MAT1332: Calculus for the Life Sciences II - Part 2

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18 Partial Derivatives

For functions with two or more variables, we don't just have one derivative, but rather a derivative for each variable. Partial derivatives keep all other variables fixed and look at the rate of change of only the variable of interest

Example 96.

$$f(x,y) = x^2 + 3xy + y^4 + 6$$
$$\frac{\partial f}{\partial x} = 2x + 3y$$
$$\frac{\partial f}{\partial y} = 3x + 4y^3$$

Example 97. Use the chain rule and product rule for $g(x,y) = x^2 e^{xy}$ to find $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$.

$$\frac{\partial g}{\partial x} = (2x + x^2 y)e^{xy} \qquad \frac{\partial g}{\partial y} = x^3 e^{xy}$$

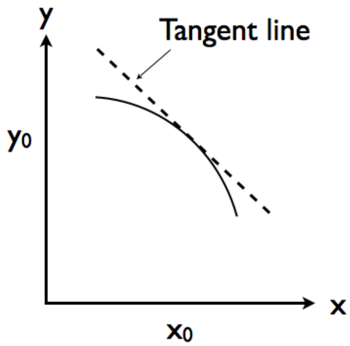
Example 98. $h(x,y) = \frac{\cos(xy)}{y^2+1}$. Find $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial y}$.

$$\frac{\partial h}{\partial x} = \frac{-\sin(xy)y}{y^2+1} = \frac{-y\sin(xy)}{y^2+1}$$
$$\frac{\partial h}{\partial y} = \frac{(y^2+1)(-\sin(xy)x) - \cos(xy)(2y)}{(y^2+1)^2}$$
$$= \frac{-x(y^2+1)\sin(xy) - 2y\cos(xy)}{(y^2+1)^2}$$

Example 99. $k(x,y,z) = e^{yz}(x^2 + z^3)$. Find all partial derivatives.

$$\frac{\partial k}{\partial x} = 2xe^{yz}$$
$$\frac{\partial k}{\partial y} = ze^{yz}(x^2 + z^3)$$
$$\frac{\partial k}{\partial z} = ye^{yz}(x^2 + z^3) + 3z^2 e^{yz}$$

Geometric Interpretation: for one-variable functions, $\frac{df}{dx}$ is the slope of the tangent line at x. The equation of the tangent line is $(y - y_0) = f'(x_0)(x - x_0)$.

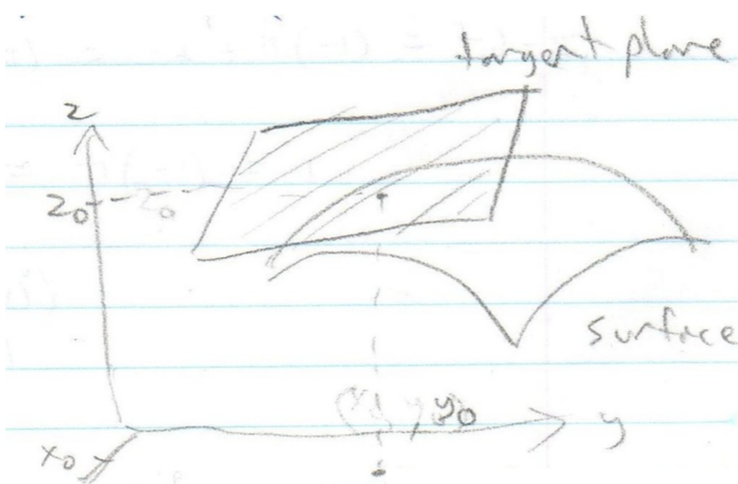


It's similar for two variables, but now instead of a tangent line, we have a tangent plane. If $z = f(x, y)$, then the tangent plane is

$$(z - z_0) = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

Note: The tangent plane might not exist if the limit does not exist. But then the tangent line might not exist either.

Example 100. Find the equation of the tangent plane to the surface $z = 4x^2 + y^2$ at the point (1,2,8).



$$\frac{\partial z}{\partial x} = 8x$$
$$\frac{\partial z}{\partial y} = 2y$$
$$\begin{aligned} z - 8 &= 8(1)(x-1) + 2(2)(y-2) \\ &= 8(x-1) + 4(y-2) \\ &= 8x - 8 + 4y - 8 \\ z &= 8x + 4y - 8 \end{aligned}$$

Example 101. Find the equation for the tangent planes to $z = x^2 + \sin(xy)$ when
a) (x, y)=(1,0) b) (x, y)=(0,1) c)(x, y)=(-1,π)

Solution:

a)

$$z(1,0) = 1^2 + \sin(0) = 1$$

$$\begin{aligned}
&\frac{\partial z}{\partial x} = 2x + \cos(xy)y \\
&\frac{\partial z}{\partial x}(1,0) = 2(1) + \cos(0)0 = 2 \\
&\frac{\partial z}{\partial y}(1,0) = \cos(xy)x \\
&\frac{\partial z}{\partial y} = \cos(0)(1) = 1 \\
&z - 1 = 2(x-1) + 1(y-0) \\
&z - 1 = 2x - 2 + y \\
&z = 2x + y - 1
\end{aligned}$$

b)

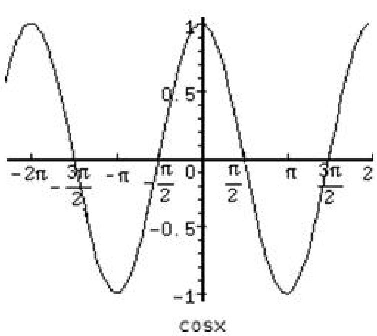
$$z(0,1) = 0^2 + \sin(0) = 0$$

$$\begin{aligned}
&\frac{\partial z}{\partial x} = 2(0) + 1\cos(0) = 1 \\
&\frac{\partial z}{\partial y} = (0)\cos(0) = 0 \\
&z - 0 = 1(x-0) + 0(y-1) \\
&z = x
\end{aligned}$$

c)

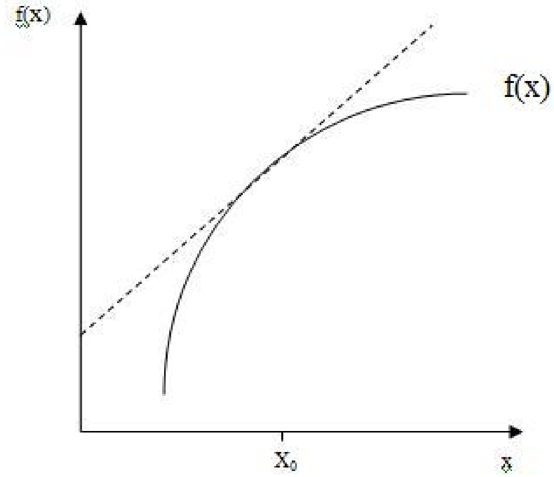
$$z(-1,\pi) = (-1)^2 + \sin(-\pi) = 1$$

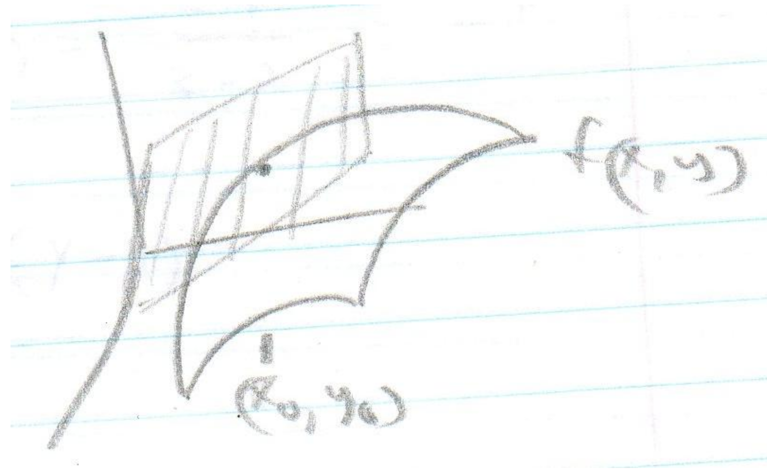
$$\begin{aligned}
&\frac{\partial z}{\partial x} = 2(-1) + \pi\cos(-\pi) = -2 + \pi(-1) = -2 - \pi \\
&\frac{\partial z}{\partial y} = (-1)\cos(-\pi) = -1(-1) = 1 \\
&z - 1 = (-2-\pi)(x+1) + 1(y-\pi) \\
&z - 1 = -(2+\pi)x - 2 - \pi + y - \pi \\
&z = -(2+\pi)x + y - 1 - 2\pi
\end{aligned}$$



18.1 Linearisation

We can approximate difficult functions by their tangent plane at the point of interest. If we're close to this point then the tangent plane is close to the function.





