Mat 1332 C DGD Notes

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The course textbook is Calculus for the Life Sciences: Modelling the Dynamics of Life by Frederick R. Adler and Miroslav Lovrić (First Canadian Edition). Throughout these notes, this textbook will be referred to as the "course text".

Disclaimer: Be aware that sometime we make changes to the questions from the course text before giving the solution.

1 Week One

1.1 Riemann Sums and Signed Area

The following are chapter questions from Section 6.3 in the course text (starting on p. 451).

Question 6. *Expand and evaluate (if possible) the expression* $\sum_{j=0}^{4}(1+2j)$.

Solution: For any sequence x_j where j runs over the integers, we can use sigma sum notation as short-hand to denote $\sum_{j=0}^{n} x_j = x_0 + x_1 + x_2 + \cdots + x_n$. For this question, we have

$$x_j = 1 + 2j \quad \text{for } j = 0, 1, 2, 3, 4.$$

Hence, we expand and evaluate:

$$\sum_{j=0}^{4}(1+2j) = 1 + 3 + 5 + 7 + 9 = 25.$$

Hence, the expression evaluates to 25. □

In mathematics, subscripts are often used to index sequences. Unless otherwise stated, subscripts never interact algebraically with the regular sized symbols (i.e. $x_2 \not\equiv 2x$ and $x_{i-1} \not\equiv x_i - 1$).

Question 10. *Let* f *be an arbitrary function. Expand and evaluate (if possible) the expression* $\sum_{i=1}^{4}(f(a_i) - f(a_{i-1}))$.

Solution: We can simply write this out and cancel like terms:

$$\sum_{i=1}^{4}(f(a_i) - f(a_{i-1})) = f(a_1) - f(a_0) + f(a_2) - f(a_1) + f(a_3) - f(a_2) + f(a_4) - f(a_3) = f(a_4) - f(a_0).$$

A more elegant solution is to use an elementary property of sums:

$$\sum_{i=1}^{n} a_i \pm b_i = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i \tag{1}$$

The second equality of the following uses the standard trick of "reindexing":

$$\sum_{i=1}^{4}(f(a_i) - f(a_{i-1})) = \sum_{i=1}^{4} f(a_i) - \sum_{i=1}^{4} f(a_{i-1}) = \sum_{i=1}^{4} f(a_i) - \sum_{i=0}^{3} f(a_i) = f(a_4) - f(a_0).$$

This solution has the advantage of being easier when 4 is replaced by a large number. □

Note in Question 10 that we do not know what f sends the elements of the sequence a_i to for any index i. This is because f is an arbitrary function. Hence, the answer that is given is sufficient.

Question 14. *Write the sum* $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{9}{10}$ *in sigma notation.*

Solution: Each summand in the expression increases by one in both the numerator and denominator and the denominator is always one more then the numerator. Hence, the following sigma notation arises:

$$\sum_{i=1}^{9} \frac{i}{i+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{9}{10}$$

□

Question 38. *An organism has* $B_i = i(6-i)$ *offspring in years* $i = 0, 1, \ldots, 6$*. Calculate the total offspring* T *that this organism has after year* 6*.*

Solution: Since the number of summands in the sum is small, we can simply add them up:

$$T = \sum_{i=0}^{6} B_i = B_0 + B_1 + B_2 + B_3 + B_4 + B_5 + B_6 = 0 + 5 + 8 + 9 + 8 + 5 + 0 = 35.$$

Alternatively, one may use the following useful formulas:

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

We employ some the summation rule (1) and the useful formulas:

$$T = \sum_{i=0}^{6} B_i = \sum_{i=0}^{6} i(6-i) = \sum_{i=0}^{6}((6i) - (i^2)) = \sum_{i=0}^{6} 6i - \sum_{i=0}^{6} i^2 = 6\sum_{i=0}^{6} i - \sum_{i=0}^{6} i^2$$

$$= 6\frac{6(6+1)}{2} - \frac{6(6+1)(12+1)}{6} = 126 - 91 = 35$$

□

Question 20. *Approximate* $\int_1^5 \ln x \, \mathrm{dx}$ *by finding* R_n*.*

Solution: We have all the ingredients: $f(x) = \ln x$ and $a = 1$ and $b = 5$.

$$\text{Since } \triangle x = \frac{b-a}{n}, \text{ we have } \triangle x = \frac{4}{n}. \quad \text{Since } x_i = a + i\triangle x, \text{ we have } x_i = 1 + \frac{4i}{n}.$$

These we plug into our formula for a right approximation:

$$R_n = \sum_{i=1}^{n} f(x_i)\triangle x = \sum_{i=1}^{n} \ln(x_i)\triangle x = \sum_{i=1}^{n} \ln(1 + \frac{4i}{n})\frac{4}{n}$$

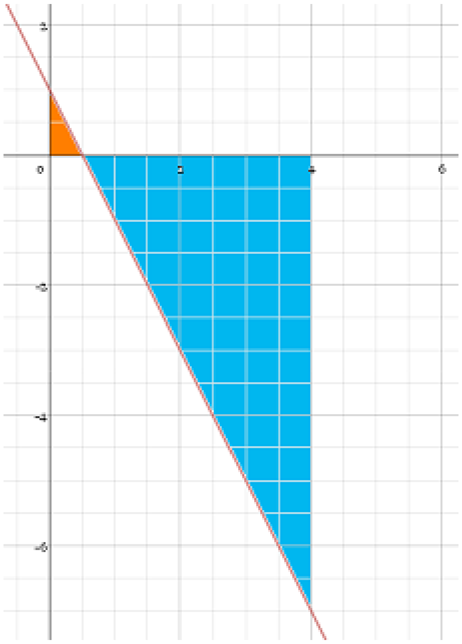
Note that R_n gives a different value for each natural number n that you plug into it. For example,

$$R_5 \doteq 4.65 \quad \text{and } R_{100} \doteq 4.08$$

As n gets bigger, the better the approximation. This makes sense because $\lim_{n\to\infty} R_n = \int_1^5 \ln x \, \mathrm{dx}$.

□

Question 24. *Calculate* $\int_0^4 1 - 2x \, \mathrm{dx}$ *by interpreting it as signed area. Alternatively, calculate it by taking the limit of* L_n*.*



Solution:
Add the orange triangle and subtract the blue triangle. We have by the forumla for areas of triangles:

$$\frac{1}{2} \cdot \frac{1}{2} \cdot 1 - \frac{1}{2} \cdot \frac{7}{2} \cdot 7 = \frac{1}{4} - \frac{49}{4} = -12.$$

Hence, the signed area is -12 which is equal to the above integral. Alternatively, $\triangle x = \frac{4}{n}$ and $x_i = \frac{4i}{n}$. Hence,

$$L_n = \sum_{i=1}^{n} f(x_{i-1}) \triangle x = \sum_{i=1}^{n} (1 - 2(\frac{4(i-1)}{n}))\frac{4}{n} = \frac{4}{n}(\sum_{i=1}^{n} 1 - \sum_{i=1}^{n} \frac{8}{n}(i-1))$$

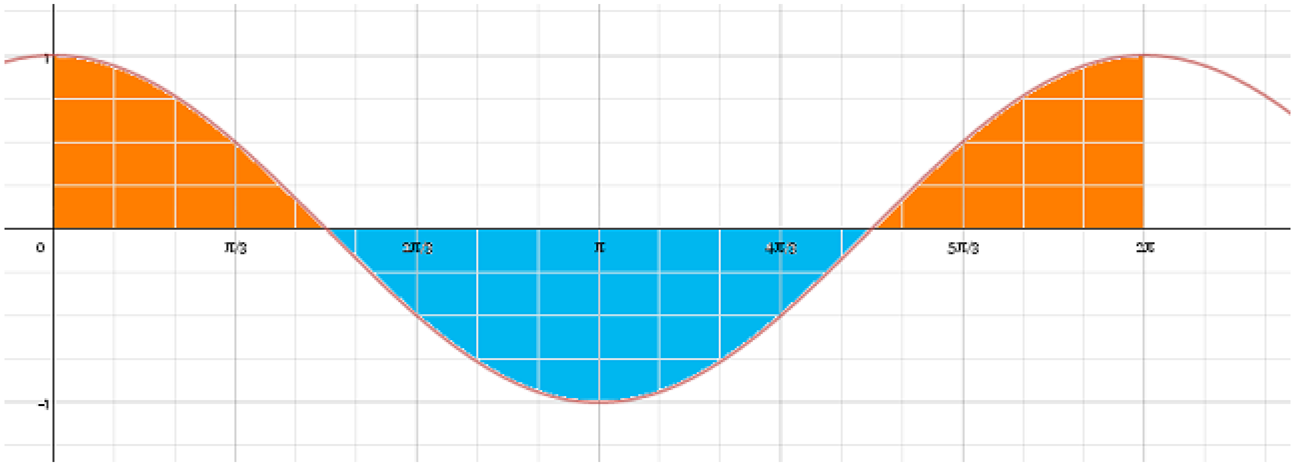
$$= 4 - \frac{32}{n^2}(\sum_{i=1}^{n} i - n) = 4 - \frac{32}{n^2}(\frac{n(n+1)}{2} - n) = 4 - 16 - \frac{16}{n}$$

Therefore,

$$\lim_{n \to \infty} L_n = 4 - 16 - 0 = -12.$$

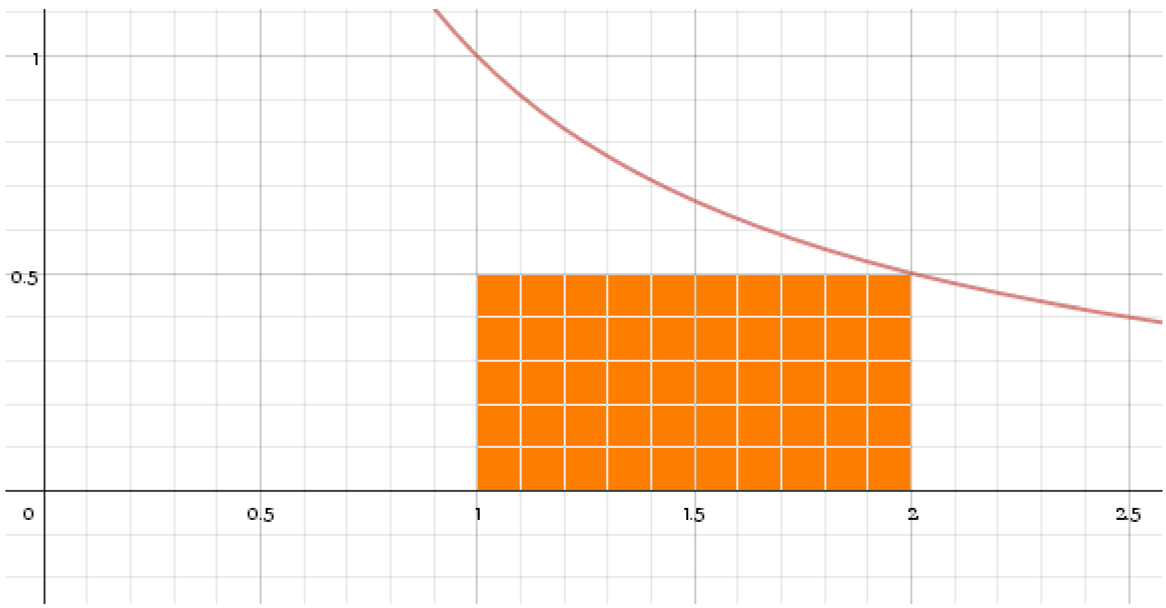
□

Question 28. *Calculate* $\int_0^{2\pi} cos x \, \mathrm{dx}$ *by interpreting it as signed area.*



Solution:
One can see from the nature of $\cos x$ (that is $\cos(x+\pi) = -\cos(x)$), that the area in orange and blue are equal and hence the signed area is zero which is equal to the above integral. □

Question 32. *Explain why the inequality* $\int_1^2 \frac{1}{x} \, \mathrm{dx} > \frac{1}{2}$ *is true.*



Solution:
The area in orange is equal to $\frac{1}{2}$. As one can see, there is more area contained under the curve hence the strict inequality. Note that this area is R_1. (Exercise: show that $R_2 > \frac{1}{2}$ and thus prove, since the function is decreasing on this interval and so R_2 must be an underestimate, that the above statement is true.) □

Question 42. *Find L_5 and R_5 for $f(x) = x^2$ from $a = 0$ to $b = 2$.*

Solution: First of all, we have $\triangle x = \frac{2-0}{5} = \frac{2}{5}$. The x_i for the approximation formula are given by $x_i = 0 + i\triangle x = \frac{2i}{5}$. Finally,

$$L_5 = \sum_{i=1}^{5} f(x_{i-1})\triangle x = \sum_{i=1}^{5}(x_{i-1})^2\frac{2}{5} = \sum_{i=1}^{5}(\frac{2(i-1)}{5})^2\frac{2}{5}$$

$$= 0 + \frac{8}{125} + \frac{32}{125} + \frac{72}{125} + \frac{128}{125} = \frac{48}{25}$$

$$R_5 = \sum_{i=1}^{5} f(x_i)\triangle x = \sum_{i=1}^{5}(x_i)^2\frac{2}{5} = \sum_{i=1}^{5}(\frac{2i}{5})^2\frac{2}{5}$$

$$= \frac{8}{125} + \frac{32}{125} + \frac{72}{125} + \frac{128}{125} + \frac{200}{125} = \frac{88}{25}$$

□

Question 48. *The speed of a bee is given by the following table:*

Time(s)	*Speed(cm/s)*
0	127
2	118
4	115
6	112
8	116
10	125

Find R_5 for the speed of the bee from 0 to 10 seconds. This is an approximation of the distance travelled by the bee in the ten seconds.

Solution: Let f be the function that gives the speed of the bee any given time. We have $\triangle x = \frac{10-0}{5} = 2$. We have $x_i = 0 + i\triangle x = 2i$. Finally,

$$R_5 = \sum_{i=1}^{5} f(x_i)\triangle x = \sum_{i=1}^{5} 2f(2i) = 236 + 230 + 224 + 232 + 250 = 1172\text{cm}.$$

□

1.2 Techniques of Integration

The following questions are from section 6.5 of the course text (starting on p. 485). We strongly recommend that you attempt these integrals (even if you saw us do them in the DGD) before looking at the solutions again. Remember, on the midterm if you have time, you can check your answer by taking the derivative of your answer.

Question 6. *Find* $\int 3^{x-5}\,\mathrm{dx}$.

Solution: $3^{x-5} = (e^{\ln 3})^{x-5} = e^{(x-5)\ln 3}$. Substitute $u = (x-5)\ln 3$. This gives us $\mathrm{dx} = \frac{\mathrm{du}}{\ln 3}$. Therefore,

$$\int 3^{x-5}\,\mathrm{dx} = \int e^u \frac{1}{\ln 3}\mathrm{du} = \frac{1}{\ln 3}\int e^u \mathrm{du} = \frac{1}{\ln 3}e^u + c = \frac{3^{x-5}}{\ln 3} + c$$

□

Question 14. *Find* $\int \frac{1}{\sqrt{1-9x^2}}\,\mathrm{dx}$.

Solution: Click here for reasoning for why $\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{dx} = \arcsin x$. Use substitution $u = 3x$. Hence, $\mathrm{du} = 3\mathrm{dx}$.

$$\int \frac{1}{\sqrt{1-9x^2}}\,\mathrm{dx} = \int \frac{1}{\sqrt{1-u^2}}\frac{1}{3}\,\mathrm{du} = \frac{1}{3}\arcsin u + c = \frac{1}{3}\arcsin 3x + c.$$

□

Question 28. *Find* $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,\mathrm{dx}$.

Solution: Use substitution $u = \sqrt{x}$. Hence, we have $\mathrm{dx} = 2\sqrt{x}\mathrm{du}$.

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,\mathrm{dx} = \int e^u 2\,\mathrm{du} = 2e^u + c = 2e^{\sqrt{x}} + c.$$

□

Question 34. *Find* $\int e^{4x}\sin e^{4x}\,\mathrm{dx}$.

