MAT1332: Calculus for the Life Sciences II - Part 1

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1 Review of integrals

1.1 Power rule

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \text{ if } n \neq -1$$

Example 1.

$$\int t^3 dt = \frac{t^4}{4} + c$$

Example 2.

$$\int y^{-3} dy = \frac{y^{-2}}{-2} + c$$

1.2 Special Functions

Exponentials:

$$\int e^x dx = e^x + c$$

Logarithms:

$$\int \frac{1}{x} dx = \ln|x| + c$$

Trigonometric functions:

$$\int \sin x dx = -\cos x + c$$

$$\int \cos x dx = \sin x + c$$

Example 3.

$$\int \frac{1}{1+y^2} dy = \arctan y + c$$

Hint:

$$\arctan(\tan x) = x$$

Let $y = \tan x$. Then, differentiating both sides with respect to x, we have

$$\begin{aligned}
\frac{d}{dx}\arctan y &= \frac{d}{dx}x \\
\text{Chain rule}: \frac{d}{dy}\arctan y \cdot \frac{d}{dx}\tan x &= 1 \\
\frac{d}{dy}\arctan y \cdot \frac{d}{dx}\frac{\sin x}{\cos x} &= 1 \\
\frac{d}{dy}\arctan y \cdot \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2(x)} &= 1 \\
\frac{d}{dy}\arctan y \cdot \frac{\cos^2 x + \sin^2 x}{\cos^2 x} &= 1 \\
\frac{d}{dy}\arctan y \cdot (1 + \tan^2 x) &= 1 \\
\frac{d}{dy}\arctan y \cdot (1 + y^2) &= 1 \\
\frac{d}{dy}\arctan y &= \frac{1}{1+y^2} \\
\therefore \int \frac{1}{1+y^2}dy &= \arctan y + c
\end{aligned}$$

1.3 Substitution

$$\begin{aligned}
\frac{d}{dx}f(g(x)) &= f'(g(x))g'(x) \\
f(g(x)) &= \int f'(g(x))g'(x)dx \\
\textit{substitute } u &= g(x) \\
\frac{du}{dx} &= g'(x) \\
dx &= \frac{du}{g'(x)} \\
\textit{then } f(g(x)) &= \int f'(u)g'(x)\frac{du}{g'(x)} \\
&= \int f'(u)du
\end{aligned}$$

Example 4. $\int 3x^2 \sin x^3 dx$

$$\begin{aligned}
\int 3x^2 \sin x^3 dx &= \int 3x^2 \sin u \cdot \frac{du}{3x^2} && u = x^3 \\
&= \int \sin u \cdot du && \frac{du}{dx} = 3x^2 \\
&= -\cos u + c && dx = \frac{du}{3x^2} \\
&= -\cos x^3 + c
\end{aligned}$$

1.4 Integration by Parts

$$\int uv' = uv - \int u'v$$

Example 5. $\int x \cdot \cos x dx$

$$\begin{aligned} u &= x & v' &= \cos x \\ u' &= 1 & v &= \sin x \\ \int x \cdot \cos x dx &= x \sin x - \int \sin x dx \\ &= x \cdot \sin x + \cos x + c \end{aligned}$$

1.5 Important properties

Integrals preserve sums: $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$

Example 6.

$$\int (\frac{1}{x^2} + \frac{1}{x})dx$$

$$\begin{aligned} \int (\frac{1}{x^2} + \frac{1}{x})dx &= \int (x^{-2} + \frac{1}{x})dx \\ &= \int x^{-2}dx + \int \frac{1}{x}dx \\ &= \frac{x^{-1}}{-1} + \ln|x| + c \\ &= -\frac{1}{x} + \ln|x| + c \end{aligned}$$

Integrals preserve constant products: $\int af(x)dx = a \int f(x)dx$

Example 7.

$$\int 6x^{-1}dx = 6 \int x^{-1}dx = 6 \ln|x| + c$$

Example 8. $\int 3xe^{x^2}dx$

Try moving the constant outside: $\int 3xe^{x^2}dx = 3 \int xe^{x^2}dx$, which is not very helpful.