Definition 18.1. *Suppose $f(x, y)$ is differentiable at (x_0, y_0). The* <u>*linearization*</u> of $f(x, y)$ at (x_0, y_0) is

$$L(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

The approximation $f(x, y) \approx L(x, y)$ is the standard linear appoximation of $f(x, y)$ at (x_0, y_0).

Example 102. Find the linear approximation of $f(x, y) = x^2y + 2xe^y$ at (2,0).

$$f(2, 0) = 0 + 4 = 4$$

$$\frac{\partial f}{\partial x} = 2xy + 2e^y$$
$$\frac{\partial f}{\partial x}(2, 0) = 0 + 2 = 2$$
$$\frac{\partial f}{\partial y} = x^2 + 2xe^y$$
$$\frac{\partial f}{\partial y}(2, 0) = 4 + 4 = 8$$

$$L(x, y) = 4 + 2(x - 2) + 8(y - 0) = 2x + 8y$$

Example 103. a) Find the linear approximation of $f(x, y) = \ln(x - 2y^2)$ at (3,1).
b) How close is this approximation to the original function at (3.05,0.95)?

a)

$$
\begin{aligned}
f(3,1) &= 0 \\
\frac{\partial f}{\partial x} &= \frac{1}{x-2y^2} \\
\frac{\partial f}{\partial x}(3,1) &= 1 \\
\frac{\partial f}{\partial y} &= \frac{-4y}{x-2y^2} \\
\frac{\partial f}{\partial y}(3,1) &= -4 \\
L(x,y) &= 0 + 1(x-3) - 4(y-1) \\
&= x - 3 - 4y + 4 \\
&= x - 4y + 1
\end{aligned}
$$

b)

$$
\begin{aligned}
f(3.05, 0.95) &= \ln(3.05 - 2(0.95)^2) = 0.2191 \\
L(3.05, 0.95) &= 3.05 - 4(0.95) + 1 = 0.25
\end{aligned}
$$

The error of approximation is $|0.25 - 0.2191| = 0.031$.

19 Vector-Valued Functions

Thus far we've looked at real-valued functions $f : \mathbb{R}^n \to \mathbb{R}$. We now consider functions whose output is m-dimensional; that is, $f : \mathbb{R}^n \to \mathbb{R}^m$.

Example 104. $f(x,y,z) = (x + y^2 z, xyz)$, $f : \underline{R}^3 \to \underline{R}^2$ Evaluate f at (1,2,0), (6,2,-1) and (3,-4,5).

$$
\begin{aligned}
f(1,2,0) &= (1,0) \\
f(6,2,-1) &= (2,-12) \\
f(3,-4,5) &= (83,-60)
\end{aligned}
$$

19.1 Linearization of Vector-Valued Function

Consider the function $h : \underline{R}^2 \to \underline{R}^2$ given by $h(x,y) = (f(x,y), g(x,y))$.
The linearization of h is the function $M = \begin{bmatrix} L_1(x,y) \\ L_2(x,y) \end{bmatrix}$ where L_1 is the linearization of $f(x,y)$ and L_2 is the linearization of $g(x,y)$.

$$
\begin{aligned}
L_1(x,y) &= f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0) \\
L_2(x,y) &= g(x_0,y_0) + \frac{\partial g}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial g}{\partial y}(x_0,y_0)(y-y_0) \\
L(x,y) &= \begin{bmatrix} f(x_0,y_0) \\ g(x_0,y_0) \end{bmatrix} + \begin{bmatrix} \frac{\partial f}{\partial x}(x_0,y_0) & \frac{\partial f}{\partial y}(x_0,y_0) \\ \frac{\partial g}{\partial x}(x_0,y_0) & \frac{\partial g}{\partial y}(x_0,y_0) \end{bmatrix} \begin{bmatrix} (x-x_0) \\ (y-y_0) \end{bmatrix}
\end{aligned}
$$

The matrix

$$J(x_0, y_0) = \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{bmatrix}$$

defined at (x_0, y_0) is called the Jacobian matrix.

Example 105. Find the Jacobian for $h(x, y) = (x^2y - y^3, 2x^3y^2 + y)$ at $(1, 2)$.

$$J = \begin{bmatrix} 2xy & x^2 - 3y^2 \\ 6x^2y^2 & 4x^3y + 1 \end{bmatrix} \qquad J(1,2) = \begin{bmatrix} 4 & -11 \\ 24 & 9 \end{bmatrix}$$

↑ (all x derivatives) ↑ (all y derivatives)

Example 106. Find the linear approximation of $K(x, y) = (ye^{-x}, \sin(x) + \cos(y))$ at $(0, 0)$. How close is this to the actual value at (0.1,-0.1)?

$$J = \begin{bmatrix} -ye^{-x} & e^{-x} \\ \cos(x) & -\sin(y) \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$K(0,0) = (0,1)$$

$$\begin{aligned} L(x,y) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x - 0 \\ y - 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} y \\ x \end{bmatrix} \\ &= \begin{bmatrix} y \\ x + 1 \end{bmatrix} \end{aligned}$$

$$K(0.1, -0.1) = (0.1e^{-0.1}, \sin(0.1) + \cos(-0.1)) = (-0.09, 1.09)$$

$$L(x,y) = \begin{bmatrix} -0.1 \\ 1.1 \end{bmatrix}$$

Therefore the approximation is quite close.

In general, if $f : \mathbb{R}^n \to \mathbb{R}^m$, $f(x_1, x_2, ..., x_n) = (f_1, f_2, ..., f_m)$ then $J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$.

20 Phase Plane Analysis

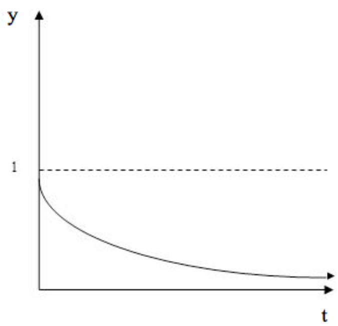
Example 107. Solve the following system with $(x_0, y_0) = (1, 0)$.

$$x' = -x$$
$$y' = x$$

Method 1. Solve x equation first, then the y equation. We can do this because the equation is independent of y:

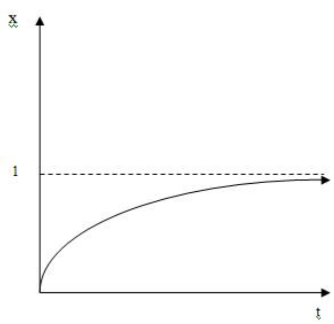
Solve the x equation first:

$$\frac{dx}{dt} = -x$$
$$\frac{dx}{x} = -dt$$
$$\ln|x| = -t + c$$
$$x = Ae^{-t}$$
$$x(0) = A = 1$$
$$x = e^{-t}$$



Then solve the y equation:

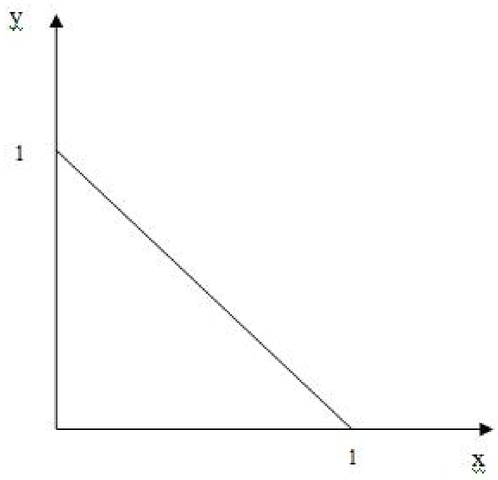
$$\frac{dy}{dt} = e^{-t}$$
$$y = -e^{-t} + c$$
$$y(0) = -1 + c = 0$$
$$\Rightarrow c = 1$$
$$\therefore y = 1 - e^{-t}$$



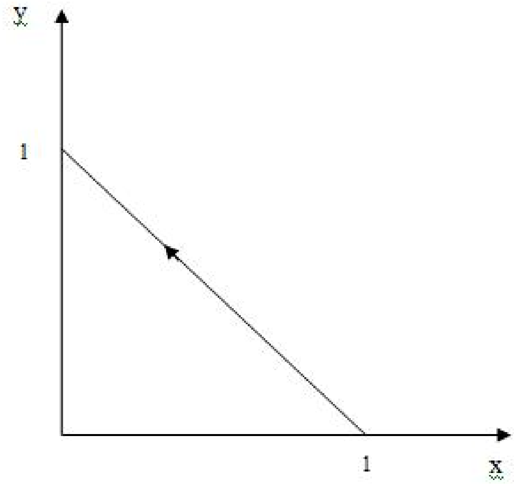
Method 2: Divide the second equation by the first.

$$\frac{y'}{x'} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx} = -1$$
$$\therefore y = -x + c$$
$$0 = -1 + c \Rightarrow c = 1$$
$$y = 1 - x$$

Now plot x-y coordinates where t is implicit, as seen in Figure 6. Which way do trajectories travel? Since $(x(0), y(0)) = (1, 0)$, that's where they start. Or we have:



(a) $y' > 0$ indicates that y is increasing



(b) $x' < 0$ indicates that x is decreasing

Question: What happens at equilibrium?
Answer: $(x, y) \to (0, 1)$

Drawing solutions in the x-y plane, with time as an implicit variable is known as the Phase Plane.