Solution: Use substitution $u = e^{4x}$. Hence, $\mathrm{dx} = \frac{\mathrm{du}}{4e^{4x}}$.

$$\int e^{4x}\sin e^{4x}\,\mathrm{dx} = \int \frac{1}{4}\sin u\,\mathrm{du} = \frac{1}{4}\int \sin u\,\mathrm{du} = -\frac{1}{4}\cos u + c = -\frac{1}{4}\cos(e^{4x}) + c$$

□

Question 44. *Find* $\int x3^x\,\mathrm{dx}$.

Solution: We have $\int x3^x\,\mathrm{dx} = \int xe^{x\ln 3}\,\mathrm{dx}$. Let $u = x$ and $dv = e^{x\ln 3}dx$. Hence, $du = dx$ and $v = \frac{1}{\ln 3}e^{x\ln 3}$. Using the formula for integration by parts:

$$\int x3^x\,\mathrm{dx} = \int u\,\mathrm{dv} = uv - \int v\,\mathrm{du} = \frac{1}{\ln 3}x3^x - \int \frac{1}{\ln 3}e^{x\ln 3}\,\mathrm{dx} = \frac{1}{\ln 3}(x3^x - \frac{1}{\ln 3}3^x) + c.$$

□

As seen in the homework, these techniques are merely tools that allow us to manipulate integrals, and do not always immediately lead us to a solution. Sometimes a combination or repetition of these techniques are required.

Question 46. *Find* $\int \sin \sqrt{x}\, \mathrm{dx}$.

Solution: Subsitution then integration by parts. Set $t = \sqrt{x}$. Hence, $\mathrm{dt} = \frac{1}{2\sqrt{x}}\, \mathrm{dx} = \frac{1}{2t}\, \mathrm{dx}$. Thus, $\mathrm{dx} = 2t\, \mathrm{dt}$. Substituting, we obtain $\int \sin \sqrt{x}\, \mathrm{dx} = \int 2t \sin t\, \mathrm{dt}$.

Next, we integrate by parts. Set $u = 2t$ and $v = -\cos t$. Thus, $\frac{\mathrm{du}}{dt} = 2$ and $\frac{\mathrm{dv}}{dt} = \sin t$. Thus, we have $\int 2t \sin t\, \mathrm{dt} = \int u\, \mathrm{dv} = uv - \int v\, \mathrm{du} = -2t \cos t + \int 2 \cos t\, \mathrm{dt} = -2\sqrt{x} \cos \sqrt{x} + 2 \sin \sqrt{x} + c$. □

Question 50. *Find* $\int e^x \sin x\, \mathrm{dx}$.

Solution: Integration by parts. Let $u = e^x$ and $dv = \sin x dx$. Then, $du = e^x dx$ and $v = -\cos x$. We have $\int e^x \sin x\, \mathrm{dx} = \int u\, \mathrm{dv} = uv - \int v\, \mathrm{du} = -e^x \cos x + \int \cos x e^x\, \mathrm{dx}$. We integrate by parts a second time. Let $u = e^x$ and $dv = \cos x dx$. Then, $du = e^x dx$ and $v = \sin x$. Hence, $\int \cos x e^x\, \mathrm{dx} = \int u\, \mathrm{dv} = uv - \int v\, \mathrm{du} = e^x \sin x - \int \sin x e^x\, \mathrm{dx}$. Together, we have

$$\int e^x \sin x\, \mathrm{dx} = -e^x \cos x + e^x \sin x - \int \sin x e^x\, \mathrm{dx}$$

Hence, combining like terms and dividing by two we have

$$\int e^x \sin x\, \mathrm{dx} = \frac{-e^x \cos x + e^x \sin x}{2}$$

□

Question 50 is an example of a problem where a student might not realize that they are done after integrating by parts twice. Always remember where you started and, if you get there again, you can move it to the "left hand side".

Question 54. *Consider the following differential equation:* $\frac{dP}{dt} = 2.5(\frac{1}{1+t} + e^{-t})$ *that describes the production of a chemical* P*. Find* $P(t)$ *given that* $P(0) = 0$*. Compute* $P(10)$*.*

Solution:

$$P(t) = \int \frac{dP}{dt}\, \mathrm{dt} = \int 2.5(\frac{1}{1+t} + e^{-t})\, \mathrm{dt}$$

$$2.5(\int \frac{1}{1+t}\, \mathrm{dt} + \int e^{-t}\, \mathrm{dt} = 2.5(\ln(t+1) - e^{-t}) + C$$

We use initial condition:

$$0 = P(0) = 2.5(\ln(1) - e^0) + C = -2.5 + C$$

Hence, $C = 2.5$. Therefore $P(t) = 2.5(\ln(t+1) - e^{-t}) + 2.5$. Furthermore, we have

$$P(10) = 2.5(\ln(11) - e^{-10}) + 2.5.$$

□

Question 62. *A population of leopards* $L(t)$ *is increasing according the formula* $L(t) = 100t$*. Leopards are being poached at rate* $B'(t) = 0.1L(t)$*. Find* $B(t)$ *with* $B(0) = 0$*.*

Solution: $B(t) = \int B'(t)\, \mathrm{dt} = \int 10t\, \mathrm{dt} = 5t^2 + C$. To find C, we see that $C = B(0) - 50^2 = B(0) = 0$. Hence, $B(t) = 5t^2$. □

2 Week Two

2.1 Definite Integrals

The following questions are from section 6.4 of the course text (starting on p. 467).

Compute each definite integral

Question 6. $\int_0^1 (10t^9 + 6t^5)dt$

Solution: From the power rule, the antiderivative of the integrand $10t^9 + 6t^5$ is the family of functions $t^{10} + t^6 + C$. By the Fundamental Theorem of Calculus, the definite integral can be evaluated as the difference of any member of this function (say the one corresponding to the choice $C = 0$) between the bounds of integration:

$$\int_0^1 (10t^9 + 6t^5)dt = (t^{10} + t^6)\Big|_0^1 = 1^{10} + 1^6 - (0^{10} + 0^6) = 2.$$

□

Question 10. $\int_1^4 3z^{\frac{3}{7}} dz$

Solution: By the power rule, one antiderivative of the integrand is the function $\frac{3}{\frac{3}{7}+1} z^{\frac{3}{7}+1} = \frac{21}{10} z^{\frac{10}{7}}$, and so

$$\int_1^4 3z^{\frac{3}{7}} dz = \frac{21}{10} z^{\frac{10}{7}}\Big|_1^4 = \frac{21}{10}(4^{\frac{10}{7}} - 1^{\frac{10}{7}}) = \frac{21}{10}(4^{\frac{10}{7}} - 1).$$

□

Question 18. $\int_{-\pi/2}^{\pi/2} [x^2 - 20\sin x] dx$

Solution: Since $\sin x$ is an odd function (i.e. $\sin(x) = -\sin(-x)$), its definite integral over any interval of the form $[-a, a]$ is 0. And since x^2 is an even function (i.e. $x^2 = (-x)^2$), its integral over any interval of the form $[-a, a]$ is $2\int_0^a x^2$. Therefore

$$\int_{-\pi/2}^{\pi/2} [x^2 - 20\sin x] dx = \int_{-\pi/2}^{\pi/2} x^2 dx - 20\int_{-\pi/2}^{\pi/2} \sin x dx = 2\int_0^{\pi/2} x^2 dx.$$

An antiderivative of x^2 is $\frac{1}{3}x^3$ and so

$$\int_{-\pi/2}^{\pi/2} [x^2 - 20\sin x] dx = 2\,\frac{1}{3}x^3\Big|_0^{\pi/2} = \frac{2}{3}\left[\left(\frac{\pi}{2}\right)^3 - 0^3\right] = \frac{\pi^3}{12}.$$

□

Question 22. $\int_{-1/2}^{1} \frac{1}{\sqrt{1-x^2}}$

Solution: Recall that one consequence of the chain rule is that $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$ and so

$$\int_{-1/2}^{1} \frac{1}{\sqrt{1-x^2}} = \arcsin x\big|_{-1/2}^{1} = \arcsin(1) - \arcsin(-1/2) = \frac{\pi}{2} - \left(-\frac{\pi}{6}\right) = \frac{2\pi}{3}.$$

□

Question 26. $\int_0^{\pi/4} \sec x \tan x dx$

Solution: By writing $\sec x \tan x = \frac{1}{\cos x}\frac{\sin x}{\cos x} = \frac{\sin x}{(\cos x)^2}$ it becomes easier to see the correct integration technique to use. Notice that the numerator of the integrand is, up to a constant, the chain rule term one obtains by differentiating the term which is squared in the denominator. This suggests that integration by substitution will work. So let $u(x) = \cos x$ so that $\frac{d}{dx}u(x) = -\sin x$, or $-du = \sin x dx$ to abuse notation. Then the indefinite integral of $\sec x \tan x$ can be computed as

$$\int \sec x \tan x dx = \int \frac{\sin x}{(\cos x)^2}dx = -\int \frac{1}{u^2}du = \frac{1}{u} + C$$

(the last line was a straightforward application of the power rule). From here one can proceed to find the definite integral in one of two ways. The first way is to find the new bounds of integration for the transformed function: the bound $x = 0$ corresponds to $u(0) = 1$ and the bound $x = \pi/4$ corresponds to $u\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, thus

$$\int_0^{\pi/4} \sec x \tan x dx = \left.\frac{1}{u}\right|_1^{1/\sqrt{2}} = \frac{1}{1/\sqrt{2}} - \frac{1}{1} = \sqrt{2} - 1.$$

The second way is to rewrite the indefinite integral found above as a function of x: $\frac{1}{u(x)} = \frac{1}{\cos x}$ and so

$$\int_0^{\pi/4} \sec x \tan x dx = \left.\frac{1}{\cos x}\right|_0^{\pi/4} = \frac{1}{\cos(\pi/4)} - \frac{1}{\cos 0} = \frac{1}{1/\sqrt{2}} - \frac{1}{1} = \sqrt{2} - 1.$$

Of course, both methods yield the same answer. □

Question 26. $\int_{-3}^0 \frac{1}{4+t}dt$

Solution: First notice that the integrand is continuous on its domain (namely $(-\infty, -4) \cup (-4, \infty)$), and that the interval of integration is contained within that domain. Next, notice that this is another situation where integration by substitution is required: letting $u(t) = 4+t$ so that $\frac{du}{dt} = 1$ or $du = dt$, one finds that the indefinite integral is given by

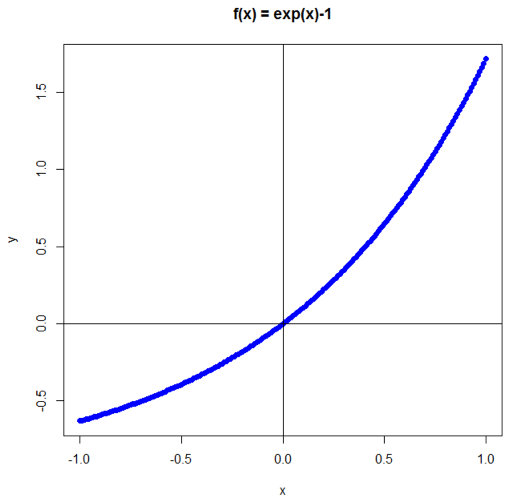
$$\int \frac{1}{4+t}dt = \int \frac{1}{u}du = \ln|u| + C.$$

Since $u(-3) = 1$ and $u(0) = 4$, it follows that the definite integral is given by

$$\int_{-3}^0 \frac{1}{4+t}dt = \ln u|_1^4 = \ln 4 - \ln 1 = \ln 4.$$

□

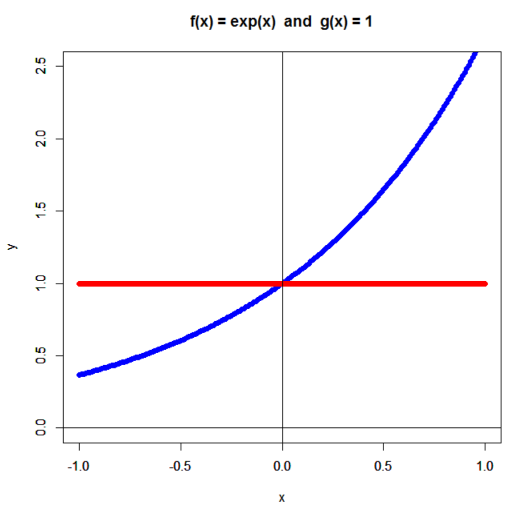
Question 34. *Evaluate* $\int_{-1}^{1}(e^x - 1)dx$*, sketch the graph of the function involved, and interpret as an area or a difference in areas.*



Solution:
Recall that the exponential function is its own derivative and hence its own antiderivative. Thus

$$\int_{-1}^{1}(e^x - 1)dx = e^x - x|_{-1}^{1} = e^1 - 1 - (e^{-1} - (-1)) = e - \frac{1}{e} - 2.$$

Now, one can see from the graph of $e^x - 1$ that the function lies below the $y - axis$ on the interval $(-\infty, 0)$. Therefore, if we interpret the definite integral as the area of this function we will have to consider some parts as "negative" area. It is more intuitive to view the final value as a difference of areas: the area under the function e^x over $[-1, 1]$ minus the area of the function 1 over $[-1, 1]$.



□

Question 40. *Compute the definite integral of* $G(t) = \frac{2}{t} + \frac{t}{2}$ *from* $t = 1$ *to* $t = 2$*, from* $t = 2$ *to* $t = 3$*, and finally from* $t = 1$ *to* $t = 3$ *to check the summation property of definite integrals.*

Solution: Using the rules of integration we have seen so far, it is easy to determine that $G(t)$ has $2\ln|t| + \frac{t^2}{4}$ as an antiderivative. Therefore

$$\int_a^b G(t)dt = 2\ln|t| + \frac{t^2}{4}\bigg|_a^b = 2\ln|b| + \frac{b^2}{4} - \left(2\ln|a| + \frac{a^2}{4}\right) = 2\ln\left|\frac{b}{a}\right| + \frac{b^2 - a^2}{4}.$$

Substituting in the relevant values of a and b then gives

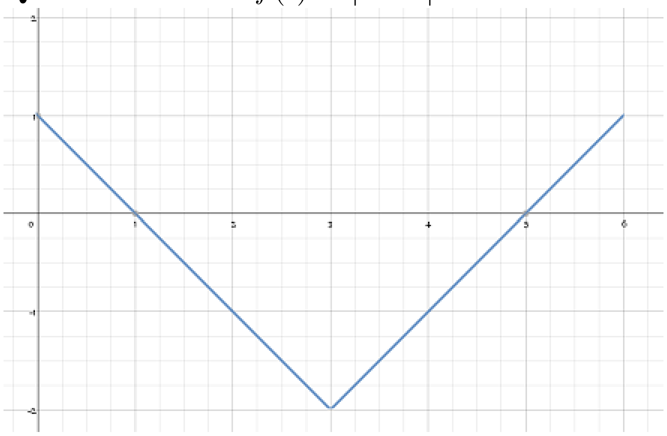
$$\int_1^2 G(t)dt = 2\ln 2 + \frac{3}{4}, \quad \int_2^3 G(t)dt = 2\ln\left(\frac{3}{2}\right) + \frac{5}{4}, \text{ and } \int_1^3 G(t)dt = 2\ln 3 + 2.$$

Of course,

$$2\ln 2 + \frac{3}{4} + 2\ln\left(\frac{3}{2}\right) + \frac{5}{4} = 2\ln\left(2\cdot\frac{3}{2}\right) + \frac{8}{4} = 2\ln 3 + 2$$

which verifies the additive property. □

Question 42. *Let* $f(t) = |t-3| - 2$ *on the domain* $[0, 6]$.



Define $g(x) = \int_0^x f(t)\,dt$ *on* $[0, 6]$. *Find intervals where* $g(x)$ *is increasing and intervals where* $g(x)$ *is decreasing. Find global maximum or minimum values of* g. *Sketch graph of* g.