Try integration by parts:

$$\begin{aligned} & u = 3x & & v' = e^{x^2} \\ & u' = 3 & & v = ?? \\ & Or \\ & u = e^{x^2} & & v' = 3x \\ & u' = 2xe^{x^2} & & v = \frac{3}{2}x^2 \end{aligned}$$

Then $\int 3xe^{x^2}dx = \frac{3}{2}x^2e^{x^2} - \int 3x^3e^{x^2}dx$ which is more complicated.

Try substitution:

$$Try\ u = e^{x^2}?$$
$$\frac{du}{dx} = 2xe^{x^2} \text{does not obviously cancel (though this actually works)}$$
$$Try\ u = 3x?$$
$$\frac{du}{dx} = 3$$
$$\int 3xe^{x^2}\frac{du}{3} = \int \frac{u}{3}e^{\frac{u^2}{9}}du \text{ is not significantly different}$$
$$Try\ u = x^2?$$
$$\frac{du}{dx} = 2x$$
$$\begin{aligned}\int 3xe^u\frac{du}{2x} &= \int \frac{3}{2}e^u du \\ &= \frac{3}{2}\int e^u du \\ &= \frac{3}{2}e^u + c \\ &= \frac{3}{2}e^{x^2} + c\end{aligned}$$

Example 9. Solve $\int (5x^4 - 2x^3 + 3)dx$

$$\int (5x^4 - 2x^3 + 3)dx = \frac{5x^5}{5} - \frac{2x^4}{4} + 3x + c = x^5 - \frac{x^4}{2} + 3x + c$$

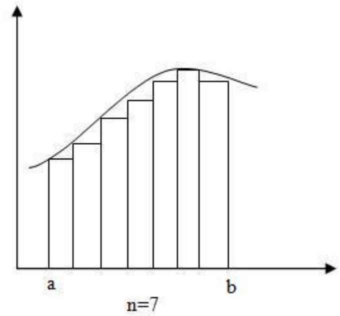
What are some applications of integrals?

- Averages
- Probabilities
- Areas
- Sums

Exercise: $\int \frac{(t+3)^2}{t}dt$
Exercise: $\int x^3e^x dx$
Exercise: $\int \ln x dx$
Exercise: $\int x \ln x dx$
Exercise: $\int x \sin x \cos x dx$
Exercise: $\int e^x \cos x dx$

2 Integrals

Definition 2.1. *The integral is defined as* $\int_a^b f(t)dt = \lim_{n\to\infty}\sum_{i=1}^{n-1} f(t_i)\Delta t$ *where the values* $t_0, ..., t_n$ *break the interval from* a *to* b *into* n *pieces, each of width* $\Delta t = \frac{b-a}{n}$.

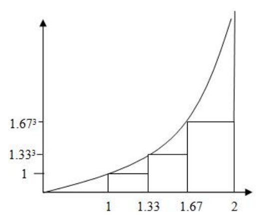


We call these Reimann Sums.

2.1 Riemann Sums

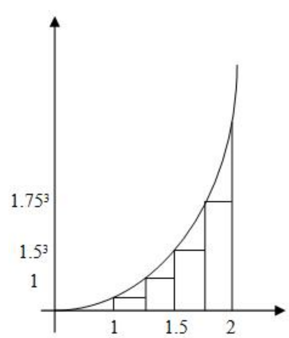
Example 10. Evaluate $\int_1^2 x^3dx$ using 3, 4 and 10 Riemann sums.

$$Area \approxeq \frac{1}{3}(1)^3 + \frac{1}{3}(1.33)^3 + \frac{1}{3}(1.67)^3 = \frac{8}{3} = 2.667$$



For 4 sums, we have:

$$Area \approxeq \frac{1}{4}(1)^3 + \frac{1}{4}(1.25)^3 + \frac{1}{4}(1.5)^3 + \frac{1}{4}(1.75)^3 = 2.922$$



For 10 sums, we have:

$$Area \approxeq 0.1(1^3 + 1.1^3 + 1.2^3 + 1.3^3 + 1.4^3 + 1.5^3 + 1.6^3 + 1.7^3 + 1.8^3 + 1.9^3) = 3.4075$$

3 Definite and Indefinite Integrals

Theorem 3.1. *(The Fundamental Theorem of Calculus) Suppose* $\frac{dF}{dt} = f(t)$*. The indefinite integral is* $\int f(t)dt = F(t) + c$*.*