Example 108.

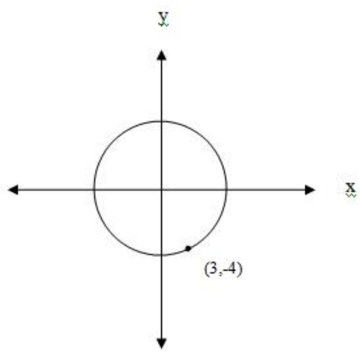
$$\begin{aligned} y' &= -x \\ x' &= y \\ (x(0), y(0)) &= (3, -4) \end{aligned}$$

Method 1: does not work because the equation for y depends on x and the equation for x depends on y.

Method 2:

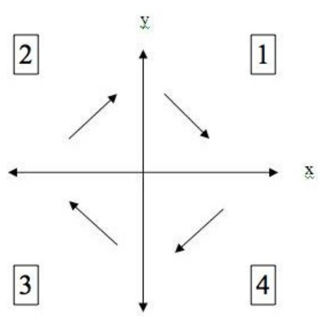
$$\frac{dy}{dx} = -\frac{x}{y}$$
$$\int y dy = -\int x dx$$
$$\frac{y^2}{2} = -\frac{x^2}{2} + c$$
$$x^2 + y^2 = 2c$$
$$9 + 16 = 2c \Rightarrow c = 25/2$$
$$x^2 + y^2 = 25$$

which is a circle of radius 5



Direction?

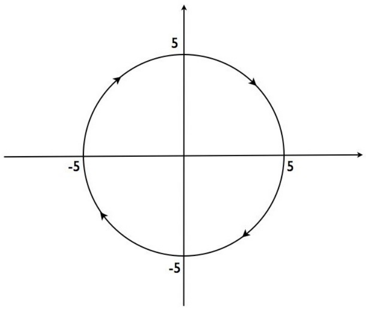
Quadrant 1 $x > 0, y > 0, y' < 0, x' > 0$
Quadrant 2 $x < 0, y > 0, y' > 0, x' > 0$
Quadrant 3 $x < 0, y < 0, y' > 0, x' < 0$
Quadrant 4 $x > 0, y < 0, y' < 0, x' < 0$



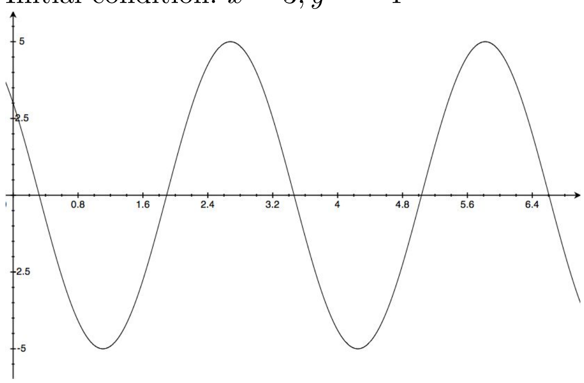
Any of these will do.

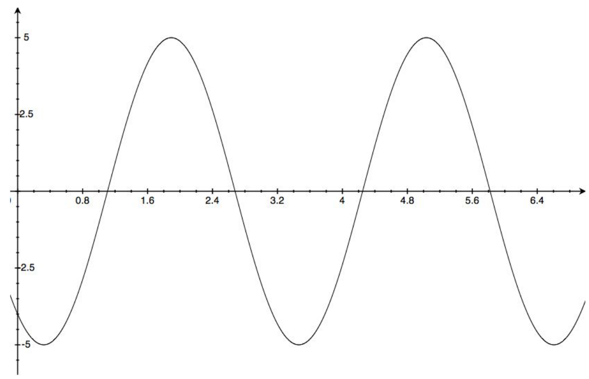
Question: What happens eventually?

Answer: Solutions cycle indefinitely.



What do solutions look like in time? They oscillate.
Initial condition: $x = 3, y = -4$





Also, both are initially decreasing.
Thus we have almost compete understanding without solving the equations.

Question: Can we guess the solutions?
Answer:

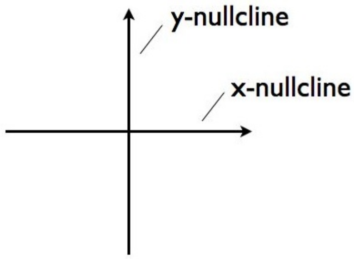
$$
\begin{aligned}
&x = 5\cos(at + b)\\
&x(0) = 5\cos b = 3\\
&x = 5\cos(at + \arccos 3/5)\\
&y = 5\cos(at + c)\\
&y(0) = 5\cos c = -4\\
&y = 5\cos(at + \pi - \arccos 4/5)
\end{aligned}
$$

Therefore we know everything from the phase plane, except how fast the solutions travel.

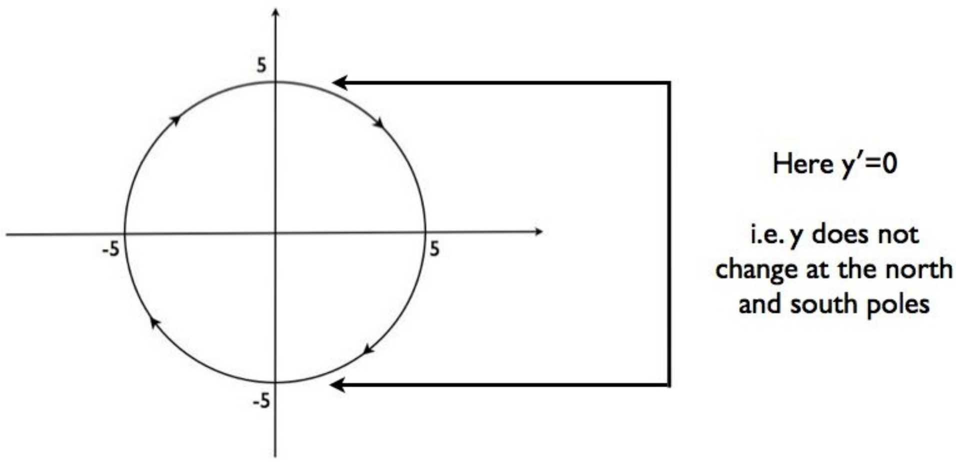
21 Nullclines

Nullclines are solutions to either $x' = 0$ or $y' = 0$

Example 109. $y' = -x$ and $x' = y$
y-nullcline: $y' = 0 \Rightarrow x = 0$
x-nullcline: $x' = 0 \Rightarrow y = 0$
When the two equations meet, we have an equilibrium.



On the y-nullcline, y does not change (although x might, and usually does.)
On the x-nullcline, x does not change (although y might, and usually does.)
Check with Figure 7:



Example 110. SIS epidemic:

$$S' = bI - aSI$$
$$I' = aSI - bI$$

We solve this before and it was a lot of work.

$$\frac{I'}{S'} = -1$$
$$\frac{dI}{dS} = -1$$
$$I = -S + c$$
$$S + I = N$$
$$I = -S + N$$

Note: if I=0, then S=N; the whole population.