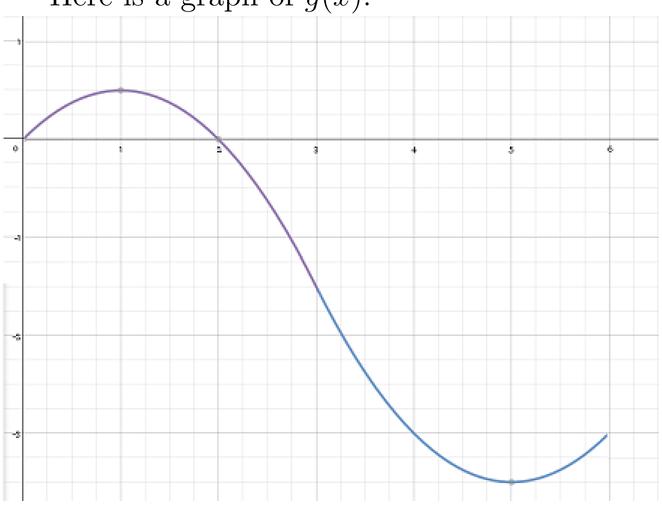
Solution: If $0 \le x \le 3$, then $g(x) = \int_0^x |t-3| - 2\,dt = \int_0^x (3-t) - 2\,dt = \int_0^x 1\,dt - \int_0^x t\,dt = t|_0^x - \frac{1}{2}t^2|_0^x = x - \frac{x^2}{2}$.

If $x > 3$, then $g(x) = \int_0^x |t-3| - 2\,dt = \int_0^3 (3-t) - 2\,dt + \int_3^x (t-3) - 2\,dt = (t - \frac{t^2}{2})|_0^3 + (\frac{t^2}{2} - 5t)|_3^x = 3 - \frac{9}{2} + \frac{x^2}{2} - 5x - \frac{9}{2} + 15 = \frac{x^2}{2} - 5x + 9$.

Note that $g'(x) = f(x)$ by (2). Hence, $g(x)$ is increasing on $[0, 1) \cup (5, 6]$ and decreasing on $(1, 5)$.

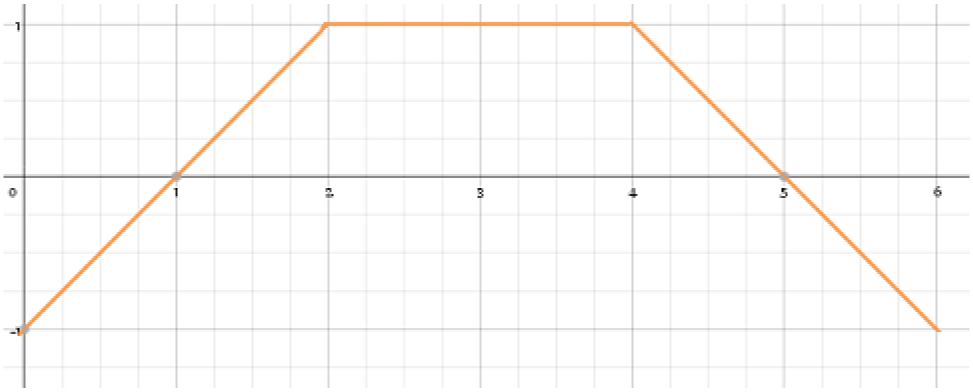
Critical points are $x = 0, 1, 5, 6$. $g(0) = 0$, $g(1) = \frac{1}{2}$, $g(5) = -\frac{7}{2}$, and $g(6) = -3$. Hence, global max value is $\frac{1}{2}$ at $x = 1$ and the global min value is $-\frac{7}{2}$ at $x = 5$.

Here is a graph of $g(x)$:



□

Question 44. *Define the function* $f(t) = \begin{cases} t-1 & \text{if } 0 \le t \le 2 \\ 1 & \text{if } 2 < t \le 4 \\ 5-t & \text{if } 4 < t \le 6 \end{cases}$ *on the domain* $[0,6]$.



Define $g(x) = \int_0^x f(t)\,dt$ *also on* $[0,6]$. *Find intervals where* $g(x)$ *is increasing and intervals where* $g(x)$ *is decreasing. Find global maximum or minimum values of* g. *Sketch graph of* g.

Solution: If $0 \le x \le 2$, then $g(x) = \int_0^x t-1\,dt = \frac{x^2}{2} - x$.

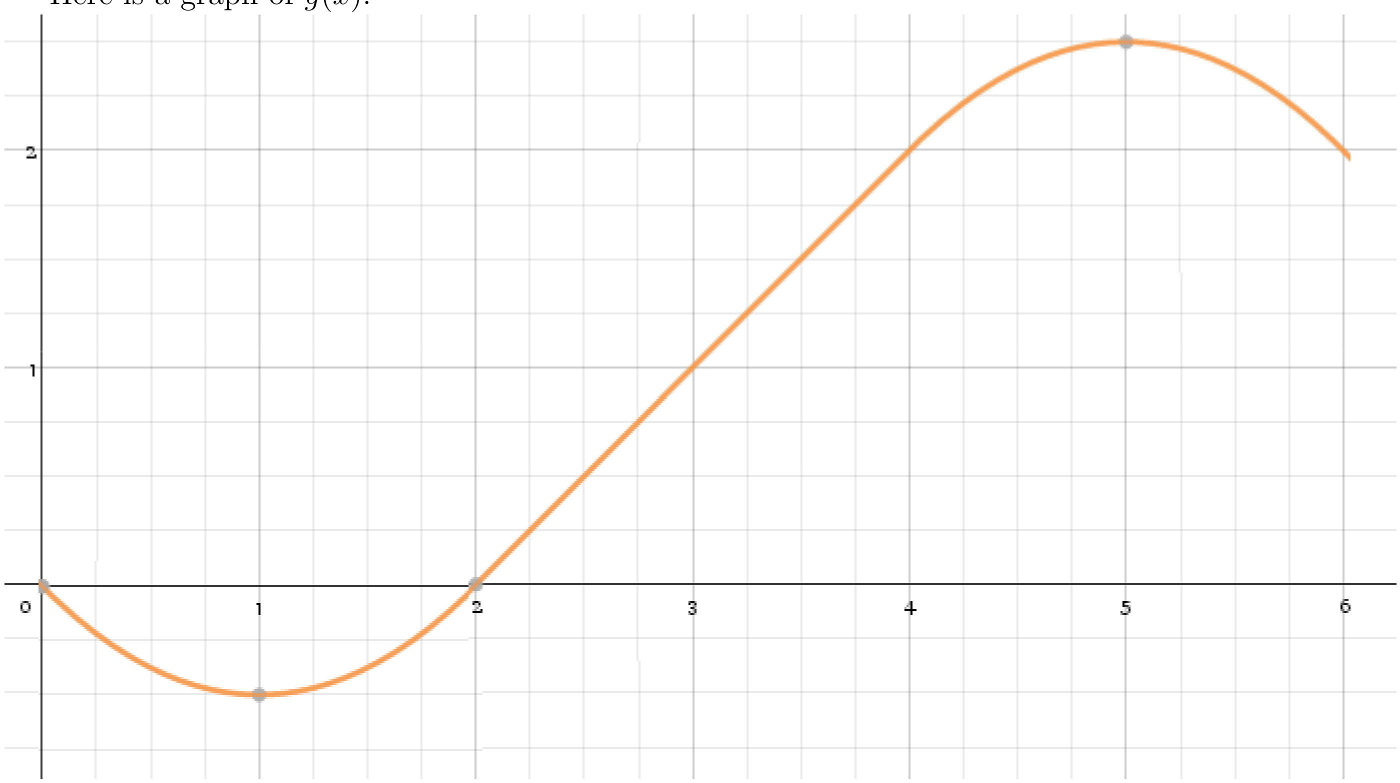
If $2 \le x \le 4$, then $g(x) = \int_0^2 t-1\,dt + \int_2^x 1\,dt = 0 + x - 2 = x - 2$.

If $4 \le x \le 6$, then $g(x) = \int_0^2 t-1\,dt + \int_2^4 1\,dt + \int_4^x 5-t\,dt = 0 + (2-0) + (5t - \frac{t^2}{2})|_4^x = 2 + (5x - \frac{x^2}{2}) - (20-8) = 5x - \frac{x^2}{2} - 10$.

Again $g'(x) = f(x)$ by (2), so we can see that $g(x)$ is increasing on $(1,5)$ and decreasing on $[0,1) \cup (5,6]$.

Find the values of the critical points $x = 0, 1, 5, 6$. $g(0) = 0$ $g(1) = -1$ $g(5) = \frac{5}{2}$ and $g(6) = 2$. Hence, The global max is $\frac{5}{2}$ at $x = 5$ and global min is -1 at $x = 1$.

Here is a graph of $g(x)$:



□

Question 46. *Find the derivative of the function* $g(x) = \int_3^x (\ln t - \frac{1}{t})dt$.

Solution: We do this brute force first:

$$g(x) = \int_3^x (\ln t - \frac{1}{t})dt = \int_3^x \ln t dt - \int_3^x \frac{1}{t} dt = [t\ln t - t]_3^x - [\ln t]_3^x = x\ln x - x - 3\ln 3 + 3 - \ln x + \ln 3$$

Hence,

$$g'(x) = (x\ln x)' - 1 - \frac{1}{x} = \ln x + 1 - 1 - \frac{1}{x} = \ln x - \frac{1}{x}.$$

Alternatively, we can quote the rule that for any value of a the following holds:

$$\frac{d}{dx}(\int_a^x f(t)dt) = f(x). \tag{2}$$

□

Question 64. *The position of a rock obeys the differential equation* $\frac{dP}{dt} = -9.8t - 5$ *with initial condition* $P(0) = 200$. *Solve the differential equation. Determine the change in position from* $t = 1$ *to* $t = 3$ *and from* $t = 3$ *to* $t = 5$, *and use this to deduce the change in position from* $t = 1$ *to* $t = 5$.

Solution: $P(t) = \int \frac{dP}{dt}\, dt = -9.8\frac{t^2}{2} - 5t + c$. From the initial condition we get $c = P(0) + 0 + 0 = 200$. Hence, $P(t) = -9.8\frac{t^2}{2} - 5t + 200$. The change in position from $t = 1$ to $t = 3$ is $\int_1^3 \frac{dP}{dt}\, dt = P(3) - P(1) = 140.9 - 190.1 = -49.2$. The change in position from $t = 3$ to $t = 5$ is $\int_3^5 \frac{dP}{dt}\, dt = P(5) - (3) = 52.5 - 140.9 = -88.4$. The change in position from $t = 1$ to $t = 5$ is thus

$$\int_1^5 \frac{dP}{dt}\, dt = \int_1^3 \frac{dP}{dt}\, dt + \int_3^5 \frac{dP}{dt}\, dt = -49.2 - 88.4 = -137.6.$$

□

Note that if an object's position obeys a differential equation $\frac{dP}{dt}$. The value $\int_a^b \frac{dP}{dt}\, dt$ measures the signed distance between the position at time a and the position at time b. If one wants to calculate the length of the route travelled including back-tracks (the mileage), simply find $\int_a^b |\frac{dP}{dt}|\, dt$.

The following is nice application of integration to physics. We obviously are ignoring concepts like drag in this question but will deal with drag in Question 32 of Week 5.

Question 70. *A rocket is shot into the air (from the ground) with an upward acceleration of* 12 m/s^2 *and has* 10s *worth of fuel. After the fuel runs out the rocket falls with an acceleration of* -9.8 m/s^2. *What is the maximal height of the rocket. Find the velocity when it hits the ground.*

Solution: The function for acceleration can be described as follows:

$$f''(t) = \begin{cases} 12 & \text{if } t \leq 10 \\ -9.8 & \text{if } t > 10 \end{cases}$$

We integrate this function on each part to get a function for the velocity:

$$f'(t) = \begin{cases} 12t + c & \text{if } t \leq 10 \\ -9.8t + d & \text{if } t \geq 10 \end{cases}$$

At time $t = 0$ we can assume the rocket has no speed so $c = f'(0) = 0$. At time $t = 10$, velocity must be continuous so we must have $12 \cdot 10 = -9.8 \cdot 10 + d$. Hence, $d = 120 + 98 = 218$. Therefore,

$$f'(t) = \begin{cases} 12t & \text{if } t \leq 10 \\ -9.8t + 218 & \text{if } t \geq 10 \end{cases}$$

The maximal height of the rocket will be when the velocity slows down to zero. That is, we need to find a $t > 10$ such that $-9.8t + 218 = 0$. We solve and get that at $t \doteq 22.2449$ the rocket is at its highest. We now solve for the height:

$$\int_0^{22.2449} f'(t)\,dt = \int_0^{10} 12t\,dt + \int_{10}^{22.2449} -9.8t + 218\,dt = 600 + (-4.9t^2 + 218t)|_{10}^{22.2449} \doteq 1234.69$$

To find how long it will take to come back down we solve:

$$\int_0^t 9.8t\,dt = 1234.69 \implies 4.9t^2 = 1234.69 \implies t \doteq 15.87$$

Hence, the time will be $t \doteq 22.2449 + 15.87 \doteq 38.11$. $f'(38.11) = -9.8(38.11) + 218 \doteq -155.5$. Therefore, the rocket will be going approximately 155.5 m/s when it hits the ground. $\square$

3 Week Three

3.1 Applications of Integrals

The following questions are from section 6.6 of the course text (starting on p. 496).

Question 10. *Find the area between $f(x) = x^2$ and $g(x) = x^3$.*

Solution: We want to find $\int_0^2 |x^2 - x^3|\,\mathrm{dx}$. One knows that on $[0,1]$, $x^2 \geq x^3$, and that on $[1,2]$, $x^2 \leq x^3$. Therefore,

$$\int_0^2 |x^2 - x^3|\,\mathrm{dx} = \int_0^1 (x^2 - x^3)\,\mathrm{dx} + \int_1^2 (x^3 - x^2)\,\mathrm{dx} = \left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Bigg|_0^1 + \left(\frac{x^4}{4} - \frac{x^3}{3}\right)\Bigg|_1^2$$
$$= \frac{1}{3} - \frac{1}{4} + \frac{2^4}{4} - \frac{2^3}{3} - \left(\frac{1}{4} - \frac{1}{3}\right) = \frac{3}{2}.$$

□

Question 16. *Find the area bounded by the functions $f(x) = |x|$ and $g(x) = 2 - x^2$.*

Solution: We have to find the points where the two graphs meet. Consider the equation $|x| = 2 - x^2$. First, consider the case where $x \leq 0$. The equation becomes $-x = 2 - x^2$. From the quadratic formula, one has that

$$x = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2} = \frac{1 \pm 3}{2} \in \{-1, 2\}.$$

Thus, in this case, $x = -1$. Secondly, consider the case where $x \geq 0$. The equation becomes $x = 2 - x^2$. Again, by thequadratic formula

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-2)}}{2} = \frac{-1 \pm 3}{2} \in \{1, -2\}.$$

Thus, in this case, $x = 1$. Therefore, we want to find $\int_{-1}^1 \big||x| - (2 - x^2)\big|\,\mathrm{dx}$. We claim that, on $[-1,1]$, $|x| \leq 2 - x^2$. Since $x \in [-1,1]$, $|x| \in [0,1]$. On the other hand,

$$x \in [-1,1] \Leftrightarrow x^2 \in [0,1] \Leftrightarrow -x^2 \in [-1,0] \Leftrightarrow 2 - x^2 \in [1,2].$$

Thus, $|x| \leq 2 - x^2$ for $x \in [-1,1]$. Therefore,

$$\int_{-1}^1 \big||x| - (2 - x^2)\big|\,\mathrm{dx} = \int_{-1}^1 (2 - x^2 - |x|)\,\mathrm{dx} = \int_{-1}^0 (2 - x^2 + x)\,\mathrm{dx} + \int_0^1 (2 - x^2 - x)\,\mathrm{dx}$$
$$= \left(2x - \frac{x^3}{3} + \frac{x^2}{2}\right)\Bigg|_{-1}^0 + \left(2x - \frac{x^3}{3} - \frac{x^2}{2}\right)\Bigg|_0^1$$
$$= -\left(-2 + \frac{1}{3} + \frac{1}{2}\right) + \left(2 - \frac{1}{3} - \frac{1}{2}\right) = 4 - \frac{2}{3} - 1 = \frac{7}{3}.$$

□

Question 26. *Find the area under the curve $g(x) = x \ln x$ between $a = 1$ and $b = 2$.*

Solution: We solve the following definite integral:

$$\int_1^2 x \ln x \, dx$$

Use integration by parts. $u = \ln x$ and $dv = xdx$. Thus, $du = \frac{1}{x}dx$ and $v = \frac{x^2}{2}$.

$$\int x \ln x \, dx = \int u \, dv = uv - \int v \, du = \ln x \frac{x^2}{2} - \int \frac{x^2}{2} \frac{1}{x} \, dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4}$$

Hence,

$$\int_1^2 x \ln x \, dx = (\frac{x^2 \ln x}{2} - \frac{x^2}{4})|_1^2 = (2\ln 2 - 1) - (0 - \frac{1}{4}) = \ln 4 - \frac{3}{4}$$

□

Question 30. *Use the fact that the perimeter of a circle of radius r is $2\pi r$ to find the area of a circle of radius R.*

Solution: We may think the circular disc as being built of little rings with width Δr, so that the area of these rings is $2\pi r \Delta r$ (Here we use the fact that the perimeter of a circle of radius r is $2\pi r$). Therefore, the area of a disc with radius R is:

$$\int_0^R 2\pi r \, \mathrm{dr} = 2\pi \left(\frac{r^2}{2}\right)\Big|_0^R = \pi R^2.$$

□

Question 32. *The natural log can be defined as* $\ln(a) := \int_1^a \frac{1}{x} \, \mathrm{dx}$. *Show that*

(i) $\ln(6) - \ln(3) = \ln(2)$,

(ii) $\ln(10^2) = 2\ln(10)$.