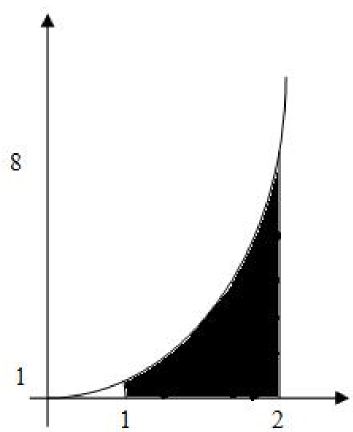
The definite integral is

$$\int_a^b f(t)dt = [F(t)+c]_{evaluated\ at\ b} - [F(t)+c]_{evaluated\ at\ a}$$
$$= (F(b)+c) - (F(a)+c) = F(b) - F(a).$$

Example 11.

$$\int_1^2 x^3 dx$$

$$\begin{aligned}\int_1^2 x^3 dx &= \frac{x^4}{4}\Big|_1^2 \\ &= \frac{2^4}{4} - \frac{1^4}{4} \\ &= \frac{16}{4} - \frac{1}{4} \\ &= 4 - \frac{1}{4} \\ &= 3\frac{3}{4} \\ &= \frac{15}{4} \\ &= 3.75\end{aligned}$$

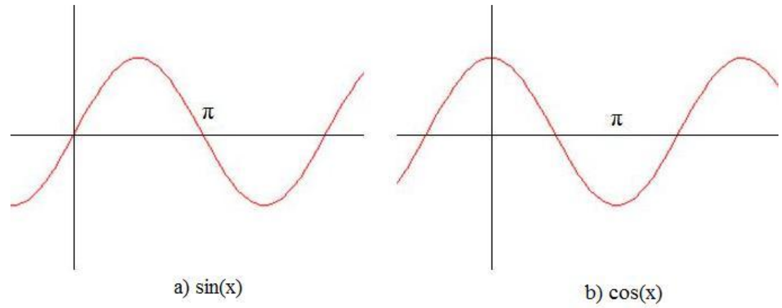


Example 12.

$$\int_0^{\pi} \sin x dx$$

$$\begin{aligned}\int_0^{\pi} \sin x dx &= \big[-\cos x\big]_0^{\pi} \\ &= (-\cos\pi) - (-\cos 0) \\ &= 1 + 1 \\ &= 2\end{aligned}$$

Exercise: Show that $\cos x$ and $\sin x$ have the same area for one whole period.



Example 13. A rock is hurled down from a building 100 m high with inital speed of 5 m/s. How far will it fall during the first second?

Facts: $a = \frac{dv}{dt} = -9.8m/s^2$ $v = \frac{dp}{dt}$

$$\begin{aligned} v &= \int a dt \\ &= \int -9.8 dt \\ &= -9.8t + v_0 \end{aligned}$$

$$\begin{aligned} \text{We know that } v(0) = -9.8(0) + v_0 = -5 &\rightarrow v_0 = -9.8 - 5 \\ v &= -9.8t - 5 \\ p &= \int_0^1 v dt \\ &= \int_0^1 (-9.8t - 5) dt \\ &= \left[\frac{-9.8t^2}{2} - 5t\right]_0^1 \\ &= \left[\frac{-9.8}{2} - 5\right] - [0 - 0] \\ &= -4.9 - 5 \\ &= -9.9 \end{aligned}$$

Therefore it falls down 9.9 metres in the first second.

Exercise: How far will it fall in 4 seconds? How far in 5 seconds? (Be careful)

Example 14. A fish grows at rate $\frac{dL}{dt} = 3e^{-0.5t}$ where t is time in years and L is length in centimetres. How much does it grow between the ages of 3 and 6?

$$\text{We must find } \int_3^6 3e^{-0.5t}dt$$

$$\text{Substitute : } u = -0.5t \qquad \frac{du}{dt} = -0.5 \qquad dt = -2du$$

$$\begin{aligned}\int_3^6 3e^{-0.5t}dt &= \int_{t=3}^{t=6} 3e^u(-2)du \\ &= -6\int_{t=3}^{t=6} e^u du \\ &= -6e^u|_{t=3}^{t=6} \\ &= -6e^{-0.5t}|_{t=3}^{t=6} \\ &= -6e^{-0.5(6)} - (-6e^{-0.5(3)}) \\ &= -6e^{-3} + 6e^{-\frac{3}{2}} \\ &= 1.04\ cm\end{aligned}$$

Exercise: How much does it grow before age 3?