Question: What happens eventually? Which direction do trajectories go in?
Answer: Not necessarily what you think.
Nullclines:

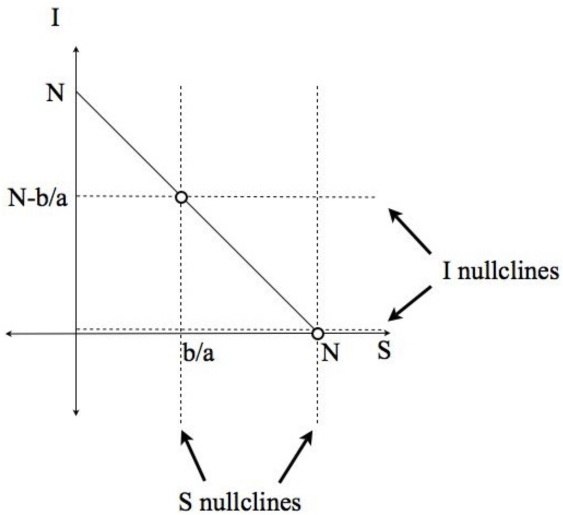
$$S' = 0 \Rightarrow \qquad (b - aS)I = 0$$
$$I' = 0 \Rightarrow \qquad -(b - aS)I = 0$$

These are the same equation with two solutions.

$$
\begin{aligned}
&b - aS = 0 \\
&S = \frac{b}{a} \\
&Or \\
&I = 0 \\
&S = N
\end{aligned}
$$

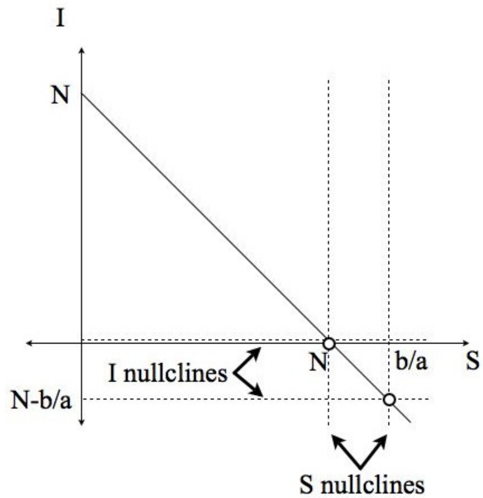
Therefore we have two equilibria: $(N, 0)$ and $(\frac{b}{a}, N - \frac{b}{a})$
Be careful: $(N, 0)$ always exists. However, $(\frac{b}{a}, N - \frac{b}{a})$ only exists if $\frac{b}{a} < N$. (Otherwise $I < 0$ which isn't biologically meaningful).

Case (i): $\frac{b}{a} < N$



Equilibria: $(N, 0)$ and $(\frac{b}{a}, N - \frac{b}{a})$

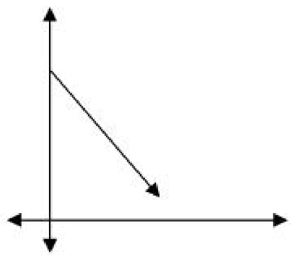
Case (ii): $\frac{b}{a} > N$



Equilibria: $(N, 0)$ only. (In the positive plane)

Directions?
If $S < \frac{b}{a}$, then

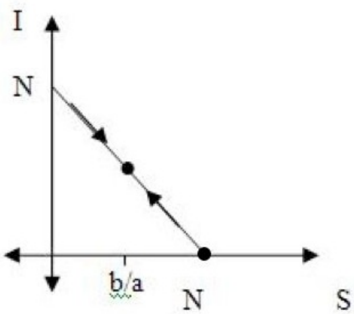
$$
\begin{aligned}
&S' = (b - as)I = a(\frac{b}{a} - S)I \\
&\therefore if\ S < \frac{b}{a}, \frac{b}{a} - S > 0\ so\ S' > 0 \\
&\Rightarrow S\ is\ increasing. \\
&I' = (aS - b)I = a(S - \frac{b}{a})I \\
&\therefore if\ S < \frac{b}{a}, S - \frac{b}{a} < 0, so\ I' < 0 \\
&\Rightarrow I\ is\ decreasing.
\end{aligned}
$$



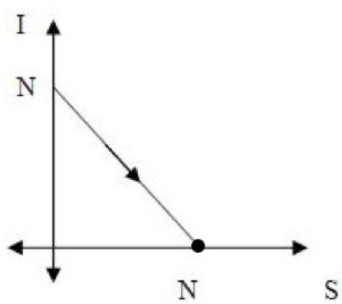
If $S > \frac{b}{a}$, (this only applies to case (i)), then

$$
\begin{aligned}
&S' = a(\frac{b}{a} - S)I \\
&\therefore if\ S > \frac{b}{a}, \frac{b}{a} - S < 0, so\ S' < 0. \\
&\Rightarrow S\ is\ decreasing. \\
&and \\
&I' = a(S - \frac{b}{a})I \\
&\therefore if\ S > \frac{b}{a}, S - \frac{b}{a} > 0, so\ I' > 0 \\
&\Rightarrow I\ is\ increasing.
\end{aligned}
$$

Case (i)



Case (ii)



Biological interpretation: Stability depends on $\frac{b}{a}$. If the recovery rate b is high compared to the infection rate a, then infected individuals recover quickly and the population moves to a population of susceptibles, clearing infection.
If the infection rate a is high compared to the recovery rate b, then infection stabilizes at an endemic equilibrium.

What if we can't solve the $\frac{dy}{dx}$ equation?

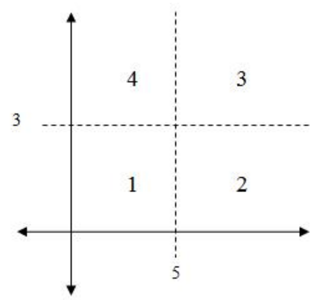
Example 111. Predator-prey dynamics:

$$y' = (15 - 3x)y$$
$$x' = (-6 + 2y)x$$

Biological explanation: The prey population increases on its own, while the predator population decreases on its own. The interaction is given by:

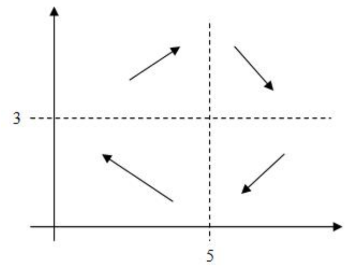
y-nullcline is given by $(15 - 3x)y = 0 \rightarrow x = 5, y = 0$
x-nullcline is given by $(-6 + 2y)x = 0 \rightarrow x = 0, y = 3$

Note: the only equilibria are (0,0) and (5,3).
Question: Why aren't (5,0) and (0,3) equilibria?
Answer: They don't satisfy both equations.
We can't solve for $\frac{dy}{dx}$ since $\frac{dy}{dx} = \frac{(15-3x)y}{(-6+2y)x}$ is too hard to solve.
But we can still get a lot of information by examining the derivatives as before by looking at each of the quadrants:



Quadrant 1:
$y' = (15 - 3x)y > 0 \; since \; x < 5$
$x' = (-6 + 2y)x < 0 \; since \; y < 3$
If we have small number of predators and few prey, then prey increases because there are few predators and predators decrease because there is little food.
Quadrant 2:
$y' = (15 - 3x)y < 0 \; since \; x > 5$
$x' = (-6 + 2y)x < 0 \; since \; y < 3$
If we have lots of predators and few prey, then prey decreases because they're eaten and predators decrease because there is little food.
Quadrant 3:
$y' = (15 - 3x)y < 0 \; since \; x > 5$
$x' = (-6 + 2y)x > 0 \; since \; y > 3$
If we have lots of predators and lots of prey, then prey decreases because they're eaten and predators increase because they have lots of food.
Quadrant 4:
$y' = (15 - 3x)y > 0 \; since \; x < 5$
$x' = (-6 + 2y)x > 0 \; since \; y > 3$
If we have few predators and lots of prey, then prey increases because there are few predators and predators increase because there is lots of food.
Therefore the dynamics will oscillate.
This was actually observed in Canada for lynx and hares by the Hudson's Bay Company over a hundred year

period. Lynx feed only on hares and nothing keeps the hare population in check other than predation by lynx.



Exercise. Draw the nullclines and phase portrait for

$$x' = 3 + 4x$$
$$y' = x + y^2.$$

Exercise. Draw the nullclines and phase portrait for

$$x' = x - y^2$$
$$y' = xy - 1.$$

22 Stability in Linear Systems

Consider the linear system:

$$x' = ax + by$$
$$y' = cx + dy$$

The only steady state is $(0, 0)$ The Jacobian is given by $J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{aligned}
\det(I - \lambda J) &= \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
&= (\lambda - a)(\lambda - d) - bc \\
&= \lambda^2 - (a + d)\lambda + ad - bc \\
\lambda &= \frac{(a + d) \pm \sqrt{(a + d)^2 - 4(ad - bc)}}{2} \\
&= \frac{1}{2}[a + d \pm \sqrt{a^2 + d^2 - 2ad + 4bc}] \\
&= \frac{1}{2}[a + d \pm \sqrt{(a - d)^2 + 4bc}]
\end{aligned}$$

If $\Delta = (a - d)^2 + 4bc < 0$ then we are taking the root of a negative.
There are two cases: $\Delta < 0, \Delta > 0$.

Our solutions depend on the terms $e^{\lambda t}$.
If $\Delta > 0$ then,
if $\lambda_1, \lambda_2 < 0$ we have a stable equilibrium.
if λ_1 or $\lambda_2 > 0$ we have an unstable equilibrium.