Solution: (i). We know that

$$\ln(6) - \ln(3) = \int_1^6 \frac{1}{x} \, \mathrm{dx} - \int_1^3 \frac{1}{x} \, \mathrm{dx} = \int_1^6 \frac{1}{x} \, \mathrm{dx} + \int_3^1 \frac{1}{x} \, \mathrm{dx} = \int_3^6 \frac{1}{x} \, \mathrm{dx}.$$

Next, we use substitution. Set $u = x/3$, so $\mathrm{du} = \frac{1}{3}\mathrm{dx}$. The integral then becomes

$$\int_3^6 \frac{1}{x} \, \mathrm{dx} = \int_3^6 \frac{1}{3x} 3 \, \mathrm{dx} = \int_1^2 \frac{1}{u} \, \mathrm{du} = \ln(2).$$

(ii). We know that $\ln(10^2) = \int_1^{10^2} \frac{1}{\mathrm{x}} \, \mathrm{dx}$. Next, we use substitution. Set $u = \sqrt{x}$, so that $\mathrm{du} = \frac{1}{2\sqrt{x}} \, \mathrm{dx} = \frac{1}{2u} \, \mathrm{dx}$. From this, we obtain

$$\int_1^{10^2} \frac{1}{x} \, \mathrm{dx} = \int_1^{10^2} \frac{1}{u^2} \, \mathrm{dx} = \int_1^{10} \frac{1}{u^2} 2u \, \mathrm{du} = 2 \int_1^{10} \frac{1}{u} \, \mathrm{du} = 2\ln(10).$$

□

Question 42. *Several 2m snakes are collected. Each has a density of $f(x) = 1+2\times 10^{-8}x^2(300-x)$ g/cm where x is the cm from the tail.*

(a) *Find minimum and maximum density of such a snake.*

(b) *Find total mass of such a snake.*

(c) *Find the average density of such a snake.*

Solution: Find critical points. We have $0 = f'(x) = 2\times 10^{-8}(x^2(300-x))' = 2\times 10^{-8}(600x - 3x^2) \implies x = 0$ or $x = 200$. These happen to be the endpoints of the snake. Hence, one must be the min and one must be the max:

$$f(0) = 1 \quad \text{ and } f(200) = 1 + 2\times 10^{-8}(200^2 \cdot 100) = \frac{27}{25}$$

Hence, the minimum density is 1 g/cm and the maximum is $\frac{27}{25}$ g/cm. The total mass of the snake is

$$\int_0^{200} f(x)\,dx = 200 + 2\times 10^{-8}\int_0^{200} 300x^2 - x^3\,dx = 200 + 2\times 10^{-8}(100x^3 - \frac{x^4}{4})|_0^{200}$$
$$= 200 + 2\times 10^{-8}(100\cdot 200^3 - \frac{200^4}{4}) = 208g$$

The average density of such a snake is 208g/200cm$= \frac{26}{25}$ g/cm. Note that in general the average density is NOT halfway between the min and max (see example 6.6.9 on pg 495 in textbook). In this case, this is due to some symmetry of the density function in this question. □

3.2 Integrating Partial Fractions

The following question is from 7.4 in the course text (starting on page 550).

Question 53. *Calculate the indefinite integral* $\int \frac{1}{x(a+bx)}\,\mathrm{dx}$.

Solution: We use the method of partial fractions. Consider the following

$$\frac{1}{x(a+bx)} = \frac{A}{x} + \frac{B}{a+bx}$$

for real numbers A and B. Note that

$$\frac{A}{x} + \frac{B}{a+bx} = \frac{A(a+bx)+Bx}{x(a+bx)} = \frac{(Ab+B)x+Aa}{x(a+bx)}.$$

Thus, $Aa = 1$ and $Ab + B = 0$. It follows that $A = \frac{1}{a}$ and $B = -Ab = -\frac{b}{a}$. Therefore,

$$\int \frac{1}{x(a+bx)} = \int \left(\frac{1/a}{x} + \frac{-b/a}{a+bx}\right) \mathrm{dx} = \frac{1}{a}\ln|\mathrm{x}| - \frac{1}{\mathrm{a}}\ln|\mathrm{a}+\mathrm{bx}| + \mathrm{C} = \frac{1}{\mathrm{a}}\ln\left|\frac{\mathrm{x}}{\mathrm{a}+\mathrm{bx}}\right| + \mathrm{C}.$$

□

The following questions are from the additional course notes on partial fractions.

Question 3. *Find* $\int \frac{1}{x^2+2x+10}\,dx$.

Solution: We complete the square:

$$\int \frac{1}{x^2+2x+10}\,dx = \int \frac{1}{x^2+2x+1-1+10}\,dx = \int \frac{1}{(x+1)^2+9}\,dx$$

To prepare for substitution we put the integral into a form that looks like the derivative of arcsin:

$$\int \frac{1}{(x+1)^2+9}\,dx = \frac{1}{9}\int \frac{1}{\frac{(x+1)^2}{9}+1}\,dx$$

To substitute we choose $u = (x+1)/3$ and, hence, $dx = 3du$.

$$\frac{1}{9}\int \frac{1}{\frac{(x+1)^2}{9}+1}\,dx = \frac{1}{3}\int \frac{1}{u^2+1}\,du = \frac{1}{3}\arctan u + c = \frac{1}{3}\arctan\left(\frac{x+1}{3}\right) + c$$

□

Question 4. *Find* $\int \frac{x-1}{x^2+7x+10}\,dx$.

Solution: The roots of $x^2+7x+10$ are $\frac{-7\pm\sqrt{49-40}}{2}$. Hence,

$$\frac{x-1}{x^2+7x+10} = \frac{x-1}{(x+5)(x+2)} = \frac{A}{x+2} + \frac{B}{x+5} \text{ for some } A \text{ and } B$$

Therefore,

$$x-1 = A(x+5)+B(x+2) \implies 1 = A+B \text{ and } -1 = 5A+2B \implies A = -1 \text{ and } B = 2$$

Hence,

$$\int \frac{x-1}{x^2+7x+10}\,dx = \int -\frac{1}{x+2} + \frac{2}{x+5}\,dx = -\ln|x+2| + 2\ln|x+5| + c$$

□

Question 9. *Find* $\int \frac{x^3+2}{x^2+4}\,dx$.

Solution: Note that:

$$\int \frac{x^3+2}{x^2+4}\,dx = \int \frac{x^3+4x}{x^2+4} + \frac{-4x+2}{x^2+4}\,dx = \int x\,dx + \int \frac{1}{2}\frac{1}{(\frac{x}{2})^2+1}\,dx - 4\int \frac{x}{x^2+4}\,dx$$

Hence, we have

$$\int \frac{x^3+2}{x^2+4}\,dx = \frac{x^2}{2} + \arctan(\frac{x}{2}) - 2\ln|x^2+4| + c$$

□

Question 10. *Find* $\int \frac{x^2+9}{x^2-9}\,dx$.

Solution: We have $\int \frac{x^2+9}{x^2-9}\,dx = \int 1 + \frac{18}{(x-3)(x+3)}\,dx$. We solve

$$\frac{18}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3} \implies 18 = A(x+3) + B(x-3) \implies A+B=0 \text{ and } 3A-3B=18$$

Hence, $A = 3$ and $B = -3$. We integrate

$$\int 1 + \frac{18}{(x-3)(x+3)}\,dx = x + \int \frac{3}{x-3}\,dx - \int \frac{3}{x+3}\,dx = x + 3\ln|x-3| - 3\ln|x+3| + c$$

□

The following question is invented by the DGD instructors. There is a different formula one uses for multiple real roots of the same number:

Question 11. *Find* $\int \frac{x}{(x-2)^2}\,dx$.

Solution: We have $\frac{x}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2} \implies x = (x-2)A + B \implies A = 1 \implies B = 2$. Hence,

$$\int \frac{x}{(x-2)^2}\,dx = \int \frac{1}{x-2}\,dx + 2\int \frac{1}{(x-2)^2}\,dx = \ln|x-2| - 2\frac{1}{x-2} + c$$

□

4 Week Four

4.1 Improper Integrals

The following questions are from 6.7

Question 8. *Find* $\int_5^\infty \frac{1}{x^2}\,dx$ *if it converges.*

Solution: This one is not too hard:

$$\int_5^\infty \frac{1}{x^2}\,dx = \lim_{n\to\infty}\int_5^n \frac{1}{x^2}\,dx = \lim_{n\to\infty}(-\frac{1}{x})|_5^n = \lim_{n\to\infty}(-\frac{1}{n}+\frac{1}{5}) = \frac{1}{5}$$

□

Question 18. $\int_0^\infty \frac{1}{1+x^2}\,\mathrm{dx}$.

Solution: We have that

$$\int_0^\infty \frac{1}{1+x^2} = \lim_{T\to\infty}\int_0^T \frac{1}{1+x^2} = \lim_{T\to\infty}\arctan(T)|_0^T = \lim_{T\to\infty}\arctan(T) = \frac{\pi}{2}.$$

□

Question 20. *Find* $\int_{-\infty}^\infty \frac{x}{1+x^2}\,dx$ *if it converges.*

Solution: By substitution, $u = 1+x^2$ and, thus, $du = 2xdx$:

$$\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln|1+x^2| + c.$$

Hence, we have

$$\int_{-\infty}^\infty \frac{x}{1+x^2}\,dx = \lim_{n\to\infty}\int_{-n}^a \frac{x}{1+x^2}\,dx + \lim_{n\to\infty}\int_a^n \frac{x}{1+x^2}\,dx. \tag{3}$$

Note that the first integral in (3)

$$\lim_{n\to\infty}\int_a^n \frac{x}{1+x^2}\,dx = \lim_{n\to\infty}(\frac{1}{2}\ln|1+x^2|)|_a^n) = \lim_{n\to\infty}\frac{1}{2}\ln|1+n^2| - \frac{1}{2}\ln|1+a^2| = \infty.$$

The criteria for convergence is that the original integral converges if and only if both integrals in the sum (3) converges. Since, one of the integrals does not converge the whole integral does not. A TRAP: it may be tempting to think that this integral equals zero by remembering one of the limit laws incorrectly. Please note that

$$\lim_{n\to\infty}(f(n)+g(n)) = \lim_{n\to\infty} f(n) + \lim_{n\to\infty} g(n) \text{ only when the limits converge to numbers!}$$

□

Question 22. $\int_1^\infty \frac{\ln(x)}{x^4}\,\mathrm{dx}$.

Solution: We know that $\int_1^\infty \frac{\ln(x)}{x^4}\,\mathrm{dx} = \lim_{T\to\infty}\int_1^T \frac{\ln(x)}{x^4}\,\mathrm{dx}$. Next, we use integration by parts. Set $u = \ln(x)$ and $v = -\frac{x^{-3}}{3}$, so that $u' = \frac{1}{x}$ and $v' = x^{-4}$. Hence,

$$\int_1^T \frac{\ln(x)}{x^4}\,\mathrm{dx} = -\ln(x)\frac{x^{-3}}{3}\Big|_1^T + \int_1^T \frac{x^{-4}}{3}\,\mathrm{dx} = -\ln(x)\frac{x^{-3}}{3}\Big|_1^T - \frac{x^{-3}}{9}\Big|_1^T = -\ln(T)\frac{T^{-3}}{3} - \frac{T^{-3}}{9} + \frac{1}{9}.$$

So, $\lim_{T\to\infty} -\ln(T)\frac{T^{-3}}{3} - \frac{T^{-3}}{9} + \frac{1}{9} = \lim_{T\to\infty} -\ln(T)\frac{T^{-3}}{3} + \frac{1}{9} = \frac{1}{9}$ since $\ln(T) \leq T$. □

Question 24. *Find* $\int_0^{\pi/2} \sec^2 x\, dx$ *if it converges.*

Solution: Note that $\tan' x = \frac{\sin x}{\cos x}' = \frac{\cos x \cdot \sin' x - \cos' x \cdot \sin x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$. Hence,

$$\int_0^{\pi/2} \sec^2 x\, dx = \lim_{n \to \frac{\pi}{2}} \int_0^n \sec^2 x\, dx = \lim_{n \to \frac{\pi}{2}} (\tan(n) - \tan(0)) \text{ which does not exist.}$$

□

Question 34. *The volume* $V(t)$ *of a cell is increasing at a rate of* $V'(t) = \frac{100}{1+t^2} \mu m^3/s$*. The initial volume is* $V(0) = 500 \mu m^3$*. Solve the differential equation. Find the long term growth in volume.*

Solution: First we solve:

$$V(t) = \int V'(t)\, dt = 100 \int \frac{1}{1+t^2}\, dt = 100 \arctan t + c.$$

Note that $500 = V(0) = c$ and so $V(t) = 100 \arctan x + 500$. The long term growth is the following:

$$\int_0^\infty V'(t)\, dt = \lim_{n \to \infty} 100 \int_0^n \frac{1}{1+t^2}\, dt = \lim_{n \to \infty} 100(\arctan(n) - \arctan(0)) = 100 \frac{\pi}{2} = 50\pi.$$

□

Question 36. *A population of bacteria* $B(t)$ *increases at a rate* $B'(t) = \frac{1000}{(2+3t)^{0.75}}$ *bacteria per hour. The population begins at* $B(0) = 10^6$ *bacteria. When will the population (if it can) reach* 2×10^6*. What is the long term growth?*

Solution: First we solve the differential equation.

$$B(t) = \int B'(t)\, dt = 1000 \int \frac{1}{(2+3t)^{0.75}}\, dt = 1000 \cdot \frac{4}{3}(2+3t)^{0.25} + c$$

Note that $10^6 = B(0) = \frac{4000}{3} 2^{0.25} + c$ and so $B(t) = \frac{4000}{3}((2+3t)^{0.25} - 2^{0.25}) + 10^6$. Hence,

$$B(t) = 2 \times 10^6 \implies (2+3t)^{0.25} - 2^{0.25} = \frac{3 \times 10^6}{4000} \implies t = \frac{1}{3}((\frac{3 \times 10^6}{4000} + 2^{0.25})^4 - 2)$$

Therefore, it takes $t = 106139272000$ hours for the population to double from its initial value. The long term growth is:

$$\int_0^\infty B'(t)\, dt = \lim_{n \to \infty} 1000 \int_0^n \frac{1}{(2+3t)^{0.75}}\, dt = \lim_{n \to \infty} (\frac{4000}{3}(2+3t)^{0.25})|_0^n = \infty$$

Hence, the long term growth is infinite which suggests both that the equation that models the growth has an expiry date. □

5 Week Five

5.1 Separation of Variables

The following are from Section 7.4 in the course text.

