Solution to Midterm Test 1(B2)

MAT 1322D, Fall 2016
Total = 20 marks

CDAB

I. Multiple-Choice Questions ($3 \times 4 = 12$ marks)

EBDA

1. The area of the region above the graph of $y = 3 - x$ and under the graph of $y = 9 - x^2$ is

(A) $\frac{125}{3}$; (B) $\frac{134}{6}$; (C) $\frac{135}{6}$; (D) $\frac{25}{3}$; (E) $\frac{125}{6}$.

Solution. (E) Let $3 - x = 9 - x^2$. $x^2 - x - 6 = 0$. $x = -2, 3$. Since $3 - 2x < 6 - x^2$ when $-1 < x < 3$, the area of the region is

$$A = \int_{-1}^{3} ((9 - x^2) - (3 - x))dx = \frac{125}{6}.$$

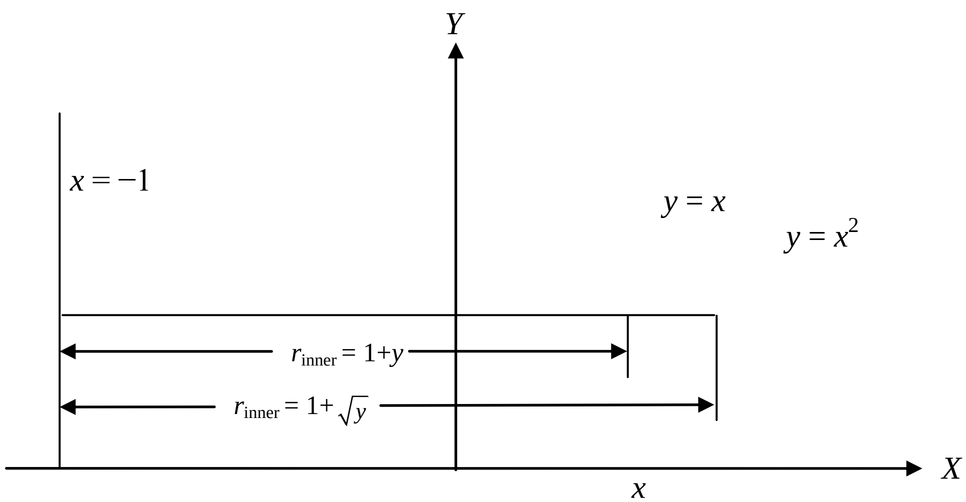
2. Let R be the region under the graph of $y = x$ and above the graph of $y = x^2$. The volume of the solid obtained by revolving R about the line $x = -1$ is given by the integral

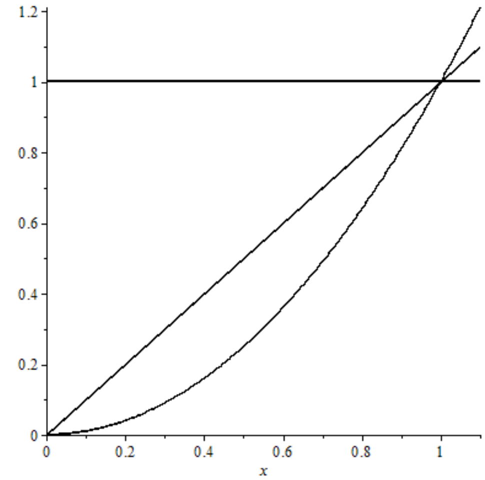
(A) $\pi\int_0^1 ((1+y)^2 - (1+\sqrt{y})^2)dy$; (B) $\pi\int_0^1 ((1+\sqrt{y})^2 - (1+y)^2)dy$;

(C) $\pi\int_0^1 ((1-y^2)^2 - (1-y)^2)dy$; (D) $\pi\int_0^1 ((1-y)^2 - (1-\sqrt{y})^2)dy$;

(E) $\pi\int_0^1 ((1-y)^2 - (1-y^2)^2)dy$.

Answer. (B) The picture is as follows:





3. Consider improper integral $\int_0^1 \frac{2-\sqrt{x}}{x}dx$. Which one of the following argument is true?

(A) Since $\frac{2-\sqrt{x}}{x} < \frac{2}{x}$, and $\int_0^1 \frac{2}{x}dx = 2\int_0^1 \frac{1}{x}dx$ diverges, improper integral $\int_0^1 \frac{2-\sqrt{x}}{x}dx$ diverges.

(B) Since $\frac{2-\sqrt{x}}{x} < \frac{2}{x}$, and $\int_0^1 \frac{2}{x}dx = 2\int_0^1 \frac{1}{x}dx$ converges, improper integral $\int_0^1 \frac{2-\sqrt{x}}{x}dx$ converges.