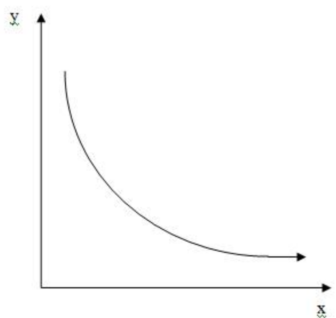
If $\Delta < 0$ then our solutions are of the form $e^{\frac{a+d}{2}t}[a_1 \sin\sqrt{-\Delta}t + a_2 \cos\sqrt{-\Delta}t]$ and stability depends on $e^{\frac{a+d}{2}t}$:
if $Re(\lambda_1)$ *and* $Re(\lambda_2) < 0$ we have a stable equilibrium.
if $Re(\lambda_1)$ *or* $Re(\lambda_2) > 0$ we have an unstable equilibrium.
Thus everything depends on the eigenvalues of the Jacobian. This is why we needed linear algebra and multivariable calculus.

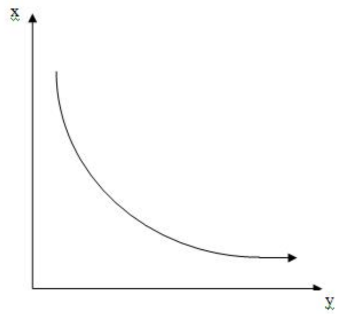
Example 112. Find the stability of the following system:

$$x' = -4x$$
$$y' = -5y$$

Solution:

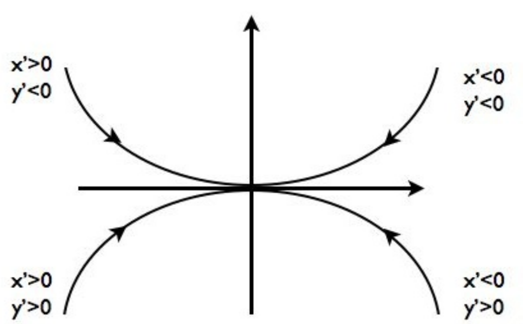
$$J = \begin{bmatrix} -4 & 0 \\ 0 & -5 \end{bmatrix}$$
$$\lambda_1 = -4, \lambda_2 = -5$$





Check:

$$x' = -4x \Rightarrow x = Ae^{-4t}$$
$$y' = -5y \Rightarrow y = Be^{-5t}$$
$$\frac{y'}{x'} = \frac{dy}{dx} = \frac{5y}{4x}$$
$$\frac{dy}{5y} = \frac{dx}{4x}$$
$$4 \ln y = 5 \ln x + c$$
$$y = kx^{5/4}$$



This is a stable sink.

Example 113. Find the stability of the following system:

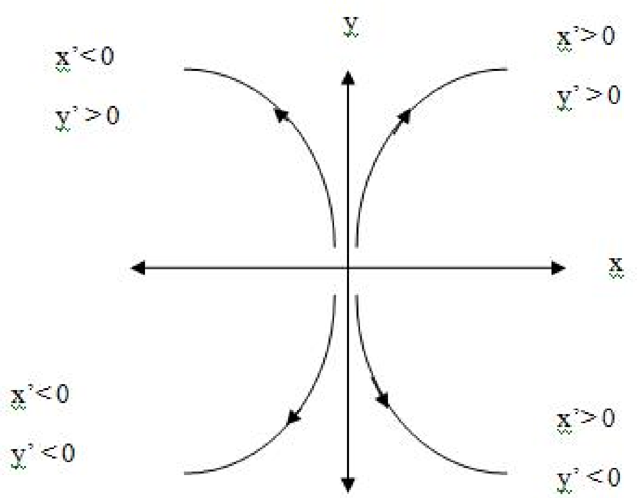
$$x' = 4x$$
$$y' = 2y$$

Solution:

$$J = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$
$$\lambda_1 = 4, \lambda_2 = 2 \rightarrow \textit{unstable equilibrium.}$$

Check:

$$x' = 4x \Rightarrow x = Ae^{4t}$$
$$y' = 2y \Rightarrow y = Be^{2t}$$
$$\frac{y'}{x'} = \frac{dy}{dx} = \frac{2y}{4x}$$
$$\frac{dy}{y} = \frac{dx}{2x}$$
$$2\ln|y| = \ln|x| + c$$
$$y = \pm A\sqrt{\pm x}$$



This is an unstable source.

Example 114. Find the stability of the following system:

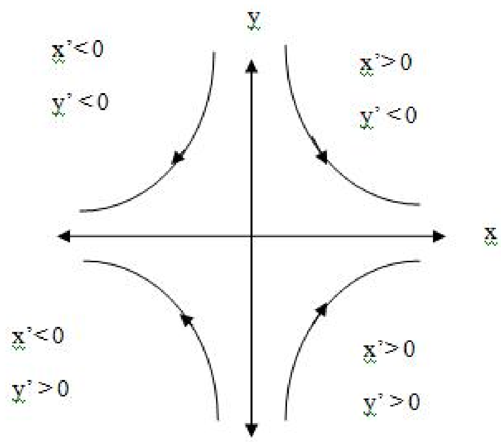
$$x' = x$$
$$y' = -3y$$

Solution:

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$$
$$\lambda_1 = 1, \lambda_2 = -3 \rightarrow \textit{unstable equilibrium.}$$

Check:

$$
\begin{aligned}
&x' = x \Rightarrow x = Ae^{t} \\
&y' = -3y \Rightarrow y = Be^{-3t} \\
&\frac{y'}{x'} = \frac{dy}{dx} = \frac{-3y}{x} \\
&\frac{dy}{-3y} = \frac{dx}{x} \\
&-\frac{1}{3}\ln|y| = \ln|x| + c \\
&y = \pm Ax^{-3}
\end{aligned}
$$



This is a saddle.

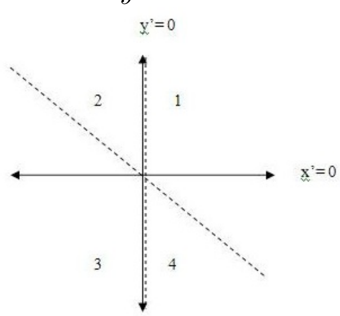
Example 115. Find the stability of the following system:

$$
\begin{aligned}
x' &= -2x - 3y \\
y' &= x
\end{aligned}
$$

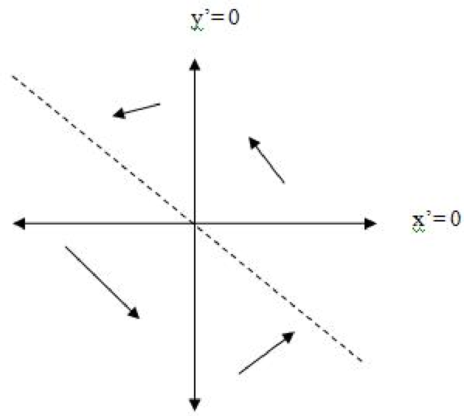
Solution:

$$
\begin{aligned}
&J = \begin{bmatrix} -2 & -3 \\ 1 & 0 \end{bmatrix} \\
&det(J - \lambda I) = \lambda(2 + \lambda) + 3 = \lambda^2 + 2\lambda + 3 \\
&\lambda = -1 \pm \sqrt{2}i
\end{aligned}
$$

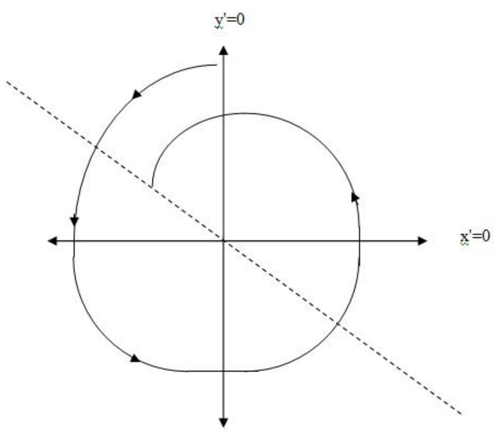
We cannot solve $\frac{dy}{dx} = \frac{-x}{2x+3y}$ directly. So we will use nullclines.
x-nullcline: $x' = 0 : -2x - 3y = 0 \rightarrow y = -\frac{2}{3}x$
y-nullcline: $y' = 0 : x = 0 \rightarrow x = 0$



$$
\begin{aligned}
&\text{Quadrant 1}: \; x' < 0,\; y' > 0\\
&\text{Quadrant 2}: \; x' < 0,\; y' < 0\\
&\text{Quadrant 3}: \; x' > 0,\; y' < 0\\
&\text{Quadrant 4}: \; x' > 0,\; y' > 0
\end{aligned}
$$



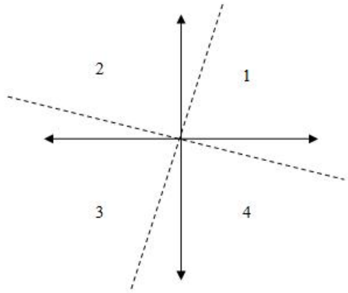
This is a stable spiral.



