University of Ottawa
MAT 1332 Second Midterm Exam

March 30, 2016, Duration: 80 Minutes.
Instructor: Petko Kitanov   Catalin Rada   Robert Smith?

Family Name:______________________

First Name:________________________

Do **not** write your student ID number on this front page. Please write your student ID number in the space provided on the second page.

Take your time to read the entire paper before you begin to write, and read each question carefully. Remember that certain questions are worth more points than others. Make a note of the questions that you feel confident you can do, and then do those first: you do not have to proceed through the paper in the order given.

- You have 80 minutes to complete this exam.
- This is a closed book exam, and no notes of any kind are allowed.
- Cellular phones, unauthorized electronic devices or course notes are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to immediately leave the exam and academic fraud allegations will be filed, which may result in you obtaining a 0 (zero) for the exam. By signing below, you acknowledge that you have ensured that you are complying with the above statement:

  Signature:______________________
- Only the Faculty approved calculators (TI-30X, TI-34X, Casio FX-260X and Casio FX-300X) are allowed. All others will be confiscated.
- The correct answer requires justification written legibly and logically: you must convince me that you know why your solution is correct. Answer these questions in the space provided. Use the backs of pages if necessary.
- If you tear off any blank pages, they have to be handed in.
- Where it is possible to check your work, do so.
- Good luck!

Student number:____________________, Total marks: ___________ out of 30

Problem	1	2	3	4	5	6	7
Marks							

Question 1. [3 points] Determine the values of a and b for which the system

$$-x_1 + 4x_2 = a$$
$$2x_1 + bx_2 = 6$$

has
a) no solutions?
b) a unique solution?
c) infinitely many solutions?

Solution.

$$\left[\begin{array}{cc|c} -1 & 4 & a \\ 2 & b & 6 \end{array}\right] \xrightarrow{2R1+R2\to R2} \left[\begin{array}{cc|c} -1 & 4 & a \\ 0 & b+8 & 2a+6 \end{array}\right].$$

(a) $b + 8 = 0$ and $2a + 6 \neq 0 \implies b = -8$ and $a \neq -3$.

(b) $b + 8 \neq 0 \implies b \neq -8$.

(c) $b + 8 = 0$ and $2a + 6 = 0 \implies b = -8$ and $a = -3$.

Question 2. [3 points] Find all values of x for which the matrix A is not invertible

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & x \\ 0 & x & -15 \end{bmatrix}$$

Solution.

The matrix A is not invertible if $\det(A) = 0$.

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 3 & x \\ 0 & x & -15 \end{vmatrix} = -45 - 2x - x^2 + 60 = -x^2 - 2x + 15$$

$$x^2 + 2x - 15 = 0$$
$$(x-3)(x+5) = 0$$

So, the values for which the matrix is not invertible are $x = 3$ and $x = -5$.

Question 3. [5 points] Consider the matrix $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 2 & -1 & 3 \end{bmatrix}$.

a) Find the inverse of A.

b) Find the solution of $Ax = b$, where $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $b = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$.

Solution.

(a)

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 2 & -1 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2R1+R3 \to R3} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-R3+R2 \to R2]{R3+R1 \to R1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 1 & -1 \\ 0 & 1 & -1 & -2 & 0 & 1 \end{array}\right] \xrightarrow{-R2+R3 \to R3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 1 & -1 \\ 0 & 0 & -1 & -4 & -1 & 2 \end{array}\right]$$

$$\xrightarrow[-R3 \to R3]{R3+R1 \to R1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -1 & 3 \\ 0 & 1 & 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & 4 & 1 & -2 \end{array}\right]$$

Therefore,

$$A^{-1} = \begin{bmatrix} -5 & -1 & 3 \\ 2 & 1 & -1 \\ 4 & 1 & -2 \end{bmatrix}.$$

(b) The solution of the system $Ax = b$ is given by $x = A^{-1}b$. So we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 & -1 & 3 \\ 2 & 1 & -1 \\ 4 & 1 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ -5 \\ -10 \end{bmatrix}.$$

Question 4. [3 points] Calculate the eigenvalues of the matrix $A = \begin{bmatrix} 6 & 4 & -4 \\ 0 & -2 & 0 \\ 1 & 10 & 2 \end{bmatrix}$.