(C) Since $\frac{2-\sqrt{x}}{x} < \frac{2}{x}$, and $\int_0^1 \frac{2}{x}dx = 2\int_0^1 \frac{1}{x}dx$ diverges, improper integral $\int_0^1 \frac{2-\sqrt{x}}{x}dx$ converges.

(D) Since $\frac{2-\sqrt{x}}{x} > \frac{1}{x}$, and $\int_0^1 \frac{1}{x}dx$ diverges, improper integral $\int_0^1 \frac{2-\sqrt{x}}{x}dx$ diverges.

(E) Since $\frac{2-\sqrt{x}}{x} > \frac{1}{x}$, and $\int_0^1 \frac{1}{x}dx$ converges, improper integral $\int_0^1 \frac{2-\sqrt{x}}{x}dx$ converges.

Answer. (D)

4. Suppose that a rope of length 20 meters and weight 10 km is hanging from the top of a building. Then the work, in Joules, needed to pull the rope to the top of the building is

(A) 100g; (B) 125g; (C) 80g; (D) 75g; (E) 50g,

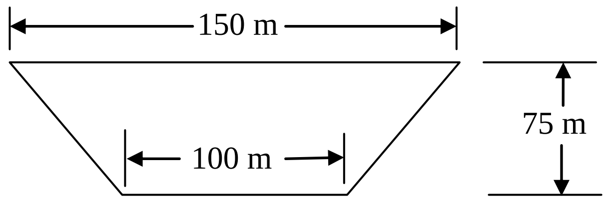
where g is the acceleration of gravity.

Solution. (A) The weight of a part of the rope x meters from the top of the building with length dx is $w(x) = 0.5gdx$. The work needed to pull this part to the top of the building is $xw(x) = 0.5gxdx$. The total work needed is

$\int_0^{20} 0.5gx\,dx = \frac{1}{4}g\left[x^2\right]_{x=0}^{20} = 100g$ Joule.

II. Detailed Answer Questions (4 × 2 = 8 marks)

5. Suppose a dam has the shape of a trapezoid as shown in the following figure.



The water level is 10 meters under the top of the dam. Construct, but not evaluate, an integral that calculates the force, in Newtons, acting on the dam. Assume the density of water is ρ kg/m^3, and the acceleration of gravity is g m/sec^2.

Solution. Look at a horizontal stripe of the dam x meters above the bottom with height dx. The area of this stripe is $A(x) = \left(\frac{2}{3}x + 100\right)dx$. The depth of this tripe is $D(x) = 65 - x$. The pressure is $\rho g D(x) = \rho g(65 - x)$. The force acting on this stripe is $dF = \rho g\left(\frac{2}{3}x + 100\right)(65 - x)dx$. The total force is

$$F = \rho g \int_0^{65} \left(\frac{2}{3}x + 100\right)(65 - x)dx.$$

Alternative solutions:

A. If you let x be the distance between a stripe of the dam and the top of the dam, than the integral is

$$F = \rho g \int_{10}^{75} \left(\frac{3}{8}(75 - x) + 100\right)(x - 10)dx.$$

B. If you let x be the distance between a stripe of the dam and the water surface, then the integral is

$$F = \rho g \int_0^{65} \left(\frac{2}{3}(65 - x) + 100\right)x\,dx.$$

6. Let R be the region between the graph of $y = \dfrac{1}{\sqrt{4-x^2}}$ and the x-axis, $0 \le x \le 1$. Assuming it has a uniform density $\rho = 1$. Find the moments of R respect to x-axis and y-axis, and coordinates of the center of mass of this region.

Solution. The moments:

$$M_x = \frac{1}{2}\int_0^1 \frac{1}{4-x^2}\,dx = \frac{1}{8}\int_0^1 \left(\frac{1}{2-x} + \frac{1}{2+x}\right)dx = \frac{1}{8}\left[\ln\frac{2+x}{2-x}\right]_{x=0}^{1} = \frac{1}{8}\ln 3,$$

$$M_y = \int_0^1 \frac{x}{\sqrt{4-x^2}}\,dx = \frac{1}{2}\int_3^4 \frac{1}{\sqrt{u}}\,du = 2-\sqrt{3}.$$

The mass of R equals its area: $m = A = \displaystyle\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx = \frac{\pi}{6}$.

Hence, $\bar{x} = \dfrac{M_y}{m} = \dfrac{6(2-\sqrt{3})}{\pi}$, $\bar{y} = \dfrac{M_x}{m} = \dfrac{3\ln 3}{4\pi}$.