Example 116. Find the stability of the following system: $x' = 3x - y, y' = x + 2y$.

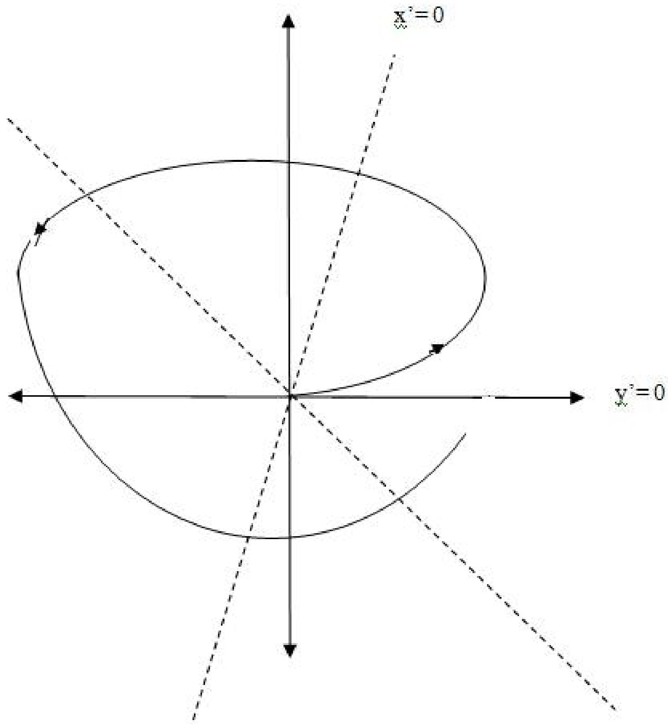
$$
\begin{aligned}
&J = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}\\
&\det(J - \lambda I) = (\lambda - 3)(\lambda - 2) + 1 = \lambda^2 - 5\lambda + 7\\
&\lambda = \frac{5 \pm \sqrt{-3}}{2}\\
&Re(\lambda) = \frac{5}{2} > 0 \therefore unstable.
\end{aligned}
$$

Again, we cannot solve the equations directly. So we will use nullclines.
x-nullcline: $x' = 0 : 3x - y = 0 \to y = 3x$
y-nullcline: $y' = 0 : x + 2y = 0 \to y = -\frac{1}{2}x$



Quadrant 1: $x' > 0, y' > 0$
Quadrant 2: $x' < 0, y' > 0$
Quadrant 3: $x' < 0, y' < 0$
Quadrant 4: $x' > 0, y' < 0$

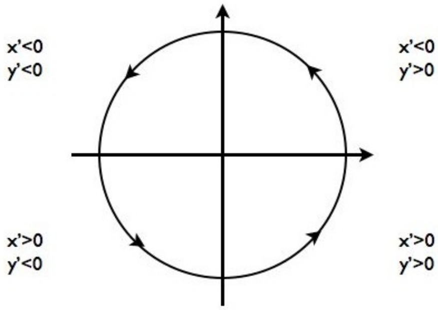
This is an unstable spiral.



Example 117. Find the stability of the following system: $x' = -y, y' = x$.

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$
$$\det(J - \lambda I) = \lambda^2 + 1$$
$$\lambda^2 + 1 = 0 \to Re(\lambda) = 0 \to \textit{We have neither stability nor instability.}$$
$$\frac{dy}{dx} = \frac{x}{-y}$$
$$-\int y dy = \int x dx$$
$$-\frac{y^2}{2} = \frac{x^2}{2} + k$$
$$x^2 + y^2 = c$$

This is a centre.



22.1 Complex Eigenvalues

Let's run through the same sort of analysis that we did before and see what happens when we have complex eigenvalues. Consider

$$x' = Ax$$

where x is a vector and A is a real valued matrix.
We look for solutions of the form

$$x = Fe^{\lambda t}$$

We find the eigenvalues by setting

$$\det(A - \lambda I) = 0$$

and the eigenvectors by solving

$$(A - \lambda I)F = 0$$

Since A is real, any complex eigenvalues must occur in conjugate pairs. For example, if there is an eigenvalue

$$\lambda_1 = \alpha + i\beta$$

then there has to be a second eigenvalue of the form

$$\lambda_2 = \alpha - i\beta$$

Suppose that the eigenvector $F^{(1)}$ that corresponds to the eigenvalue λ_1 is of the form

$$F^{(1)} = a + ib$$

where a and b are both real vectors. Then one solution is

$$x = (a + ib)e^{(\alpha+i\beta)t}$$

and so

$$\begin{aligned} x &= (a + ib)e^{\alpha t}(\cos(\beta t) + i\sin(\beta t)) \\ &= a\cos(\beta t)e^{\alpha t} - b\sin(\beta t)e^{\alpha t} + ia\sin(\beta t)e^{\alpha t} + ib\cos(\beta t)e^{\alpha t} \end{aligned}$$

We can get two solutions by taking real and imaginary parts. It might not be obvious but $\det(A - \lambda I) = 0$ is always a polynomial of degree n (where A is an $n \times n$ matrix). To solve an arbitrary polynomial, we need complex numbers because we can then find exactly n roots over the complex field.

Example 118. Solve

$$x' = -y, \qquad y' = x$$

We find the eigenvalues:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{aligned} \det(J - \lambda I) &= \det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} \\ &= \lambda^2 + 1 = 0 \\ \lambda &= \pm i \end{aligned}$$

We find an eigenvector for $\lambda_1 = i$:

$$\begin{aligned} \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ f_1 - i f_2 &= 0 \qquad \text{(both equations)} \\ f_1 &= i f_2 \end{aligned}$$

Therefore, $f = \begin{pmatrix} i \\ 1 \end{pmatrix}$ is an eigenvector.

We find an eigenvector for $\lambda_2 = -i$:

$$\begin{aligned} \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ f_1 + i f_2 &= 0 \\ f_1 &= -i f_2 \end{aligned}$$

Therefore, $f = \begin{pmatrix} -i \\ 1 \end{pmatrix}$ is an eigenvector.

The general solution is

$$x = A \begin{pmatrix} i \\ 1 \end{pmatrix} e^{it} + B \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{-it}$$

But suppose we want two real solutions? (Since A is real, this seems reasonable.) Two independent solutions are

$$\vec{u} = \begin{pmatrix} i \\ 1 \end{pmatrix} e^{it} \qquad \text{and} \qquad \vec{v} = \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{-it}$$

We can rearrange $\vec{u}$ and $\vec{v}$:

$$\begin{aligned} \vec{u} &= \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] (\cos(t) + i \sin(t)) \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(t) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(t) + i \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(t) \right] \\ \vec{v} &= \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] (\cos(t) - i \sin(t)) \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(t) + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \sin(t) + i \left[\begin{pmatrix} -1 \\ 0 \end{pmatrix} \cos(t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(t) \right] \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(t) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(t) - i \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(t) \right] \end{aligned}$$

Any linear combination of $\vec{u}$ and $\vec{v}$ is also a solution so

$$\frac{1}{2}(\vec{u}+\vec{v}) = Re(\vec{u}) \qquad \text{and} \qquad \frac{1}{2i}(\vec{u}-\vec{v}) = Im(\vec{u})$$

are also solutions (i.e., we choose $A = \frac{1}{2}$ and $B = \frac{1}{2}$ in the first, and we choose $A = \frac{1}{2i}$ and $B = -\frac{1}{2i}$ in the second).