Solution.

$$\det(A - \lambda I) = \begin{vmatrix} 6-\lambda & 4 & -4 \\ 0 & -2-\lambda & 0 \\ 1 & 10 & 2-\lambda \end{vmatrix} = (6-\lambda)(-2-\lambda)(2-\lambda) + 4(-2-\lambda)$$

$$= (-2-\lambda)[(6-\lambda)(2-\lambda)+4] = (-2-\lambda)(\lambda^2 - 8\lambda + 16)$$

$$= (-2-\lambda)(\lambda-4)^2.$$

The eigenvalues are $\lambda = -2$ and $\lambda = 4$.

Question 5. [2 points] Consider the matrix $A = \begin{bmatrix} 5 & -5 \\ 2 & -1 \end{bmatrix}$.

a) Show that $2+i$ is an eigenvalue of A and find the other eigenvalue.
b) (**Bonus** 2 points) Find the eigenvectors corresponding to $2+i$.

Solution.

(a) $\begin{vmatrix} 5-\lambda & -5 \\ 2 & -1-\lambda \end{vmatrix} = (5-\lambda)(-1-\lambda) + 10 = 0.$

$\lambda^2 - 4\lambda + 5 = 0 \quad \Longrightarrow \quad \lambda = \frac{1}{2}\left(4 \pm \sqrt{16-20}\right) = 2 \pm i.$

So, the other eigenvalue is $\lambda = 2 - i$.

(b)

$$\begin{vmatrix} 5-\lambda & -5 \\ 2 & -1-\lambda \end{vmatrix} \longrightarrow \begin{vmatrix} 3-2-i & -5 \\ 2 & -1-2-i \end{vmatrix} \longrightarrow \begin{vmatrix} 3-i & -5 \\ 2 & -3-i \end{vmatrix}$$

$$\xrightarrow{(3+i)R1 \to R1} \begin{vmatrix} 10 & -15-5i \\ 2 & -3-i \end{vmatrix} \xrightarrow{(1/5)R1 \to R1} \begin{vmatrix} 2 & -3-i \\ 2 & -3-i \end{vmatrix}. \xrightarrow{-R1+R2 \to R2} \begin{vmatrix} 2 & -3-i \\ 0 & 0 \end{vmatrix}.$$

$v_2 = s \quad \text{(free)} \quad \Longrightarrow \quad v_1 = \dfrac{3+i}{2} s = (3/2 + 1/2\, i)s.$

The eigenvectors corresponding to $1+2\,i$ are $s \begin{bmatrix} 3/2 + i/2 \\ 1 \end{bmatrix}$, where s is a nonzero complex number.

Question 6. [8 points] Consider the system of differential equations

$$\begin{aligned} x_1' &= x_1 - 4x_2 \\ x_2' &= 3x_1 - 6x_2 \end{aligned}$$

a) Calculate the eigenvalues and corresponding eigenvectors of the coefficient matrix.
b) Give the general solution of the system.
c) Find the unique solution with initial conditions $x_1(0) = 0,\ x_2(0) = 1$.

Solution.

(a) First we find the eigenvalues and the corresponding eigenvectors.

$$\begin{vmatrix} 1-\lambda & -4 \\ 3 & -6-\lambda \end{vmatrix} = (1-\lambda)(-6-\lambda) + 12 = \lambda^2 + 5\lambda + 6$$

$$\lambda^2 + 5\lambda + 6 = 0 \quad \Longrightarrow \quad \lambda = -2 \quad \text{and} \quad \lambda = -3.$$

For $\lambda = -2$.

$$\begin{bmatrix} 3 & -4 \\ 3 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & -4 \\ 0 & 0 \end{bmatrix}$$

$$v_2 = s \text{ and } v_1 = \frac{4}{3}s \quad \Longrightarrow \quad v = \begin{bmatrix} 4/3s \\ s \end{bmatrix} = s\begin{bmatrix} 4/3 \\ 1 \end{bmatrix},\ s \in \mathbb{R} \text{ and } s \neq 0.$$