Question 6. *Solve* $\frac{dP}{dt} = 1 + P^2$ *where* $P(0) = 1$.

Solution: Note that $\frac{1}{1+P^2}\frac{dP}{dt} = 1$. Hence,

$$\arctan P = \int \frac{1}{1+P^2} dP = \int \frac{1}{1+P^2}\frac{dP}{dt} dt = \int dt = t + C.$$

Hence, $P(t) = \tan(t + C)$. To solve for C we use initial condition:

$$1 = P(0) = \tan(C) \implies C = \frac{\pi}{4}.$$

□

Question 14. *Solve* $\frac{dy}{dt} = -8ty^2$ *where* $y(0) = 3$.

Solution: We rearrange the expression to get

$$\frac{1}{y^2}\frac{dy}{dt} = -8t.$$

Hence, we have

$$-\frac{1}{y} = \int \frac{1}{y^2} dy = \int \frac{1}{y^2}\frac{dy}{dt} dt = \int -8t dt = -4t^2 + c.$$

Technically, there should be a plus c on both sides of the equality, but we absorb them together. Hence $y(t) = \frac{1}{4t^2 - c}$. Hence, $3 = y(0) = -\frac{1}{c}$. Therefore,

$$y(t) = \frac{1}{4t^2 + \frac{1}{3}}.$$

□

Question 26. *Solve* $\frac{db}{dt} = \frac{1}{1+t} b$ *where* $b(0) = 10^6$.

Solution: Rearrange: $\frac{db}{dt} = \frac{1}{1+t} b \iff \frac{1}{b}\frac{db}{dt} = \frac{1}{1+t}$. Employ method of separation of variables:

$$\ln|b| = \int \frac{1}{b} db = \int \frac{1}{b}\frac{db}{dt} dt = \int \frac{1}{1+t} dt = \ln|1+t| + c.$$

Plug in $b(0)$ to get c:

$$\ln|10^6| = 0 + c.$$

Isolate $b(t)$ best we can:

$$|b(t)| = 10^6 |1+t|.$$

□

5.2 Differential Equations

The following are from section 7.1 in the course text.

Question 4. *Consider the differential equation* $\frac{\mathrm{d}m}{\mathrm{d}t} = \frac{e^{\alpha m} m^2}{mt - \lambda}$. *Identify the state variable. Is this DE pure-time, autonomous or nonautonomous?*

Solution: The state variabe is m. This rate of change depends on both m and t, and is therefore nonautonomous. □

Question 6. *Consider the differential equation* $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x^2}{x-\lambda}$. *When* $x = 1$ *and* $\lambda = 2$, *indicate whether the state variable is increasing, decreasing and remains unchanged.*

Solution: We obtain $x'(t) = \frac{1}{1-2} = -1$. So, x is decreasing. □

Question 12. *Describe the following system with a differential equation. The rate of change of thickness of the ice on a lake is inversely proportional to the square root of the thickness. Initially, the ice is 3 mm thick.*

Solution: First, we call T the thickness of the ice, so the rate of change of thickness is $\frac{\mathrm{d}T}{\mathrm{d}t}$. We know that the rate of change is inversely proportional to the square root of the thickness T itself. This means that $\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\alpha}{\sqrt{T}}$ for some real number α. □

Question 18. *Solve the pure-time differential equation* $\frac{\mathrm{d}p}{\mathrm{d}t} = t$, *with initial condition* $p(1) = 1$.

Solution: To solve it, we simply integrate. We have that $p(t) = \int \frac{\mathrm{d}p}{\mathrm{d}t}\,\mathrm{d}t = \int t\,\mathrm{d}t = \frac{t^2}{2} + C$. Since $p(1) = 1$, $C = \frac{1}{2}$. Therefore, $p(t) = \frac{x^2}{2} + \frac{1}{2}$. □

Question 26. *Consider the DE* $\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{1}{z-1}$ *with initial condition* $z(0) = 2$. *Check that* $z(t) = 1 + \sqrt{1+2t}$ *is a solution.*

Solution: First, we see that $z(0) = 1 + \sqrt{1 + 2 \cdot 0} = 2$. Now, we plug $z(t)$ into the DE to see

$$\frac{\mathrm{d}}{\mathrm{d}t}(1 + \sqrt{1+2t}) = \frac{2}{2\sqrt{1+2t}} = \frac{1}{\sqrt{1+2t}},$$

$$\frac{1}{z-1} = \frac{1}{1 + \sqrt{1+2t} - 1} = \frac{1}{\sqrt{1+2t}}.$$

Therefore, $z(t)$ is indeed **a** solution (another is $z(t) = 1 - \sqrt{1+2t}$). □

Question 36. *Suppose that* $\frac{\mathrm{d}x}{\mathrm{d}t} = x + x^2$. *Set* $y = \frac{1}{x}$. *Find a differential equation for* y.

Solution: Note: The resulting equation is a linear differential equation while the original is not. We have that $xy = 1$.

$$x'y + xy' = 0 \iff (x + x^2)y + xy' = 0 \iff \left(1 + \frac{1}{y}\right) + \frac{y'}{y} = 0 \iff y' = -y - 1.$$

□

Some models use the formula $\frac{db}{dt} = \lambda(b)b$. That is, the per-capita production is dependent on the population size.

Question 39. *Using the above formula, set $\lambda(b) = 1 - 0.002b$. This is an example of a model for competition. Is $b(t)$ increasing at $b = 10$ and is it increasing at $b = 1000$?*

Solution: Note that $\frac{db}{dt} = (1 - 0.002b)b$. Hence, for $b = 10$, we have $\frac{db}{dt} = (1 - 0.02)(10) > 0$ so $b(t)$ is increasing. For $b = 1000$, we have $\frac{db}{dt} = (1 - 2)(1000) < 0$ so $b(t)$ is, in fact, decreasing. Please note that determining $b(t)$ was not required for this question. □

Question 49. *Two (nonzero) bacteria population a and b grow according to the the differential equations:*

$$\frac{da}{dt} = 2(1 - p)a \quad \textit{and} \quad \frac{db}{dt} = 1.5(1 - p)b.$$

where $p = \frac{a}{a+b}$. Determine $\frac{dp}{dt}$ and use it to decide the possibility of population of a taking over.

Solution: We have the following:

$$\frac{dp}{dt} = \frac{d}{dt}\frac{a}{a+b} = \frac{(a+b)a' - a(a+b)'}{(a+b)^2} = \frac{ba' - ab'}{(a+b)^2} = \frac{2b(1-p)a - 1.5a(1-p)b}{(a+b)^2} = \frac{ba(1-p)}{2(a+b)^2}.$$

Noting that $1 - p = \frac{b}{a+b}$, we have $\frac{dp}{dt} = \frac{p(1-p)^2}{2}$. Note that p is the fraction of a over the sum of a and b. Hence, we consider p getting close to the value 1 as a "taking over". For high values of p (like 0.99) we see that $\frac{dp}{dt} > 0$. Therefore, it is possible for a to take over! □

Question 50. *Newton's law of cooling is the differential equation $\frac{dT}{dt} = \alpha(A - T)$ where A denotes the ambient temperature and α is a positive constant that depends on the ability of an object to retain heat. Verify that a solution to this differential equation is $T(t) = A + (T(0) - A)e^{-\alpha t}$. What is the value of $\lim_{t\to\infty} T(t)$?*

Solution: We verify the solution:

$$\frac{dT}{dt} = \frac{d}{dt}(A + (T(0) - A)e^{-\alpha t}) = -\alpha(T(0) - A)e^{-\alpha t} = -\alpha(T - A) = \alpha(A - T).$$

We verify the initial condition:

$$T(0) = A + (T(0) - A) = T(0).$$

The limit is what we would expect:

$$\lim_{t\to\infty} T(t) = A + (T(0) - A)\lim_{t\to\infty} e^{-\alpha t} = A + 0 = A.$$

□

The following question is made up.

Question 51. *Detectives find a body at 2pm. The temperature of the body is approximately 33.06531 degrees Celsius. Six hours later at 8pm, the morgue receives the body and quickly checks the temperature before bringing it inside and finds it to be approximately 28.35335 degrees Celsius. The temperature was a constant 27 degrees Celsius during the day. When was the victim killed?*

Solution: Let T be the temperature of the victim. Note that $T(0) = 37$ (temperature of alive person) and $A = 27$. Let t_0 be the time in hours until the body was discovered at 2pm. We have:

$$33.06531 = T(t_0) = 27 + (37 - 27) * e^{-\alpha t_0}$$

Hence,

$$\alpha t_0 = -\ln(e^{-\alpha t_0}) = -\ln(\frac{33.06531 - 27}{10}) = 0.5.$$

Similarly,

$$28.35335 = T(t_0 + 6) = 27 + (37 - 27) * e^{-\alpha(t_0+6)}$$

Therefore,

$$\alpha(t_0 + 6) = -\ln(\frac{28.35335 - 27}{10}) = 2.$$

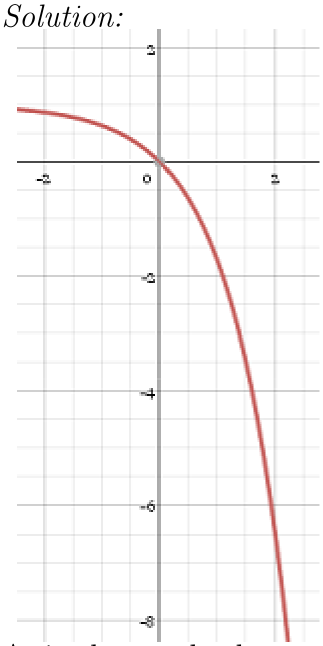
One sees that $\alpha t_0 = 0.5 \implies 2 = \alpha(t_0 + 6) = 0.5 + 6\alpha \implies \alpha = 0.25$. Thus, $t_0 = 2$. We conclude that the murder happened at noon. □

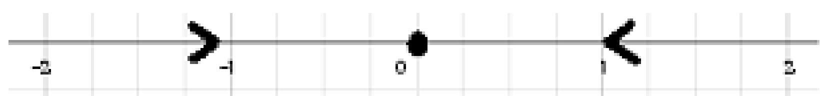
5.3 Steady States and Phase-lines

These questions are from 7.2 from the course text.

Question 16. *Determine the phase-line for* $\frac{dx}{dt} = 1 - e^x$ *from* -2 *to* 2. *Verify the (in)stability of the steady states by the stability theorem.*

Solution:



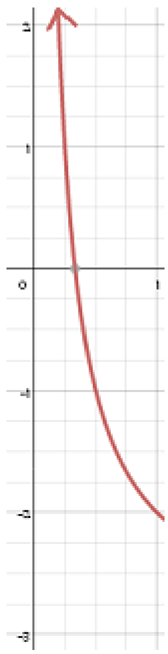


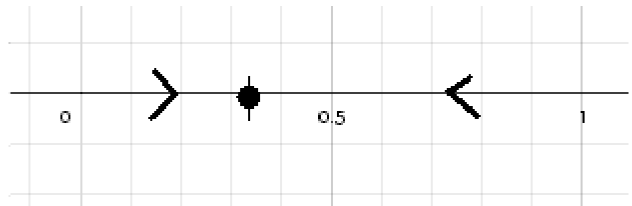
We first graph the equation as a function of the state variable. The red line in the graph is the function $x'(t) = 1 - e^x$. Making a phase-line diagram essentially simplifies the graph into the pertinent information concerning the equilibria. We put dots for steady states (at $x = 0$ in this case), and then arrows to denote whether there is a positive or negative rate of change on an interval.

As in the graph, the steady state of the DE is where $1 - e^x = 0$. It follows that the only steady state is at $x = 0$. Since our DE is autonomous, we can apply the stability theorem. Now, $\frac{\mathrm{d}}{\mathrm{d}x}(1 - e^x) = -e^x$. So, by stability theorem, $-e^0 = -1 < 0$ implies that the steady state is stable. □

Question 18. *Determine the phase-line for* $\frac{dz}{dt} = \frac{1}{z} - 3$ *on* $0 < z \leq 1$. *Use the stability theorem to verify the (in)stability of the steady states.*

Solution:



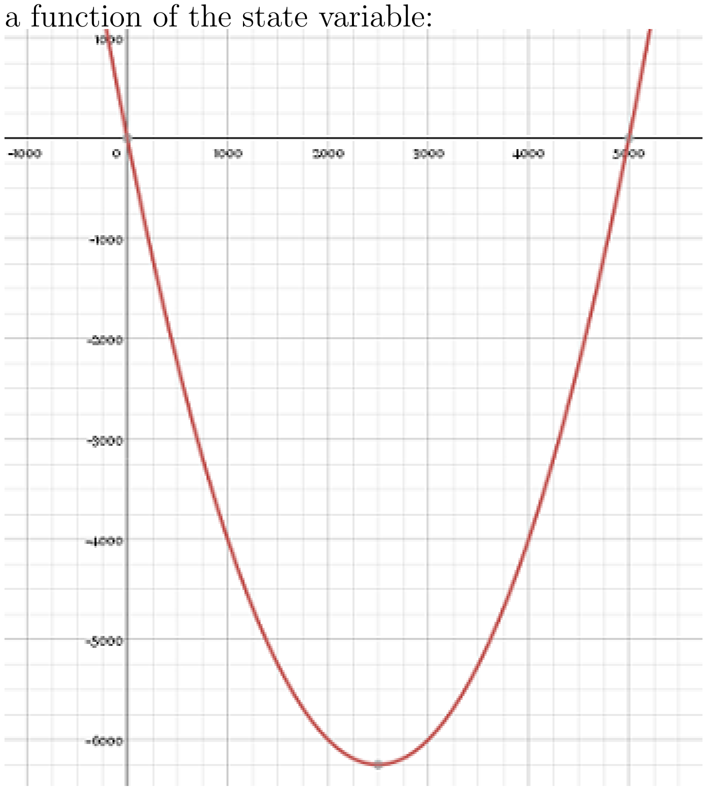


This time there is an equilibrium point (an z-intercept of $z'(t)$) at $z = \frac{1}{3}$. Since the rate of change of $z(t)$ is positive before $\frac{1}{3}$ we put an arrowhead to the right. Since the rate of change of $z(t)$ is negative after $\frac{1}{3}$ we put an arrowhead to the left.