Example 15. The rate at which you learn math is $\frac{dC}{dt} = 50t^2e^{-t}$ where C is a measure of comprehension and t is time in weeks. How much will you learn in 1 week, 6 weeks, 13 weeks?

General solution:

$$C = 50\int t^2e^{-t}dt$$

$$Let\ u = t^2 \qquad v' = e^{-t}$$

$$u' = 2t \qquad v = -e^{-t}$$

$$C = 50[-t^2e^{-t} + 2\int te^{-t}dt]$$

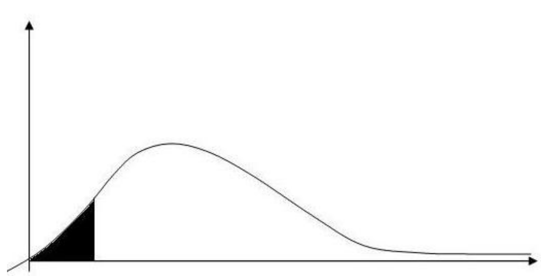
$$Let\ u = 2t \qquad v' = e^{-t}$$

$$u' = 2 \qquad v = -e^{-t}$$

$$\begin{aligned}C &= 50[-t^2e^{-t} - 2te^{-t} + 2\int e^{-t}dt] \\ &= 50[-t^2e^{-t} - 2te^{-t} - 2e^{-t}]\end{aligned}$$

In one week:

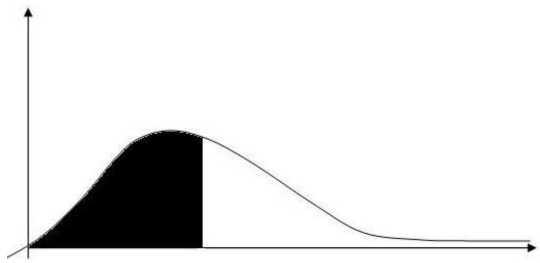
$$\begin{aligned}C &= 50[-t^2e^{-t} - 2te^{-t} - 2e^{-t}]_0^1 \\ &= 50[(-e^{-1} - 2e^{-1} - 2e^{-1}) - (0 - 0 - 2)] \\ &= 50[2 - 5e^{-1}] \\ &= 100 - 250e^{-1}\end{aligned}$$



$$= 8.03\%$$

In 6 weeks:

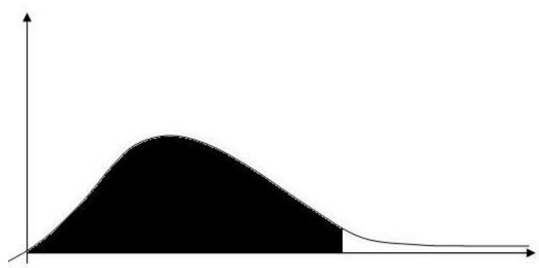
$$\begin{aligned} C &= 50[-t^2e^{-t} - 2te^{-t} - 2e^{-t}]_0^6 \\ &= 50[(-36e^{-6} - 12e^{-6} - 2e^{-6}) - (0 - 0 - 2)] \\ &= 50[-50e^{-6} + 2] \end{aligned}$$



$= 93.8\%$

In 13 weeks:

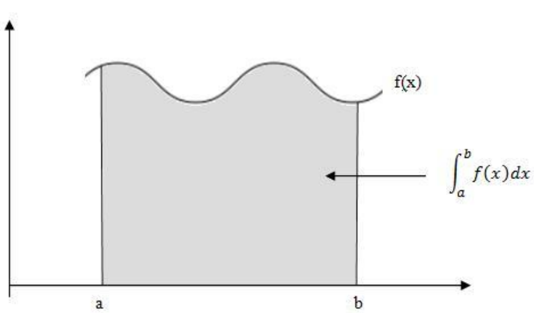
$$\begin{aligned} C &= 50[-t^2e^{-t} - 2te^{-t} - 2e^{-t}]_0^{13} \\ &= 50[(-169e^{-13} - 26e^{-13} - 2e^{-13}) - (0 - 0 - 2)] \\ &= 50[2 - 197e^{-13}] \end{aligned}$$