For $\lambda = -3$.

$$\begin{bmatrix} 4 & -4 \\ 3 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$w_2 = s \text{ and } w_1 = s \quad \Longrightarrow \quad w = \begin{bmatrix} s \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \end{bmatrix},\ s \in \mathbb{R} \text{ and } s \neq 0.$$

(b) The general solution is

$$x(t) = C_1 e^{-2t}\begin{bmatrix} 4/3 \\ 1 \end{bmatrix} + C_2 e^{-3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(c)

$$\begin{aligned} x_1(0) &= \frac{4}{3}C_1 + C_2 = 0 \\ x_2(0) &= C_1 + C_2 = 1 \end{aligned}$$

$$\Longrightarrow C_1 = -3 \quad \text{and} \quad C_2 = 4$$

Therefore, the solution of the initial value problem is

$$x(t) = -3e^{-2t}\begin{bmatrix} 4/3 \\ 1 \end{bmatrix} + 4e^{-3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Question 7. [6 points] Consider the function $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2-4}}$.

a) Determine the domain of the function and sketch it in the xy-plane.
b) Determine the range of the function.
c) Sketch the level curves $f(x,y) = k$ for $k = \sqrt{1/5}$ and $k = 1/3$. Label your coordinate system and clearly indicate which curve to which k corresponds.
d) Find the equation of the tangent plane to the surface, given by $f(x,y)$, at the point $(x,y) = (2,2)$. **Do not** simplify your answer.

Solution.

(a) $x^2+y^2-4>0 \implies x^2+y^2>4$. The domain is $\{(x,y)\in\mathbb{R}^2 : x^2+y^2>4\}$. This is the region in $xy-$plane outside of the circle of radius 2.

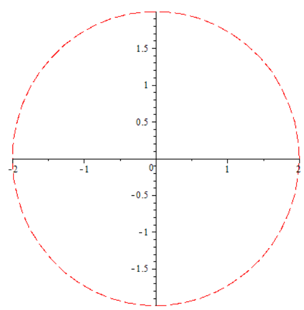

Figure 1: The domain is the region outside of the circle

(b) $\{f \in \mathbb{R} | f > 0\}$

(c) $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2-4}} = \dfrac{1}{\sqrt{5}} \implies \sqrt{x^2+y^2-4} = \sqrt{5} \implies x^2+y^2 = 9.$

$f(x,y) = \dfrac{1}{\sqrt{x^2+y^2-4}} = \dfrac{1}{3} \implies \sqrt{x^2+y^2-4} = 3 \implies x^2+y^2 = 13.$

So, the level curves are circles with radius 3 and $\sqrt{13}$, respectively, centered at the origin.

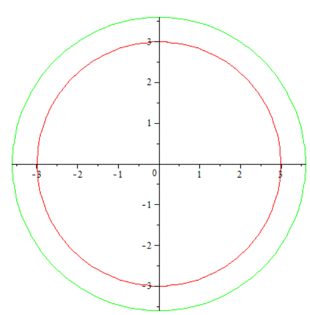

Figure 2: Level curves: $R=3$, $R=\sqrt{13}$

(d) The equation of tangent plane at (x_0,y_0) is $z-z_0 = f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$.

$$z_0 = f(2,2) = \frac{1}{\sqrt{2^2+2^2-4}} \quad \Longrightarrow \quad z_0 = \frac{1}{2}.$$

$$f_x(x,y) = -\frac{x}{\left(\sqrt{x^2+y^2-4}\right)^3} \quad \Longrightarrow \quad f_x(2,2) = -\frac{2}{\left(\sqrt{2^2+2^2-4}\right)^3} = -1.$$

$$f_y(x,y) = -\frac{y}{\left(\sqrt{x^2+y^2-4}\right)^3} \quad \Longrightarrow \quad f_y(2,2) = -\frac{2}{\left(\sqrt{2^2+2^2-4}\right)^3} = -1.$$

The equation of the tangent plane at $(2,2)$ is $z - \frac{1}{2} = -(x-2) - (y-2)$.