begin by finding, for the production function, the dot-product of its partial derivatives with its variables:

$$\begin{aligned} \begin{bmatrix} f'_1 \\ f'_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \frac{2kx_2}{[2 + x_1x_2]^2} \cdot x_1 + \frac{2kx_1}{[2 + x_1x_2]^2} \cdot x_2 \\ &= \frac{4kx_1x_2}{[2 + x_1x_2]^2} \end{aligned}$$

(6 pts) The elasticity of scale is equal to this dot-product, divided by the production function:

$$\mu = \frac{4kx_1x_2}{[2 + x_1x_2]^2} \bigg/ \frac{kx_1x_2}{2 + x_1x_2} = \frac{4}{2 + x_1x_2}$$

(c) (8 pts) Local increasing returns to scale

Find the range of output q over which this technology exhibits local increasing returns to scale.

Local increasing returns to scale implies that $\mu > 1$. First, we simplify this inequality:

$$\begin{aligned} \mu &= \frac{4}{2 + x_1x_2} > 1 \\ \implies 4 &> 2 + x_1x_2 \\ \implies 2 &> x_1x_2 \end{aligned}$$

Next, we invert the production function with respect to (x_1x_2) :

$$\begin{aligned} q &= \frac{kx_1x_2}{2 + x_1x_2} \\ \implies q(2 + x_1x_2) &= kx_1x_2 \\ \implies 2q + qx_1x_2 &= kx_1x_2 \\ \implies 2q &= (k - q)x_1x_2 \\ \implies x_1x_2 &= \frac{2q}{k - q} \end{aligned}$$

Finally, we substitute the above result into $(2 > x_1 x_2)$:

$$2 > \frac{2q}{k - q}$$

$(k - q > 0$, so the inequality sign does not change if we multiply both sides with it)

$$\implies 2(k - q) > 2q$$

$$\implies k > 2q$$

$$\implies q < k/2$$

Hence, this technology exhibits local increasing returns to scale when output is less than half of k .