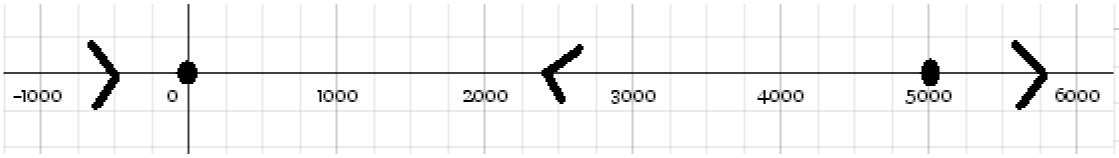
As seen in the graph, the steady state is where $\frac{1}{z} - 3 = 0$. It follows that the only steady state is at $z = \frac{1}{3}$. Since our DE is autonomous, we can apply the stability theorem. Now, $\frac{\mathrm{d}}{\mathrm{d}z}(\frac{1}{z} - 3) = -\frac{1}{z^2}$. So, by stability theorem, $-\frac{1}{(1/3)^2} = -9 < 0$ implies that the equilibrium is stable. □

Question 24. *Determine the equilibria and phase-line for* $\frac{db}{dt} = (-5 + 0.001b)b$. *Check that points* $b = 1000$ *and* $b = 3000$ *are consistent with the phase-line.*

Solution: Equilibria are where $\frac{db}{dt} = 0$ so $b = 0$ and $b = 5000$. Here is the graph of the equation as a function of the state variable:



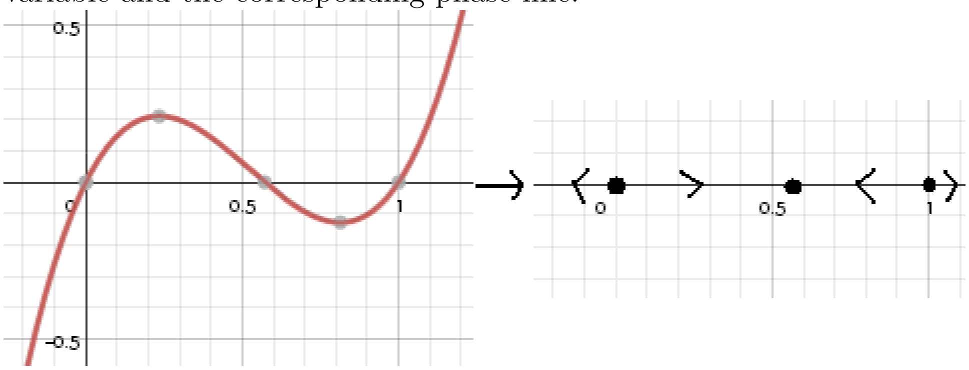
This translates into a phase-line as follows:



Note that $b(t) = 1000 \implies \frac{db}{dt} = -4000 < 0$ and $b(t) = 3000 \implies \frac{db}{dt} = -6000$ which is consistent with our phase-line. □

Question 28. *Determine the equilibria and phase-line for* $\frac{dp}{dt} = (2(1-p) - 1.5p)p(1-p)$*. What happens to a solution for the differential equation with a very small positive initial condition?*

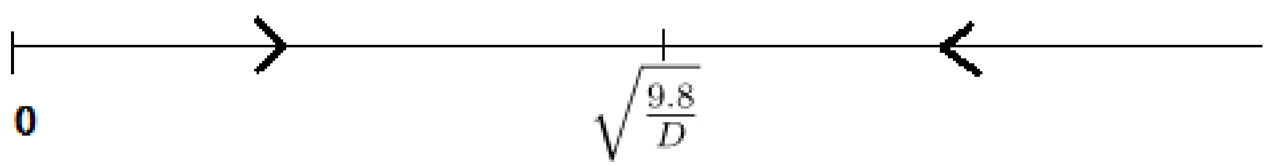
Solution: Equilibria are where $\frac{dp}{dt} = 0$. We note $(2(1-p) - 1.5p)p(1-p) = (2 - 3.5p)p(1-p)$. So the equilibria are $p = 0, \frac{4}{7}, 1$. Here is a graph of the differential equation as a function of the state variable and the corresponding phase-line:



A solution with a very small positive initial condition will tend toward $\frac{4}{7}$. □

Question 32. *The speed* $v(t)$ *m/s of a skydiver satisfies the differential equation* $\frac{dv}{dt} = 9.8 - Dv^2$ *where* $D = 0.0032$ *if the skydiver is simply falling and* $D = 0.00048$ *if the skydiver is pointing his/her toes. Compare equilibrium speeds.*

Solution: We are investigating terminal velocity. Our rocket example in Week Two did not consider drag. Drag can eventually kill acceleration. Equilibria of $v(t)$ are where $9.8 - Dv^2 = 0$ so $v = \pm\sqrt{\frac{9.8}{D}}$. We only care about positive values of $v(t)$ because we can assume the skydiver will be going down the whole trip. When $0 \leq v(t) < \sqrt{\frac{9.8}{D}}$, we have $\frac{dv}{dt} > 0$. When $v(t) > \sqrt{\frac{9.8}{D}}$, we have $\frac{dv}{dt} < 0$. Here is the phase line:



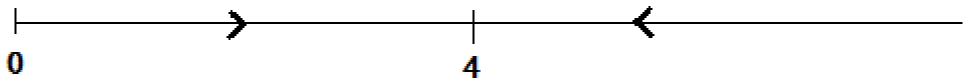
We can assume that the skydiver starts at an initial velocity of 0 and so his/her velocity will tend to $\sqrt{\frac{9.8}{D}}$. Hence, the terminal velocity of normal skydiving is approximately 55.3 m/s whereas the terminal velocity of pointed toe skydiving is approximately 142.8 m/s. □

Question 34. *According to Torricelli's law of draining, the rate at which a fluid flows out of a cylinder through a hole in the bottom is proportional to the square root of the depth of the water. Let* y *represent the depth of water. Suppose we have a cylinder whose proportional constant is* 2 *and receives water at a rate of* 4 *cm/s. Our differential equation is:*

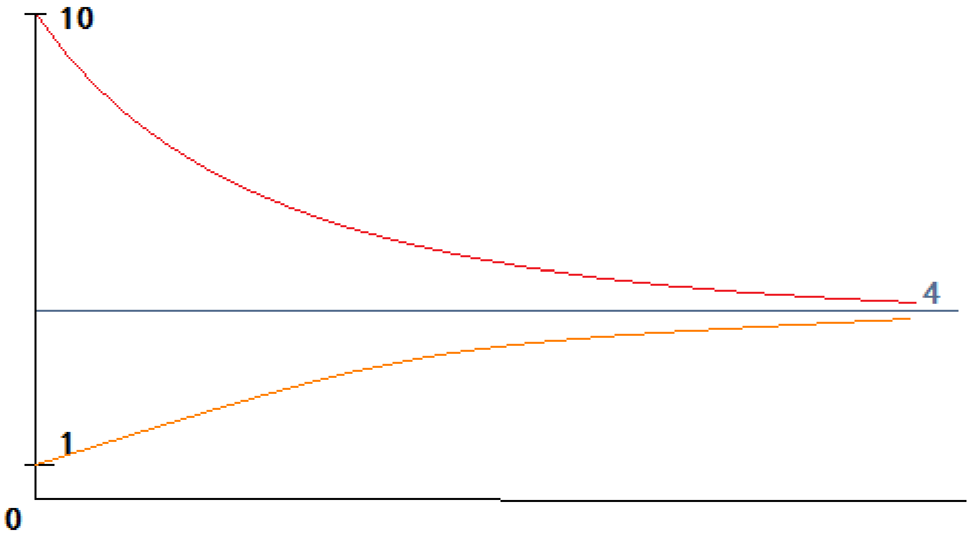
$$\frac{dy}{dt} = 4 - 2\sqrt{y}.$$

Use a phase line diagram to sketch solutions starting from $y = 10$ and $y = 1$.

Solution: We have $0 = \frac{dy}{dt} = 4 - 2\sqrt{y} \implies y^* = 4$. Note that $0 \le y < 4 \implies \frac{dy}{dt} > 0$ and $y > 4 \implies \frac{dy}{dt} < 0$. We summarize this information in a phase line diagram:



The following is a genuine sketch of the two solutions:



$\square$

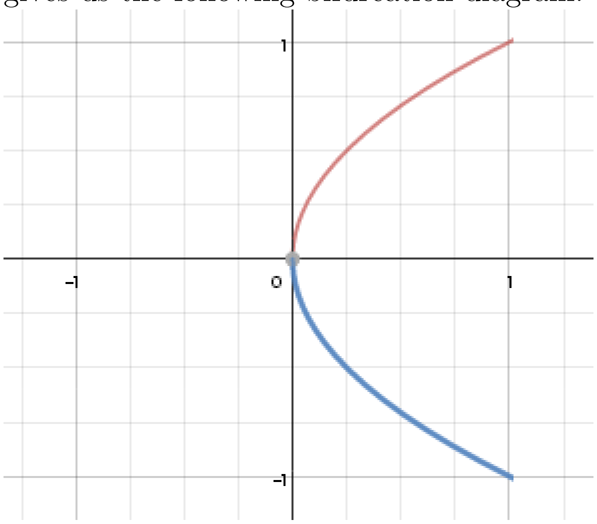
6 Week Six

6.1 More Stability: Bifurcation

The following are chapter questions from Section 7.3.

Question 18. *Consider the equation* $\frac{dx}{dt} = a - x^2$ *for all real values of* x*. Graph the steady states as over the parameter value* a *for* $-1 \leq a \leq 1$*. Use a red line where steady states are stable and a blue line where steady states are unstable. The result is a bifurcation diagram.*

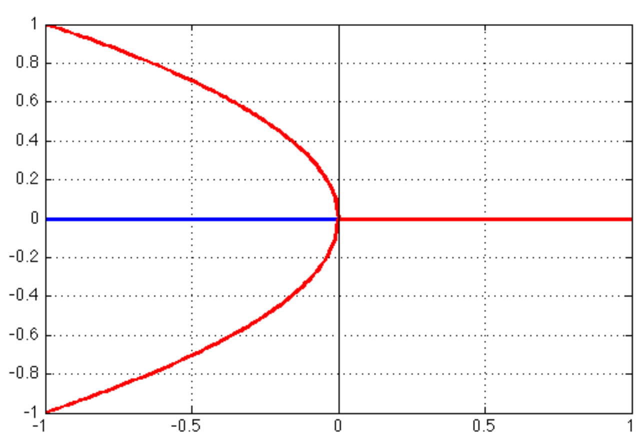
Solution: Let $f(x) = a - x^2$. Steady states are where $f(x) = 0$. Hence, when $a \geq 0$, the steady states are $\pm\sqrt{a}$. When $a < 0$, there are no steady states. Suppose $a > 0$. By the stability theorem, we have $f'(\sqrt{a}) = -2\sqrt{a} < 0$ so $\sqrt{a}$ is stable and $f'(-\sqrt{a}) = 2\sqrt{a} > 0$ so $-\sqrt{a}$ is unstable. This gives us the following bifurcation diagram:



□

Question 20. *Consider the equation* $\frac{dx}{dt} = ax + x^3$*. Draw the bifurcation diagram for* $-1 \leq a \leq 1$*.*

Solution: Let $f(x) = ax + x^3$. Note that $x = 0$ is always a steady state. There are at most two other steady states which correspond to the roots of $a + x^2$. If $a \geq 0$, then there are no more roots and, hence, no steady states. If $a \leq 0$, then $\pm\sqrt{-a}$ are steady states. We have $f'(x) = a + 3x^2$. If $a > 0$, then $f'(0) = a > 0$ so $x = 0$ is unstable. If $a < 0$, then $f'(0) < 0$ so $x = 0$ is stable. When $a < 0$, we have $f'(\sqrt{-a}) = a + 3(-a) = -2a > 0$ so $x = \sqrt{-a}$ is unstable. Similarly, when $a < 0$, we have $f'(-\sqrt{-a}) = a + 3(-a) = -2a > 0$ so $x = -\sqrt{-a}$ is also unstable. We have the following bifurcation diagram:



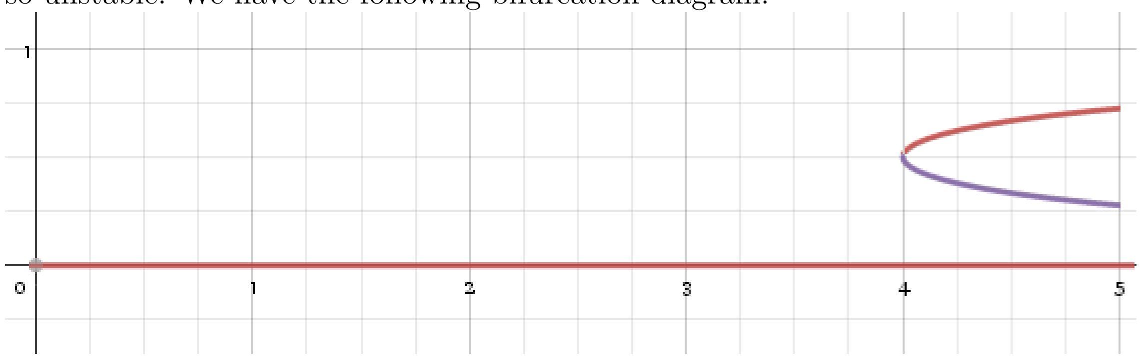
□

Question 45. *Consider a variant of the basic disease model given by* $\frac{dI}{dt} = \alpha I^2(1-I) - I$. *Draw the bifurcation diagram for* $0 \le \alpha \le 5$.

Solution: Let $f(x) = \alpha x^2(1-x) - x = x((-\alpha)x^2 + \alpha x - 1)$. Clearly, there is always a $x = 0$ steady state for all α. To find the other roots of $f(x)$, we use the quadratic formula.

$$x = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\alpha}}{-2\alpha} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{\alpha}}$$

Note that for $\alpha < 4$ there is no real solution. If $\alpha > 4$, then we have two. Note that $f'(x) = \alpha(2x(1-x) - x^2) - 1$. For any α, we have $f'(0) = -1 < 0$ so $x = 0$ is stable. For $\alpha > 4$, let us use the fact that stability only changes at bifurcation points (see graph) and pick $\alpha = 4.5$ to check stability. For $x = 0.52\overline{77}$, we have $f'(0.52\overline{77}) < 0$ so stable. For $x = 0.47\overline{22}$, we have $f'(0.47\overline{22}) > 0$ so unstable. We have the following bifurcation diagram:



□

6.2 Complex Numbers

Question 1. *Put in the form* $a + ib$.

(a) $\frac{3+2i}{1-4i}$

(b) i^{100}

(c) $\sqrt{-3}\sqrt{-12}$.

(d) $\frac{1+4i}{3+2i}$

Solution: (a) $\frac{3+2i}{1-4i} = \frac{1+4i}{1+4i} \cdot \frac{3+2i}{1-4i} = \frac{3-8+(2+12)i}{1+16} = -\frac{5}{17} + \frac{14}{17}i$.

(b) $i^{100} = (-1)^{50} = 1^{25} = 1 + 0i$.

(c) $\sqrt{-3}\sqrt{-12} = i^2\sqrt{(3)(12)} = -\sqrt{36} + 0i$.

(d) $\frac{1+4i}{3+2i} = \frac{3-2i}{3-2i} \cdot \frac{1+4i}{3+2i} = \frac{3+8+(12-2)i}{9+4} = \frac{11}{13} + \frac{10}{13}i$. □

Question 2. *Find the complex conjugate and modulus of* $-1 + 2\sqrt{2}i$.

Solution: The complex conjugate of $a+ib$ is $\overline{a+ib} = a - ib$. In our case, $\overline{-1+2\sqrt{2}i} = -1 - 2\sqrt{2}i$.

The modulus of $a+ib$ is $|a+ib| = \sqrt{a^2+b^2}$. In our case, $|-1+2\sqrt{2}i| = \sqrt{-1+2\sqrt{2}i \cdot \overline{-1+2\sqrt{2}i}} = \sqrt{1+8} = 3$. □

Question 3. *Find all solutions to the equation* $2x^2 - 2x + 1 = 0$.

Solution: By quadratic formula, we have $x = \frac{2 \pm \sqrt{4-8}}{4}$. Hence, $x = \frac{1}{2} \pm \frac{\sqrt{-1}\sqrt{4}}{4} = \frac{1}{2} \pm \frac{i}{2}$. □

Question 4. *Find all solutions to the equation* $x^5 = 5$.

Solution: Note that $x^5 = 1$ has solutions according to the fifth roots of unity. That is, $e^{\frac{2\pi i k}{5}}$ is a root for $k = 0, 1, 2, 3, 4$. By the distributivity property of powers, we have $\sqrt[5]{5} \cdot e^{\frac{2\pi i k}{5}}$ is a root for the polynomial in question for $k = 0, 1, 2, 3, 4$. It is well known that the number of roots cannot exceed the degree of the polynomial, so we are done. □

Question 5. *Put* $3 + 4i$ *in polar form.*

Solution: (Insert point on complex plane). The modulus: $R = |3 + 4i| = \sqrt{9 + 16} = 5$. The argument $\theta = \arctan(\frac{4}{3})$. The polar form is $R(\cos\theta + i\sin\theta) = 5(\cos(\arctan(\frac{4}{3})) + i\sin(\arctan(\frac{4}{3})))$.
□

Question 6. *Find the polar form of* z/w, zw *and* $1/z$ *for* $z = \sqrt{3} + i$ *and* $w = 1 + \sqrt{3}i$.