$= 99.98\%$

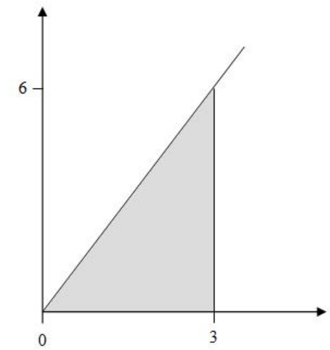
4 Applications of integrals

Definition 4.1. *A definite integral is an area under the curve between the limits.*



Example 16. Find the area under the line $f(x) = 2x$ between 0 and 3.

$$
\begin{aligned}
Area &= \int_0^3 2x dx \\
&= \left[\frac{2x^2}{2}\right]_0^3
\end{aligned}
$$



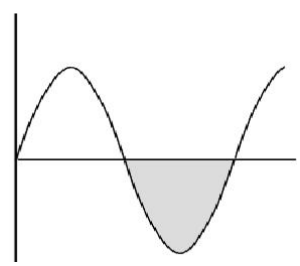
$$= 9 \ units^2$$

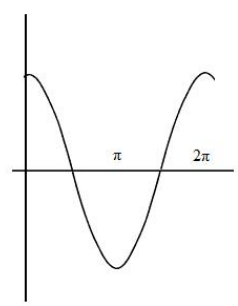
But this area is a triangle with base=3 and height=6:

$$\therefore \ Area = \frac{b \cdot h}{2} = \frac{3 \cdot 6}{2} = 9$$

Example 17. Find the area under the curve $f(x) = \sin x$ between π and 2π.

$$
\begin{aligned}
Area &= \int_\pi^{2\pi} \sin x dx \\
&= [-\cos x]_\pi^{2\pi} \\
&= -\cos 2\pi - (-\cos \pi) \\
&= -1 - (-(-1))
\end{aligned}
$$

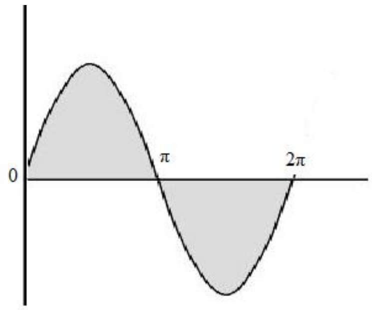




$$= -2$$

How can this be? The definite integral gives a positive area if the curve is above the x-axis. The answer will be negative if it is below.

Example 18. Find the total shaded area for one period of $\sin x$.



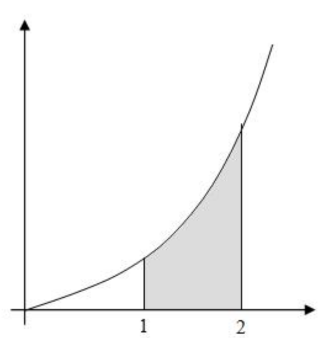
$$
\begin{aligned}
\int_0^{2\pi} \sin x dx &= [-\cos x]_0^{2\pi} \\
&= -\cos 2\pi - (-\cos 0) \\
&= -1 - (-1) \\
&= 0 \qquad \text{which is clearly not the right answer.}
\end{aligned}
$$

Try again using absolute value:

$$
\begin{aligned}
\int_0^{2\pi} |\sin x| dx &= \int_0^{\pi} |\sin x| dx + \int_{\pi}^{2\pi} |\sin x| dx \\
&= \int_0^{\pi} \sin x dx + \int_{\pi}^{2\pi} -\sin x dx \\
&= [-\cos x]_0^{\pi} + [\cos x]_{\pi}^{2\pi} \\
&= [-\cos \pi - (-\cos 0)] + [\cos 2\pi - \cos \pi] \\
&= -(-1) - (-1) + 1 - (-1) \\
&= 4 \; units^2
\end{aligned}
$$

Example 19. Find $\int_2^1 4x^3 dx$.

$$
\begin{aligned}
\int_2^1 4x^3 dx &= [x^4]\,_2^1 \\
&= 1^4 - 2^4 \\
&= 1 - 16
\end{aligned}
$$



$$= -15$$

The answer will be negative if the limits are in the wrong order; that is, the area is positive if the curve is above the area and the limits go from left to right.

