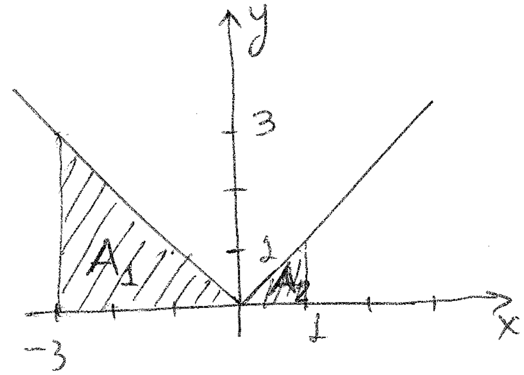
Math-205

Sample Midterm Test
Solutions

1. (a) Evaluate the integral $\int_{-3}^{1} |x|\,dx$ by interpreting it in terms of area:



The graph shows that both triangles are above the x-axis, so the signs of

$A_1 = \int_{-3}^{0} |x|\,dx$ and

$A_2 = \int_{0}^{1} |x|\,dx$ are posidive. Therefore

$$A_1 = +\frac{1}{2}\,3\cdot 3 = \frac{9}{2}$$

$$A_2 = +\frac{1}{2}\,1\cdot 1 = \frac{1}{2}$$

$$\Rightarrow A = \int_{-3}^{1} |x|\,dx = \int_{-3}^{0} |x|\,dx + \int_{0}^{1} |x|\,dx = \frac{9}{2} + \frac{1}{2} = 5$$

(b) Find the derivative of the function

$$F(x) = \int_{x^2-1}^{0} \frac{\sin(t+1)}{t+1}\,dt \quad \left(= -\int_{0}^{x^2-1} \frac{\sin(t+1)}{t+1}\,dt \right)$$

Solution: Using Fundamental Theorem of calculus

$$\frac{dF}{dx} = -\left.\frac{\sin(t+1)}{t+1}\right|_{t=x^2-1} \frac{d}{dx}(x^2-1) = -\frac{\sin(x^2)}{x^2}\cdot 2x = -2\,\frac{\sin(x^2)}{x}$$

2. Find the antiderivative $F(x)$ of $f(x) = xe^{-x^2}$ such that $F(0) = 3$.

Solution:

$$F(x) = \int x e^{-x^2} dx =$$

using substitution
$g = x^2$
$dg = 2x\,dx$

$$= \frac{1}{2}\int e^{-g} dg = \frac{-1}{2} e^{-g} + C =$$

$$= -\frac{1}{2} e^{-x^2} + C.$$

$$F(0) = -\frac{1}{2} e^{-0} + C = -\frac{1}{2} + C = 3, \Rightarrow C = \frac{7}{2}$$

answer: $F(x) = -\frac{1}{2} e^{-x^2} + \frac{7}{2}$

3. Calculate the following indefinite integrals:

(a) $$F(x) = \int \frac{(\sqrt{2x} - 1)^2}{x} dx = \int \frac{2x - 2\sqrt{2}\sqrt{x} + 1}{x} dx$$

$$= \int \left(2 - 2\sqrt{2}\, x^{-\frac{1}{2}} + x^{-1}\right) dx = 2x - \frac{2^{3/2}}{-\frac{1}{2}+1} x^{-\frac{1}{2}+1} + \ln|x| + C$$

$$= 2x - 2^{5/2} x^{1/2} + \ln|x| + C.$$

(b) $$\int 4t^2 \ln(t)\, dt =$$

integration by parts:
$u = t^3 \quad du = 3t^2 dt$
$v = \ln t$:
$\int v\,du = vu - \int u\,dv$

$$= 4\int \ln t \, d\left(\frac{t^3}{3}\right) =$$

$$= \frac{4}{3} t^3 \ln t - \frac{4}{3}\int t^3 \, d\ln t =$$

$$= \frac{4}{3} t^3 \ln t - \frac{4}{3}\int t^3 \frac{1}{t} dt = \frac{4}{3} t^3 \ln t - \frac{4}{9} t^3 + C$$