Solution: Let us first find the polar form of z. $R = |\sqrt{3} + i| = \sqrt{3+1} = 2$. $\theta = \arctan(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$. Hence, $z = 2(\cos(\frac{\pi}{6}) + i\sin(\frac{\pi}{6}))$. Now we find the polar form for w. $R = |1 + \sqrt{3}i| = 2$. $\theta = \arctan(\sqrt{3}) = \frac{\pi}{3}$. Hence, $w = 2(\cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3}))$. Hence,

$$z/w = \frac{2}{2} \cdot \frac{\cos(\frac{\pi}{6}) + i\sin(\frac{\pi}{6})}{\cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3})} = e^{i\frac{\pi}{6}}e^{-i\frac{\pi}{3}} = e^{-i\frac{\pi}{6}} = e^{i\frac{11\pi}{6}} = \cos(\frac{11\pi}{6}) + i\sin(\frac{11\pi}{6})$$

$$zw = 4 \cdot (\cos(\frac{\pi}{6}) + i\sin(\frac{\pi}{6})) \cdot (\cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3})) = 4e^{i\frac{\pi}{6}}e^{i\frac{\pi}{3}} = 4e^{i\frac{\pi}{2}} = 4(\cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2}))$$

$$1/z = \frac{1}{2}e^{-i\frac{\pi}{6}} = \frac{1}{2}e^{i\frac{11\pi}{6}} = \frac{1}{2}(\cos(\frac{11\pi}{6}) + i\sin(\frac{11\pi}{6}))$$

□

Question 7. *Write* $e^{-\pi i}$ *in the form* $a + ib$.

Solution: $e^{-\pi i} = \cos(-\pi) + i\sin(-\pi) = -1 + 0i$. □

Question 8. *Find* $(1 + i)^{20}$.

Solution: Do in polar form. $R = |1 + i| = \sqrt{1+1} = \sqrt{2}$. $\theta = \arctan(1) = \frac{\pi}{4}$.

$$(1 + i)^{20} = (\sqrt{2}e^{i\frac{\pi}{4}})^{20} = 2^{10}e^{i5\pi} = 2^{10}e^{i\pi} = -2^{10}.$$

□

7 Week Seven

7.1 Linear Systems and the Gauss Algorithm

Question 1.

Find the reduced row echelon form of the coefficient matrix $A = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

Solution: We use the row reduction algorithm. For step 1, we note that the first column is a leading column. For step 2, we select the first entry in the first column as our leading term. Then we use the replacement row operation and add the negative of row 1 to row 2 and replace row 2 with the result:

$$(-R_1 + R_2 \to R_2) \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

All values below the leading term are zero now. So, we now ignore the first row (so I color it blue) and repeat the process. We select column two as the leading column and the third entry as the leading term. We interchange row 2 and row 3 to bring this to the top (ignoring the first row):

$$(R_2 \leftrightarrow R_3) \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Since all entries below the leading term are zero, we are done with this leading term and we now ignore the first row and the second row. The third column is now the first nonzero column, our leading column. The first nonzero entry that we care about is in the third row. This is our leading term. We use the replacement row operation and add the negative of row 3 to row 4:

$$(-R_3 + R_4 \to R_4) \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We now ignore the third row, and notice that we are done. We will have a matrix is row-echelon form:

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We now back-substitute using our coefficient matrix to get the matrix in reduced row-echelon form. Here are the row operations:

$$(R_3 + R_2 \to R_2 \;\&\; -R_3 + R_1 \to R_1) \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(-R_2 + R_1 \to R_1) \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This resulting matrix is in reduced row-echelon form and we are done. □

Question 2. *Find the general solution for the following system of equations:*

$$x + y - z = 0, \quad 2x + 4y - z = 0, \quad \textit{and} \quad 6y + 3z = 0.$$

Solution: We use the row reduction algorithm on the corresponding augmented matrix. Fill in the row operations.

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & 4 & -1 & 0 \\ 0 & 6 & 3 & 0 \end{array}\right) \mapsto \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 6 & 3 & 0 \end{array}\right) \mapsto \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 6 & 3 & 0 \end{array}\right)$$

$$\mapsto \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \mapsto \left(\begin{array}{ccc|c} 1 & 0 & -\frac{3}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

To get our general solution we set $z = s$ since z is a free variable (no leading term). Then $y = -\frac{1}{2}s$ and $x = \frac{3}{2}s$. This gives us the general solution:

$$x = \frac{3}{2}s, \quad y = -\frac{1}{2}s, \quad \text{and} \quad z = s \quad s \in \mathbb{R}.$$

□

Question 3. *What condition must we impose on* (a, b, c) *such that the following system of equations is consistent?*

$$x - 2y + 2z = a, \quad -2x + y + z = b, \quad \textit{and} \quad x - 5y + 7z = a + c.$$

Solution: We use the row reduction algorithm. Fill in the steps.

$$\left(\begin{array}{ccc|c} 1 & -2 & 2 & a \\ -2 & 1 & 1 & b \\ 1 & -5 & 7 & a+c \end{array}\right) \mapsto \left(\begin{array}{ccc|c} 1 & -2 & 2 & a \\ 0 & -3 & 5 & b+2a \\ 0 & -3 & 5 & c \end{array}\right) \mapsto \left(\begin{array}{ccc|c} 1 & -2 & 2 & a \\ 0 & -3 & 5 & b+2a \\ 0 & -3 & 5 & c \end{array}\right) \mapsto \left(\begin{array}{ccc|c} 1 & -2 & 2 & a \\ 0 & -3 & 5 & b+2a \\ 0 & 0 & 0 & c-b-2a \end{array}\right)$$

The system is consistent if and only if the rank (number of leading terms in RE) of the augmented matrix must equal the rank of coefficient matrix. Therefore, we must have $2a + b - c = 0$. □

Here is an application of system of linear equations in optimization. The following is a question from a Midterm Exam in MAT1341.

Question 4. *A parrot must take 14 units of vitamin A, 15 units of vitamin D, and 29 units of vitamin E. Brand I, II, and III come in capsules that contain varying amounts of each vitamin according to the following chart:*

	I	*II*	*III*
A	2	1	1
D	3	3	0
E	5	4	1

Find a general solution to the corresponding system of equations and use this to deduce how many of each brand the parrot needs. Suppose brand I costs 7 cents, brand II costs 3 cents, and brand III costs 2 cents. Find the optimal solution.

Solution: Let $x =$ the number of capsules from brand I, let $y =$ the number of capsules from brand II, and let $z =$ the number of capsules from brand III. Note that $x, y, z \geq 0$ are integers as capsules should not be be split. We have the following system of equations:

$$2x + y + z = 14, \quad 3x + 3y = 15, \quad \text{and} \quad 5x + 4y + z = 29.$$

We use the Gaussian algorithm.

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 14 \\ 3 & 3 & 0 & 15 \\ 5 & 4 & 1 & 29 \end{array}\right) \mapsto \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 7 \\ 3 & 3 & 0 & 15 \\ 5 & 4 & 1 & 29 \end{array}\right) \mapsto \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 7 \\ 0 & \frac{3}{2} & -\frac{3}{2} & -6 \\ 0 & \frac{3}{2} & -\frac{3}{2} & -6 \end{array}\right) \mapsto \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 7 \\ 0 & 1 & -1 & -4 \\ 0 & \frac{3}{2} & -\frac{3}{2} & -6 \end{array}\right)$$

$$\mapsto \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 7 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right) \mapsto \left(\begin{array}{ccc|c} 1 & 0 & 1 & 9 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Suppose $z = s$. Then, $x = 9 - s$ and $y = s - 4$. This is our general solutions:

$$x = 9 - s, \quad y = s - 4, \quad \text{and} \quad z = s \qquad s \in \mathbb{R}$$

Note that $x, y, z \geq 0$. Hence $4 \leq s \leq 9$. To find the optimal solution we note that the total cost in cents is given by the following equation and then we plug in our general solution:

$$7x + 3y + 2z = 7(9 - s) + 3(s - 4) + 2s = 51 - 2s.$$

The smallest we can make this number is when $s = 9$. Our optimal solution is no brand I, 5 brand II, and 9 brand III. $\square$

Question 5. *Find numbers A, B, and C such that* $\frac{x^2+x+2}{(x-2)^2(x-3)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-3}$ *if they exist.*

Solution: Times both sides of the equality by $(x-2)^2(x-3)$ and we get:

$$x^2 + x + 2 = A(x-2)(x-3) + B(x-3) + C(x-2)^2$$

Collect like terms:

$$x^2 + x + 2 = (A + C)x^2 + (-5A + B - 4C)x + (6A - 3B + 4C)$$

This equality holds if and only if:

$$A + C = 1, \quad -5A + B - 4C = 1, \quad \text{and} \quad 6A - 3B + 4C = 2.$$

This is a system of linear equations. Let's solve it.

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ -5 & 1 & -4 & 1 \\ 6 & -3 & 4 & 2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 6 \\ 0 & -3 & -2 & -4 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 14 \end{pmatrix}$$

$$\mapsto \begin{pmatrix} 1 & 0 & 0 & -13 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 14 \end{pmatrix}$$

Hence, we have $A = -13$, $B = -8$ and $C = 14$. □

The following questions are from the additional course notes:

Question 6. *For what values of* (a, b) *does the system*

$$\begin{cases} x_1 + ax_2 = 1 \\ 2x_1 + 3x_2 = b \end{cases}$$

have

(a) *a unique solution*

(b) *infinitely many solutions*

(c) *no solution.*

Solution: We row-reduce the augmented matrix:

$$\left(\begin{array}{cc|c} 1 & a & 1 \\ 2 & 3 & b \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & a & 1 \\ 0 & 3-2a & b-2 \end{array}\right)$$

(a) If $a \neq \frac{3}{2}$, then both the augmented matrix and coefficient matrix have rank 2 which is the number of variables and so there is a unique solution.
(b) If $a = \frac{3}{2}$ and $b = 2$, then the system is consistent as the rank of the coefficient matrix and augmented matrix coincide. In fact, since the rank of the coefficient matrix is less than the number of variables, the system has infinitely many solutions.
(c) If $a = \frac{3}{2}$ and $b \neq 2$, then the coefficient matrix has rank 1 whereas the augmented matrix has rank 2. This means that the system would have no solutions.
We have exhausted all choices of a and b. □

Question 7. *Insects of two species are reared on two types of food. Species 1 consumes 5 units of food A and 3 units of food B per day. Species 2 consumes 2 units of A and 4 units of B, respectively. Every day, 900 units of food A and 960 units of food B are provided. How many individuals of each species are reared?*

Solution: Let x be the number of species 1 and y be the number of species 2. We create two linear equations to model this problem:

$$\begin{cases} 5x + 2y = 900 \\ 3x + 4y = 960 \end{cases}$$

We turn this into an augmented matrix and solve:

$$\left(\begin{array}{cc|c} 5 & 2 & 900 \\ 3 & 4 & 960 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & \frac{2}{5} & 180 \\ 1 & \frac{4}{3} & 320 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & \frac{2}{5} & 180 \\ 0 & 1 & 150 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 120 \\ 0 & 1 & 150 \end{array}\right)$$

Hence, there are 120 of species 1 and 150 of species 2. □

Question 8. *Explain why the system*

$$\begin{cases} x_1 + x_2 + 2x_3 = a \\ x_1 \quad\ + x_3 = b \\ 2x_1 + x_2 + 3x_3 = c \end{cases}$$

is consistent if and only if $c = a + b$.

Solution:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 1 & 0 & 1 & b \\ 2 & 1 & 3 & c \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & -1 & -1 & b-a \\ 0 & -1 & -1 & c-2a \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 1 & 1 & a-b \\ 0 & 0 & 0 & c-a-b \end{array}\right)$$

We are in REF. Hence, the system is consistent if the augmented matrix and the coefficient matrix have the same rank (number of leading terms). This occurs if and only if $c - a - b = 0$. □

Sometimes general solution has two parameters...

Question 9. *Solve the system*

$$\begin{cases} 7x_1 + 2x_2 + x_3 - 3x_4 = 5 \\ x_1 + 2x_2 + 4x_3 = 1 \end{cases}$$

Solution: We use row reduction:

$$\left(\begin{array}{cccc|c} 7 & 2 & 1 & -3 & 5 \\ 1 & 2 & 4 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 4 & 0 & 1 \\ 7 & 2 & 1 & -3 & 5 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 4 & 0 & 1 \\ 0 & -12 & -27 & -3 & -2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 4 & 0 & 1 \\ 0 & 1 & \frac{9}{4} & \frac{1}{4} & \frac{1}{6} \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{3} \\ 0 & 1 & \frac{9}{4} & \frac{1}{4} & \frac{1}{6} \end{array}\right)$$

The general solution is thus:

$$x_1 = \frac{s}{2} + \frac{t}{2} - \frac{1}{3} \quad x_2 = -\frac{9s}{4} - \frac{t}{4} + \frac{1}{6} \quad x_3 = s \quad x_4 = t \quad s, t \in \mathbb{R}$$

□

8 Week Eight

8.1 Matrix Operations

The notes for this DGD were not typed. Refer to additional course notes.

8.2 Determinants and Inverses

The notes for this DGD were not typed. Refer to additional course notes.

9 Week Nine

9.1 Eigenvalues and Eigenvectors

Question 10. *Find the Eigenvalues/Eigenvectors for* $A = \begin{pmatrix} 1 & 0.5 \\ 2 & 1 \end{pmatrix}$.

Solution: Let $v \neq 0$ (this is important! 0 is never an Eigenvector!). Recall that $Av = \lambda v \implies (A - \lambda I)v = 0 \implies (A - \lambda I)^{-1}$ does not exist $\implies \det(A - \lambda I) = 0$. Hence,

$$0 = \det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 0.5 \\ 2 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 1 = \lambda(2-\lambda).$$

Hence, the Eigenvalues of A are 2 and 0. Note that an Eigen*value* can be zero. Now we determine what the Eigenvector v could be for each Eigenvalue. <u>For $\lambda = 2$</u>:

$$Av = 2v \implies (A - 2I)v = 0 \implies v \text{ is a solution of } \left(\begin{array}{cc|c} 1-2 & 0.5 & 0 \\ 2 & 1-2 & 0 \end{array}\right).$$

Simply row reduce and find the general solution:

$$\left(\begin{array}{cc|c} -1 & 0.5 & 0 \\ 2 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -0.5 & 0 \\ 0 & 0 & 0 \end{array}\right) \implies v \in \{s\begin{pmatrix} 0.5 \\ 1 \end{pmatrix} \mid s \in \mathbb{R} \setminus \{0\}\}.$$

Anything in this set is an Eigenvector. For example, we can choose the vector $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ to be our Eigenvector. It is a good idea to check this:

$$Av = \begin{pmatrix} 1 & 0.5 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \lambda v$$

<u>For $\lambda = 0$</u>:

$$Av = 0v \implies Av = 0 \implies v \text{ is a solution of } \left(\begin{array}{cc|c} 1 & 0.5 & 0 \\ 2 & 1 & 0 \end{array}\right).$$

Simply row reduce and find the general solution:

$$\left(\begin{array}{cc|c} 1 & 0.5 & 0 \\ 2 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0.5 & 0 \\ 0 & 0 & 0 \end{array}\right) \implies v \in \{s\begin{pmatrix} -0.5 \\ 1 \end{pmatrix} \mid s \in \mathbb{R} \setminus \{0\}\}.$$

Anything in this set is an Eigenvector. For example, we can choose the vector $v = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ to be our Eigenvector. It is a good idea to check this:

$$Av = \begin{pmatrix} 1 & 0.5 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \lambda v$$

□

Question 11. *Find the complex Eigenvalues for* $A = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$.