In particular

$$\int_b^a f(t) dt = -\int_a^b f(t) dt$$

Example 20. $\int_0^1 (5-2t-t^2)(1+t)dt$. We can try three methods: Substitution, multiplying out and integration by parts.

Try Substitution:

$$\begin{aligned} u &= 5-2t-t^2 \\ \frac{du}{dt} &= -2-2t \\ dt &= \frac{du}{-2(1+t)} \end{aligned}$$

Method 1: Hold limits until the end

$$\begin{aligned} \int_0^1 (5-2t-t^2)(1+t)dt &= \int_{t=0}^{t=1} (u)(1+t)\frac{du}{-2(1+t)} \\ &= -\frac{1}{2}\int_{t=0}^{t=1} udu \\ &= -\frac{1}{2}\left[\frac{u^2}{2}\right]_{t=0}^{t=1} \quad \leftarrow \text{Do not substitute!} \\ &= -\frac{1}{2}\left[\frac{(5-2t-t^2)^2}{2}\right]_{t=0}^{t=1} \\ &= -\frac{1}{4}[(5-2-1)^2-(5-0-0)^2] \\ &= -\frac{1}{4}(2^2-5^2) \\ &= -\frac{1}{4}(4-25) \\ &= -\frac{1}{4}(-21) \\ &= \frac{21}{4} \end{aligned}$$

Method 2: Change limits for new variable

$$\begin{aligned} \text{From the substitution we find } t=0 &\rightarrow u=5-0-0=5 \\ t=1 &\rightarrow u=5-2-1=2 \\ \int_0^1 (5-2t-t^2)(1+t)dt &= \int_{u=5}^{u=2} u(1+t)\frac{du}{-2(1+t)} \\ &= -\frac{1}{2}\int_5^2 udu \\ &= -\frac{1}{2}\left[\frac{u^2}{2}\right]_5^2 \\ &= -\frac{1}{2}\left[\frac{2^2}{2}-\frac{5^2}{2}\right] \\ &= -\frac{1}{2}\left[\frac{4-25}{2}\right] \\ &= \frac{21}{4} \end{aligned}$$

Either of these methods are fine, but you have to do one of them. Do **not** put the original limits in the answer for the new variable.

Exercise: Try multiplying out and integration by parts.

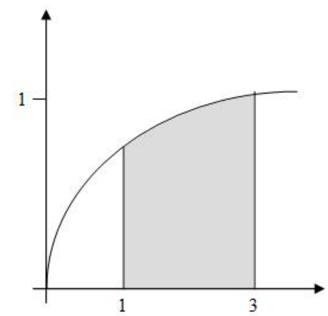
4.1 Integration and Averages

Recall that a rate is an amount per time. That is, *average rate* $= \frac{\text{total amount}}{\text{total time}}$.

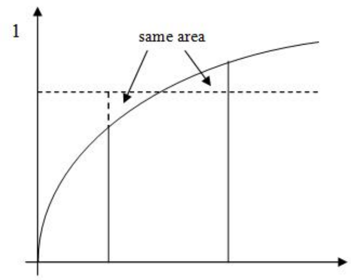
Example 21. Water flows into a vessel at a rate of $1 - e^{-t}\ cm^3/s$. What is the average rate at which water enters between $t = 1$ and $t = 3$?