3(c). $F(x) = \int \frac{x-1}{x^2-7x+12}\,dx$: integration by partial fractions:

$$f(x) \equiv \frac{x-1}{(x-3)(x-4)} = \frac{A}{x-3} + \frac{B}{x-4} \Bigg| \Rightarrow$$

$$x-1 = A(x-4) + B(x-3) : \Rightarrow \text{at } x=3$$

$$2 = A(3-4);\ \underline{A=-2}$$

at $x=4$

$$3 = B(4-3) = B$$

$$\Rightarrow \frac{x-1}{(x-3)(x-4)} = -\frac{2}{x-3} + \frac{3}{x-4}$$

$$F(x) = \int\left(\frac{3}{x-4} - \frac{2}{x-3}\right)dx = 3\ln|x-4| - 2\ln|x-3| + C$$

N4 Evaluate the following definite integrals:

(a) $\int_0^3 \frac{1+\arctan(x/3)}{9+x^2}\,dx$

$u = \arctan\left(\frac{x}{3}\right)$

$du = \frac{3}{x^2+9}\,dx$

$u(0) = 0$

$u(3) = \frac{\pi}{4}$

$$= \int_0^{\pi/4} (1+u)\frac{1}{3}\,du = \frac{1}{3}\left(u + \frac{u^2}{2}\right)\Bigg]_0^{\pi/4} =$$

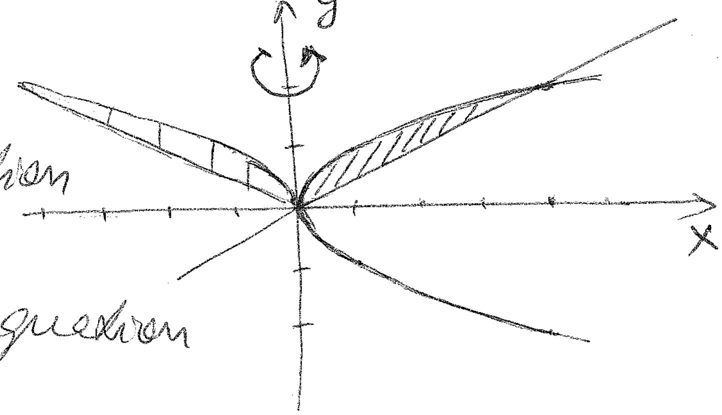
$$= \frac{1}{3}\left(\frac{\pi}{4} + \frac{\pi^2}{32}\right)$$

(b): $\int_0^1 x e^{-x}\,dx = -\int_0^1 x\,de^{-x} = -xe^{-x}\Big|_0^1 + \int_0^1 e^{-x}\,dx$

$$= -e^{-1} + 0 - e^{-x}\Big|_0^1 =$$

$$= -e^{-1} - e^{-1} + e^{0} = 1 - 2e^{-1}.$$

5. Find the volume of the solid obtained by rotating the region bounded by $y^2 = x$ and $x = 2y$ about the y-axis:



Solution:

The points of intersection of these curves are found from the equation

$x = (2y = y^2) \Rightarrow$

$y_1 = 0;\ y_2 = 2.$

Integration is done along y-axis using disks:

$$V = \int_0^2 \pi\left[(2y)^2 - (y^2)^2\right] dy = \pi \int_0^2 (4y^2 - y^4)\, dy$$

$$= \pi\left(\frac{4}{3}y^3 - \frac{1}{5}y^5\right)\Big|_0^2 = \frac{64}{15}\pi.$$

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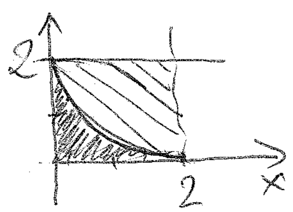
6. Bonus: Calculate the definite integral:

$$A = \int_0^2 \left[2 - \sqrt{(2-x)(2+x)}\right] dx:$$

Solution: Note that $\sqrt{(2-x)(2+x)} = \sqrt{2^2 - x^2}$ is a circle with radius $r = 2$. So the graph is



a curve representing the difference between the square and the circle.

By definition of the definite integral,

$$A = A_{\square} - \frac{1}{4}A_{\circ} = 2^2 - \frac{1}{4}\pi 2^2 = 4 - \pi.$$