Thus two independent real solutions are

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(t) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(t) \qquad \text{and} \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(t)$$

Example 119. Solve

$$\begin{aligned} x' &= -x - y \qquad & x(0) &= 3 \\ y' &= 4x - y \qquad & y(0) &= 4 \end{aligned}$$

Find the eigenvalues:

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} -1-\lambda & -1 \\ 4 & -1-\lambda \end{pmatrix} \\ &= (-1-\lambda)^2 + 4 \\ &= \lambda^2 + 2\lambda + 5 \\ \lambda &= \frac{-2 \pm \sqrt{4-20}}{2} \\ &= -1 \pm 2i \end{aligned}$$

Find the eigenvectors corresponding to $\lambda_1 = -1 + 2i$:

$$A - \lambda_1 I = \left(\begin{array}{cc|c} -2i & -1 & 0 \\ 4 & -2i & 0 \end{array}\right) \xrightarrow{R_2 - 2iR_1} \left(\begin{array}{cc|c} -2i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$-2iv_1 - v_2 = 0$$

$$v_1 = s, \qquad v_2 = -2is$$

$$\vec{v_1} = \begin{pmatrix} 1 \\ -2i \end{pmatrix} s, \qquad s \neq 0$$

$$\begin{aligned} \therefore\ \vec{x_1} &= \begin{pmatrix} 1 \\ -2i \end{pmatrix} e^{(-1+2i)t} \qquad \text{is a complex solution} \\ &= \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ -2 \end{pmatrix} \right] e^{-t}(\cos(2t) + i\sin(2t)) \\ &= e^{-t} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2t) - \begin{pmatrix} 0 \\ -2 \end{pmatrix} \sin(2t) \right] + ie^{-t} \left[\begin{pmatrix} 0 \\ -2 \end{pmatrix} \cos(2t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(2t) \right] \end{aligned}$$

But $Re(\vec{x_1})$ and $Im(\vec{x_1})$ are themselves solutions.

$$\begin{aligned} \therefore\ \vec{w_1} &= e^{-t} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2t) - \begin{pmatrix} 0 \\ -2 \end{pmatrix} \sin(2t) \right] \qquad \text{is a solution} \\ \vec{w_2} &= e^{-t} \left[\begin{pmatrix} 0 \\ -2 \end{pmatrix} \cos(2t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(2t) \right] \qquad \text{is a solution} \end{aligned}$$

Therefore we have two solutions which is enough (no need to do $\lambda = -1 - 2i$). A general solution is thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-t}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos(2t) - \begin{pmatrix} 0 \\ -2 \end{pmatrix}\sin(2t)\right] + c_2 e^{-t}\left[\begin{pmatrix} 0 \\ -2 \end{pmatrix}\cos(2t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin(2t)\right]$$

Finally, we need to substitute the initial conditions:

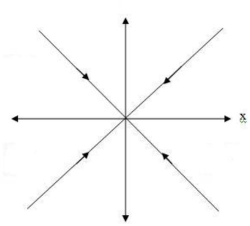
$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$c_1 = 3, \qquad -2c_2 = 4 \to c_2 = -2$$

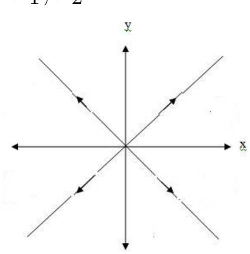
$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= 3e^{-t}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos(2t) - \begin{pmatrix} 0 \\ -2 \end{pmatrix}\sin(2t)\right] - 2e^{-t}\left[\begin{pmatrix} 0 \\ -2 \end{pmatrix}\cos(2t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin(2t)\right] \\ &= 3e^{-t}\begin{pmatrix} \cos(2t) \\ 2\sin(2t) \end{pmatrix} - 2e^{-t}\begin{pmatrix} \sin(2t) \\ -2\cos(2t) \end{pmatrix} \\ &= \begin{pmatrix} 3e^{-t}\cos(2t) - 2e^{-t}\sin(2t) \\ 6e^{-t}\sin(2t) + 4e^{-t}\cos(2t) \end{pmatrix} \end{aligned}$$

23 Stability in nonlinear systems

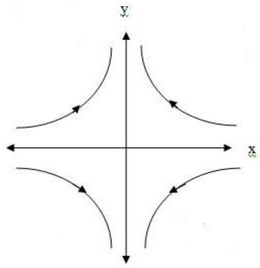
Recall that the Jacobian came from the linearization of the system. Therefore, for nonlinear systems, the stability of an equilibrium is the same as for the linearization, therefore we still look at the eigenvalues of J. If λ_1, λ_2 are both real and negative - stable system



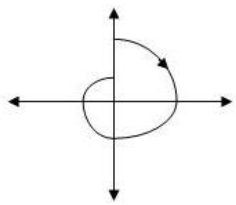
If λ_1, λ_2 are both real and positive - unstable system



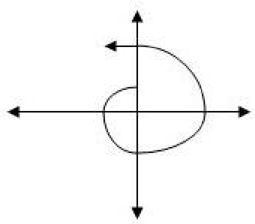
If λ_1, λ_2 are both real and have different signs - saddle



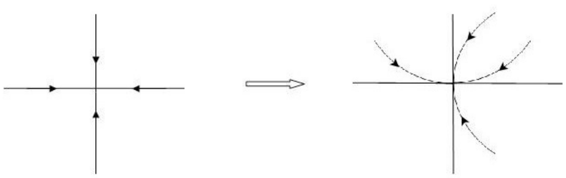
If λ_1, λ_2 are both complex and $Re(\lambda) < 0$ - stable spiral



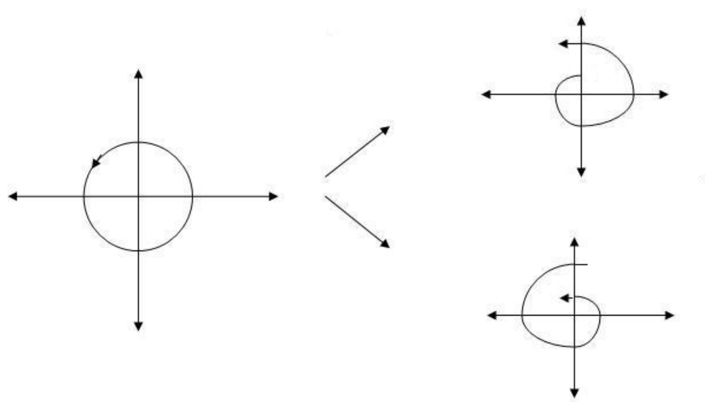
If λ_1, λ_2 are both complex and $Re(\lambda) > 0$ - unstable spiral



For a non-linear system, we have:



One exception: A centre in a linear system may become a spiral - stable or unstable.



Example 120. Find all equilibria for the following system and determine their stability:

$$x' = xy^2 + x - 3y^2 - 3 \tag{1}$$
$$y' = x^4y + y - 2x^4 - 2 \tag{2}$$

$$\begin{aligned}
&\textit{Rearranging equation } (1):\\
&x(y^2+1) - 3(y^2+1) = 0\\
&(x-3)(y^2+1) = 0 \rightarrow x = 3\\
&\textit{Rearranging equation } (2):\\
&(x^4+1)y - 2(x^4+1) = 0\\
&(x^4+1)(y-2) = 0 \rightarrow y = 2\\
&\therefore (3,2) \textit{is the only equilibrium.}\\
&J = \begin{bmatrix} y^2+1 & 2xy-6y \\ 4x^3y-8x^3 & x^4+1 \end{bmatrix}\\
&J(3,2) = \begin{bmatrix} 5 & 0 \\ 0 & 82 \end{bmatrix}\\
&\lambda = 5, 82 \therefore \textit{unstable source.}
\end{aligned}$$

Example 121. Find all equilibria for the following system and determine their stability:

$$x' = xy - 2x - 5y + 10 \tag{3}$$
$$y' = xy - 4x + y - 4 \tag{4}$$

Rearranging equation (3) :

$x(y-2) - 5(y-2) = 0$

$(x-5)(y-2) = 0 \rightarrow x = 5, y = 2$

Rearranging equation (4) :

$x(y-4) + y - 4 = 0$

$(x+1)(y-4) = 0 \rightarrow x = -1, y = 4$

$\therefore$ *the equilibria are* $(5,4), (-1,2)$

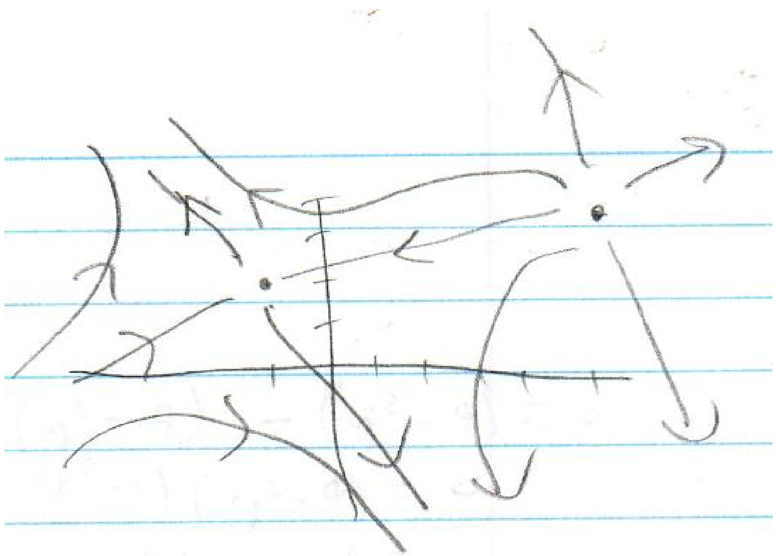
$$J = \begin{bmatrix} y-2 & x-5 \\ y-4 & x+1 \end{bmatrix}$$

$$J(5,4) = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$$

$\lambda = 2, 6 \therefore$ *unstable source.*

$$J(-1,2) = \begin{bmatrix} 0 & -6 \\ -2 & 0 \end{bmatrix}$$

$\det(J - \lambda I) = \lambda^2 - 12$



$\lambda = \pm\sqrt{12} \therefore$ *unstable saddle.*

Example 122. Find all equilibria for the following system and determine their stability:

$$x' = xy^2 - x - 3y^2 + 3 \quad (5)$$
$$y' = xy - 7x - 4y + 28 \quad (6)$$

Rearranging equation (5) :