Solution: The Eigenvalues are the complex roots for $\det(A - \lambda I)$ over the variable λ. We have

$$0 = \det(A - \lambda I) = (3-\lambda)^2 + 1 = \lambda^2 - 6\lambda + 10.$$

Using the quadratic formula we see that

$$\lambda = \frac{6 \pm \sqrt{36 - 40}}{2} = 3 \pm i$$

□

Question 12. *Find the Eigenvalues/Eigenvectors for* $A = \begin{pmatrix} 3 & 2 & -6 \\ 0 & -2 & 1 \\ 0 & 4 & 1 \end{pmatrix}$.

Solution: Let $v \neq 0$. Recall that $Av = \lambda v \implies (A - \lambda I)v = 0 \implies (A - \lambda I)^{-1}$ DNE $\implies \det(A - \lambda I) = 0$. Hence,

$$0 = \det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & 2 & -6 \\ 0 & -2-\lambda & 1 \\ 0 & 4 & 1-\lambda \end{pmatrix} = (3-\lambda)((-2-\lambda)(1-\lambda)-4) = (3-\lambda)(3+\lambda)(\lambda-2).$$

Therefore, the three Eigenvalues of A are $\lambda = 2, 3$, and -3. Now we determine what the Eigenvector v could be for each Eigenvalue. <u>For $\lambda = 3$</u>:

$$Av = 3v \implies (A - 3I)v = 0 \implies v \text{ is a solution to the SLE } \left(\begin{array}{ccc|c} 3-3 & 2 & -6 & 0 \\ 0 & -2-3 & 1 & 0 \\ 0 & 4 & 1-3 & 0 \end{array}\right).$$

Simply row reduce and find the general solution:

$$\left(\begin{array}{ccc|c} 0 & 2 & -6 & 0 \\ 0 & -5 & 1 & 0 \\ 0 & 4 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \implies v \in \{s \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mid s \in \mathbb{R} \setminus \{0\}\}.$$

Anything in this set is an Eigenvector. For example, we can choose the vector $v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ to be our Eigenvector. It is a good idea to check:

$$Av = \begin{pmatrix} 3 & 2 & -6 \\ 0 & -2 & 1 \\ 0 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \lambda v$$

<u>For $\lambda = -3$</u>:

$$Av = -3v \implies (A + 3I)v = 0 \implies v \text{ is a solution to the SLE } \left(\begin{array}{ccc|c} 3+3 & 2 & -6 & 0 \\ 0 & -2+3 & 1 & 0 \\ 0 & 4 & 1+3 & 0 \end{array}\right).$$

Simply row reduce and find the general solution:

$$\left(\begin{array}{ccc|c} 6 & 2 & -6 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 4 & 4 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \implies v \in \{s \begin{pmatrix} \frac{4}{3} \\ -1 \\ 1 \end{pmatrix} \mid s \in \mathbb{R} \setminus \{0\}\}.$$

Anything in this set is an Eigenvector. For example, we can choose the vector $v = \begin{pmatrix} \frac{4}{3} \\ -1 \\ 1 \end{pmatrix}$ to be our Eigenvector. It is a good idea to check:

$$Av = \begin{pmatrix} 3 & 2 & -6 \\ 0 & -2 & 1 \\ 0 & 4 & 1 \end{pmatrix} \begin{pmatrix} \frac{4}{3} \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ -3 \end{pmatrix} = -3 \begin{pmatrix} \frac{4}{3} \\ -1 \\ 1 \end{pmatrix} = \lambda v.$$

<u>For $\lambda = 2$</u>:

$$Av = 2v \implies (A - 2I)v = 0 \implies v \text{ is a solution to the SLE } \left(\begin{array}{ccc|c} 3-2 & 2 & -6 & 0 \\ 0 & -2-2 & 1 & 0 \\ 0 & 4 & 1-2 & 0 \end{array}\right).$$

Simply row reduce and find the general solution:

$$\left(\begin{array}{ccc|c} 1 & 2 & -6 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 4 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -\frac{11}{2} & 0 \\ 0 & 1 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \implies v \in \{s \begin{pmatrix} \frac{11}{2} \\ \frac{1}{4} \\ 1 \end{pmatrix} \mid s \in \mathbb{R} \setminus \{0\}\}.$$

Anything in this set is an Eigenvector. For example, we can choose the vector $v = \begin{pmatrix} \frac{11}{2} \\ \frac{1}{4} \\ 1 \end{pmatrix}$ to be our Eigenvector. It is a good idea to check:

$$Av = \begin{pmatrix} 3 & 2 & -6 \\ 0 & -2 & 1 \\ 0 & 4 & 1 \end{pmatrix} \begin{pmatrix} \frac{11}{2} \\ \frac{1}{4} \\ 1 \end{pmatrix} = \begin{pmatrix} 11 \\ \frac{1}{2} \\ 2 \end{pmatrix} = 2 \begin{pmatrix} \frac{11}{2} \\ \frac{1}{4} \\ 1 \end{pmatrix} = \lambda v.$$

□

9.2 Domain and Range

Question 13. *Determine the domain and range for the function $f(x) = \cos^2(\sqrt{x})$.*

Solution: For us, the domain of a function is all real values for which the function gives us a well-defined real output. The function $\cos(x)$ is defined for all real numbers, so we do not worry about it. The function $\sqrt{x}$ is only defined for real numbers ≥ 0. The domain of a composition of functions $g(f(x))$ is all real values x such that $f(x)$ is a well-defined real number and $g(f(x))$ is a well-defined real number. Hence, $D_f = \{x \in \mathbb{R} \mid x \geq 0\}$.

The range is all the real values that are given by the function acting on a member of the domain. We know that $R_{\cos(x)} = \{\cos(x) \mid x \in \mathbb{R}\} = [-1, 1]$ the interval from -1 to 1 including the endpoints. Note that $R_{\sqrt{x}} = \{\sqrt{x} \mid x \in \mathbb{R} \text{ and } x \geq 0\} = [0, \infty)$. By the periodicity of cos, we have $R_{\cos(\sqrt{x})} = \{\cos(\sqrt{x}) \mid x \in \mathbb{R} \text{ and } x \geq 0\} = \{\cos(x) \mid x \in [0, \infty)\} = [-1, 1]$ as well. Hence, $R_f = \{f(x) \mid x \in \mathbb{R} \text{ and } x \geq 0\} = [0, 1]$ the interval from 0 to 1 including the endpoints. □

Question 14. *Determine the domain and range for the function $g(x) = \frac{x-2}{x-2}$.*

Solution: At first look, one might mistake g for the constant function that returns the number 1 for each input; however, the domain is different than that function. We have $D_g = \{x \in \mathbb{R} \mid x \neq 2\}$ and, of course, $R_g = \{1\}$. So, g is a function that has a constant value 1 except for a missing point at $x = 2$. □

Question 15. *Determine the domain and range for the function* $h(x, y) = \sqrt{x^2 + y^2} + 1$*. Draw the level curves (if non-empty) for* $c = 0, 1$ *and* 2*.*

Solution: Clearly, we have $D_h = \mathbb{R}^2$ because the sum of squares is always positive. We have $R_f = [1, \infty)$. The level curve of h for $c = 0$ is $L_0(f) = \{(x, y) \in \mathbb{R}^2 \mid \sqrt{x^2 + y^2} + 1 = 0\} = \varnothing$ the empty set. The level curve of h for $c = 1$ is $L_1(f) = \{(x, y) \in \mathbb{R}^2 \mid \sqrt{x^2 + y^2} + 1 = 0\} = \{0\}$ the set containing just the origin. We have $L_2(f)$ is the unit circle. □

Question 16. *Determine the domain for the function* $k(x, y) = \ln(x - \sqrt{2 - y})$ *and sketch it on a graph.*

Solution: For $k(x, y)$ to be a well-defined real number, we need $x - \sqrt{2 - y} > 0$ and $2 - y \geq 0$. Hence, $x^2 > 2 - y$ and $2 \geq y$. Therefore, $D_k = \{(x, y) \in \mathbb{R}^2 \mid 2 - x^2 < y \leq 2\}$. Sketch graph. □

10 Week Ten

10.1 Limits in two variables

Question 17. *Compute*

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6 + y^2}$$

Solution: We employ the two "line" (a better word would be curve) test. Let $y = x$ then as $x \to 0$ we have $y \to 0$. Hence,

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6 + y^2} = \lim_{x\to 0} \frac{x^4}{x^6 + x^2} = \lim_{x\to 0} \frac{x^2}{x^4 + 1} = 0$$

The last equality is because the inside of the limit is a function that we know to be continuous away from -1. On the other hand, let $y = x^3$, as $x \to 0$ we have $y \to 0$. Hence,

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6 + y^2} = \lim_{x\to 0} \frac{x^6}{x^6 + x^6} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

By lack of uniqueness, the limit does not exist. □

Question 18. *Compute*

$$\lim_{(x,y)\to(0,0)} \frac{xy}{|xy|}.$$

Solution: We again employ the two line test. Let $y = x$ then as $x \to 0$ we have $y \to 0$. Hence,

$$\lim_{(x,y)\to(0,0)} \frac{xy}{|xy|} = \lim_{x\to 0} \frac{x^2}{|x^2|} = \lim_{x\to 0} 1 = 1.$$

On the other hand, let $y = -x$, as $x \to 0$ we have $y \to 0$. Hence,

$$\lim_{(x,y)\to(0,0)} \frac{xy}{|xy|} = \lim_{x\to 0} \frac{-x^2}{|-x^2|} = \lim_{x\to 0} -1 = -1.$$

By lack of uniqueness, the limit does not exist. □

Question 19. *Compute*

$$\lim_{(x,y)\to(0,1)} \ln(\frac{2x+1}{y}).$$

Solution: The function $f(x,y) = \frac{2x+1}{y}$ is continuous wherever $y \neq 0$. Hence,

$$\lim_{(x,y)\to(0,1)} (\frac{2x+1}{y}) = \frac{0+1}{1} = 1.$$

Furthermore, since $g(x) = \ln(x)$ is continuous at numbers greater than 0, we have the following:

$$\lim_{(x,y)\to(0,1)} \ln(\frac{2x+1}{y}) = \ln(1) = 0.$$

□

10.2 Partial Derivatives

Question 20. *Let* $f(x,y,z) = 2xyz + 2x^2 + 3y + 4y^2$. *Compute:*

$$\frac{\partial}{\partial x} f(x,y,z) \quad \frac{\partial}{\partial y} f(x,y,z) \quad \text{and} \quad \frac{\partial}{\partial z} f(x,y,z).$$

Solution: Differentiate with respect to the variable, treat the other variables as constants. We have $\frac{\partial}{\partial x} f(x,y,z) = 2yz + 4x$. We have $\frac{\partial}{\partial y} f(x,y,z) = 2xz + 8y + 3$. We have $\frac{\partial}{\partial z} f(x,y,z) = 2xy$. □

Question 21. *Let* $f(x,y) = \frac{cosx+siny}{3y}$. *Find the first order partial derivatives.*

Solution: We have $\frac{\partial}{\partial x} f(x,y) = -\frac{sinx}{3y}$ whereas we have the following:

$$\frac{\partial}{\partial y} f(x,y) = -\frac{cosx}{3y^2} + \frac{\partial}{\partial y}\frac{siny}{3y} = -\frac{cosx}{3y^2} + \frac{3ycosy - 3siny}{9y^2} = \frac{-cosx + ycosy - siny}{3y^2}$$

□

Question 22. *Let* $f(x,y) = x^{y^2+1}$. *Find the first order partial derivatives.*

Solution: We have $\frac{\partial}{\partial x} f(x,y) = (y^2+1)x^{y^2}$ by the power rule since $y^2 + 1 \neq 0$ for any choice of y. On the other hand we have the following:

$$\frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y} e^{(y^2+1)\ln|x|} = e^{(y^2+1)\ln|x|} 2y \ln|x| = \ln|x| 2y x^{y^2+1}$$

□

10.3 Linear Approximation, Jacobian Matrix

See pages 64-69 in the Addition Course Notes.

11 Week Eleven

11.1 Systems of Linear Differential Equations and their Stability

Question 23. *Solve the system of linear differential equations:*

$$x'(t) = 4x(t) + 3y(t) \quad x(0) = 0$$

$$y'(t) = 3x(t) - 4y(t) \quad y(0) = 4$$

Solution: For your information, one can rewrite the question as the following:

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}.$$

To solve this LSDE, we use Eigenvalues and Eigenvectors. Suppose $v \neq 0$. We have $Av = \lambda v \implies \det(A - \lambda I) = 0$ and so we calculate the following:

$$0 = \det\begin{pmatrix} 4-\lambda & 3 \\ 3 & -4-\lambda \end{pmatrix} = (4-\lambda)(-4-\lambda) - 9 = \lambda^2 - 25 = (\lambda - 5)(\lambda + 5).$$

Therefore, the two Eigenvalues are $\lambda = 5$ and $\mu = -5$ Now we determine what the Eigenvector v could be for each Eigenvalue. <u>For $\lambda = 5$</u>:

$$Av = 5v \implies (A - 5I)v = 0 \implies v \text{ is a solution to the SLE } \left(\begin{array}{cc|c} 4-5 & 3 & 0 \\ 3 & -4-5 & 0 \end{array}\right).$$

Simply row reduce and find the general solution:

$$\left(\begin{array}{cc|c} -1 & 3 & 0 \\ 3 & -9 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right) \implies v \in \{s\begin{pmatrix} 3 \\ 1 \end{pmatrix} \mid s \in \mathbb{R} \setminus \{0\}\}.$$

Anything in this set is an Eigenvector. For example, we can choose the vector $v = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ to be our Eigenvector. It is a good idea to check:

$$Av = \begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 15 \\ 5 \end{pmatrix} = 5\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \lambda v$$

<u>For $\mu = -5$</u>:

$$Aw = -5w \implies (A + 5I)w = 0 \implies w \text{ is a solution to the SLE } \left(\begin{array}{cc|c} 4+5 & 3 & 0 \\ 3 & -4+5 & 0 \end{array}\right).$$

Simply row reduce and find the general solution:

$$\left(\begin{array}{cc|c} 9 & 3 & 0 \\ 3 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 \end{array}\right) \implies w \in \{s\begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix} \mid s \in \mathbb{R} \setminus \{0\}\}.$$

Anything in this set is an Eigenvector. For example, we can choose the vector $w = \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix}$ to be our Eigenvector. It is a good idea to check:

$$Aw = \begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ -5 \end{pmatrix} = -5\begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix} = \mu v$$

Now to solve the differential equations, we use a formula:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = ce^{\lambda t}v + de^{\mu t}w$$

where v is the Eigenvector with Eigenvalue λ and w is the Eigenvector with Eigenvalue μ and c and d are arbitrary scalar numbers in $\mathbb{R}$. Hence, in our case we have the following:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = ce^{5t}\begin{pmatrix} 3 \\ 1 \end{pmatrix} + de^{-5t}\begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix}.$$

If an initial value had not been given this would be a sufficient answer - it is our general solution. However, we can deduce the scalars c and d using the initial value and the fact that $e^0 = 1$:

$$\begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = c\begin{pmatrix} 3 \\ 1 \end{pmatrix} + d\begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix}.$$

This is a SLE that can be solved for $c = 1$ and $d = 3$. Hence the solution is the following:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{5t}\begin{pmatrix} 3 \\ 1 \end{pmatrix} + 3e^{-5t}\begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix}.$$

Note that the since the coefficient matrix has a positive eigenvalue, the general solution has a summand that tends to infinity. Hence, the long term values of $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ is divergent, and looking at the general solution we see that $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \to \begin{pmatrix} +\infty \\ +\infty \end{pmatrix}$ as $t \to \infty$. □

For other problems like this see pages 75-79 and question 19 on page 93 in "Additional Course Notes".

12 Week Twelve

12.1 Nullclines and Stability of Non-linear Systems of Differential Equations

The notes for this DGD were not typed. Good examples:
`http://homepage.univie.ac.at/manfred.fuellsack/textbook/sw2/phpl_examples.html`