$x(y^2-1)-3(y^2-1)=0$

$x=3, y=\pm1$

Rearranging equation (6) :

$x(y-7)-4(y-7)=0$

$x=4, y=7$

Equilibria : $(3,7),(4,1),(4,-1)$

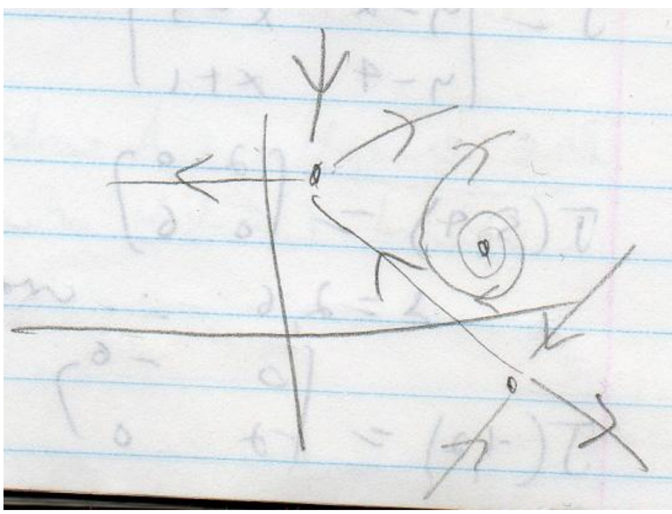
$$J=\begin{bmatrix} y^2-1 & 2y(x-3) \\ y-7 & x-4 \end{bmatrix}$$

$$J(3,7)=\begin{bmatrix} 48 & 0 \\ 0 & -1 \end{bmatrix} \qquad \lambda = 48, -1 \rightarrow \textit{saddle}$$

$$J(4,1)=\begin{bmatrix} 0 & 2 \\ -6 & 0 \end{bmatrix} \qquad \lambda = \pm\sqrt{12}i \rightarrow \textit{centre}$$

$$J(4,-1)=\begin{bmatrix} 0 & -2 \\ -8 & 0 \end{bmatrix} \qquad \lambda = \pm 4 \rightarrow \textit{saddle}$$

The result is two saddles and either a stable or an unstable spiral.

Be careful: solutions must be consistent. Neighbouring trajectories cannot have opposite directions. Trajectories cannot cross or suddenly change direction, except at an equilibrium.

Example 123. Find all equilibria for the following system and determine their stability:

$$x' = xy - 7x - 3y + 21 \tag{7}$$
$$y' = x^2y^2 - 4x^2 - y^2 + 4 \tag{8}$$

Rearranging equation (7) :
$x(y-7) - 3(y-7) = 0$
$(x-3)(y-7) = 0$
$x = 3, y = 7$

Next, substitute these values into equation (8). When $x = 3$, we have

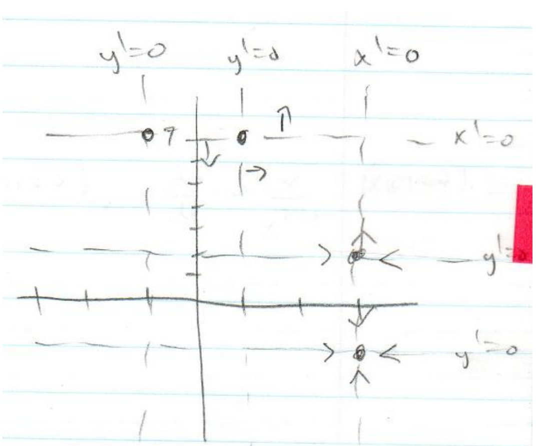
$$9y^2 - 4(9) - y^2 + 4 = 0$$
$$8y^2 = 32$$
$$y = \pm 2$$

Thus $(x, y) = (3, 2)$ and $(x, y) = (3, -2)$ are both equilibria.
Finally, when $y = 7$, we have

$$49x^2 - 4x^2 - 49 + 4 = 0$$
$$45x^2 = 45$$
$$x = \pm 1$$

Thus the remaining equilibria are $(x, y) = (1, 7)$ and $(x, y) = (-1, 7)$.
(Note: (1,2), (3,7) etc are not equilibria)
Nullclines:

x-nullclines: $(x-3)(y-7) = 0$
y-nullclines: $(x-1)(x+1)(y-2)(y+2) = 0$



$$J = \begin{bmatrix} y-7 & x-3 \\ 2x(y^2-4) & 2y(x^2-1) \end{bmatrix}$$

$$J(3,2) = \begin{bmatrix} -5 & 0 \\ 0 & 32 \end{bmatrix} \qquad \lambda = -5, 32 \to \textit{saddle}$$

$$J(3,-2) = \begin{bmatrix} -9 & 0 \\ 0 & -32 \end{bmatrix} \qquad \lambda = -9, -32 \to \textit{sink}$$

$$J(1,7) = \begin{bmatrix} 0 & -2 \\ 90 & 0 \end{bmatrix} \qquad \lambda = \pm\sqrt{180}i \to \textit{centre}$$

$$J(-1,7) = \begin{bmatrix} 0 & -4 \\ -90 & 0 \end{bmatrix} \qquad \lambda = \pm\sqrt{360} \to \textit{saddle}$$

Exercise. Find all equilibria for the following system and determine their stability:

$$x' = y^2 - 4$$
$$y' = x^2 - 1$$

24 Review List

Integration

- Definite/indefinite integrals
- Fundamental theorem of calculus
- Applications
- Improper integrals
- Partial fractions and volumes
- Examples:

$$\int_0^1 \frac{1}{x^2}\,dx$$
$$\int_0^\infty \frac{1}{\sqrt[3]{x}}\,dx$$
$$\int \frac{x^2+3x+1}{x^2+x}$$

Differential equations (one variable)

- Separation of variables
- Equilibria
- Stability
- Phase line diagrams
- Examples: - Logistic equation
  - SIS
  - $\frac{dy}{dt} = \frac{yt}{1+t}$ (y(0)=1)
  - $\frac{dx}{dt} = \frac{x}{x^2-1}$ (x(0)=1)
  - $\frac{dy}{dt} = y \sin y$ (Hint: don't solve, but draw the diagram.)

Complex Numbers

- Vectors
- Matrices
- Row reduction
- Eigenvalues and Eigenvectors
- Markov Chains
- Examples

$$Solve\ Ax = b\ for:$$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & -2 \\ 2 & 4 & 4 \end{bmatrix},\ b = \begin{bmatrix} 6 \\ 10 \\ 8 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -2 & -4 \\ 3 & 7 & 5 \\ 4 & 5 & 1 \end{bmatrix},\ b = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$$

$$Find\ eigenvalues\ and\ eigenvectors\ of\ A = \begin{bmatrix} 3 & -1 & 6 \\ 0 & 2 & 1 \\ 0 & 0 & -4 \end{bmatrix}$$

Multivariate Calculus

- Domain and Range
- Limits
- Partial derivatives
- Tangent lines and planes
- Linearisation

- Examples
  - Find the domain, range and derivatives of $z = \sin\sqrt{2x - y}$
  - Sketch the domain and three level curves
  - Find the domain, range, derivatives, and tangent plane at (2,1) for the equation $z = \ln xy$

Systems of differential equations

- Equilibria
- Stability
- Nullclines
- Phase plane
- Jacobian Matrix
- Stability in linear/nonlinear systems
- Examples

$$
\begin{aligned}
1.\ x' &= y^2 - 4 \\
y' &= x^2 - 1 \\
2.\ x' &= xy - y \\
y' &= 2x - xy \\
3.\ x' &= xy - 7x - 3y + 21 \\
y' &= x^2y^2 - 4x^2 - y^2 + 4
\end{aligned}
$$