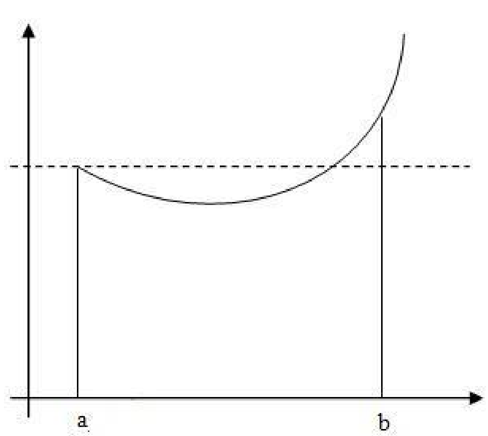
$$\begin{aligned} \textit{Total water entering} &= \int_1^3 (1 - e^{-t})dt \\ &= \left[t - \frac{e^{-t}}{-1}\right]_1^3 \\ &= \left[t + e^{-t}\right]_1^3 \\ &= (3 + e^{-3}) - (1 + e^{-1}) \\ &= 1.682 \end{aligned}$$

$$\textit{Total time} = 3 - 1 = 2$$



$$\therefore\ \textit{average rate} = \frac{1.682}{2} = 0.841\ cm^3/s.$$





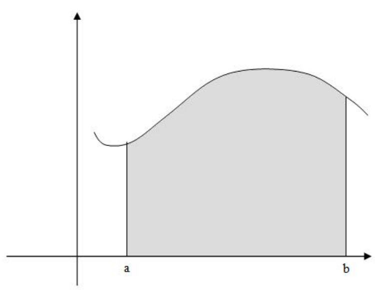
In general, the average value of $f = \frac{1}{b-a}\int_a^b f(x)dx$. The area under the average (between a and b) is equal to the area under the curve (between a and b).

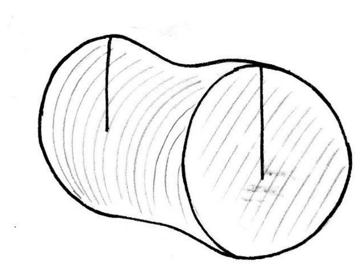
Example 22. In its first decade, the number of AIDS cases in the U.S. followed the formula $\frac{dA}{dt} = 523.8t^2$ where t is the time in years. How many people, on average, were infected each day during this decade?

$$\begin{aligned}
\overline{A} &= \frac{1}{10-0}\int_0^{10} 523.8t^2 dt \\
&= \frac{1}{10} 523.8 \frac{t^3}{3}\Big|_0^{10} \\
&= \frac{1}{10} 523.8 \frac{(10^3)}{3} \\
&= 17,460 \textit{ per year} \\
\textit{Average} &= \frac{17,460}{365} = 47.8 \textit{ per day}
\end{aligned}$$

5 Volumes of Revolution

If we rotate an area under a function around the x-axis, it forms a 3-dimensional solid, called a volume of revolution.



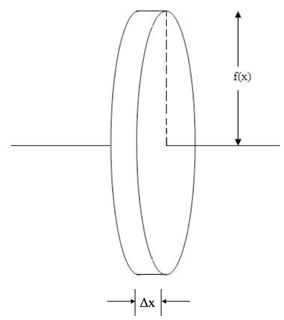


How can we find the volume of such an object?
Consider a small section. If it has width Δx (height) and height $f(x)$ (radius).

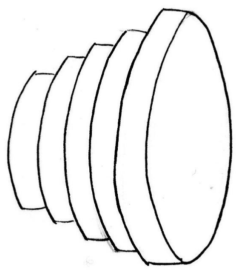
The volume of a cylinder is $V = \pi r^2 h = \pi f(x)^2 \Delta x$. If we have many of these, the volume is approximately

$$V \approx \sum_{i=1}^{n} \pi f(x_i)^2 \Delta x$$

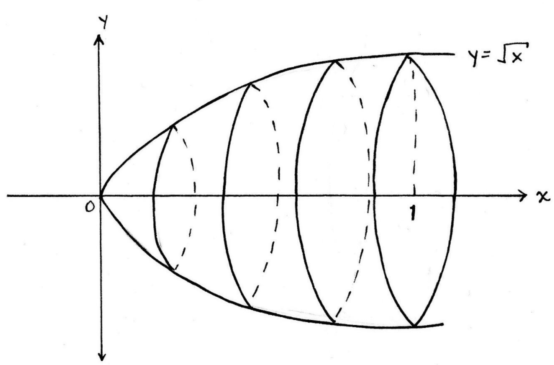
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \pi f(x_i)^2 \Delta x$$



$$= \int_a^b \pi f(x)^2 dx$$

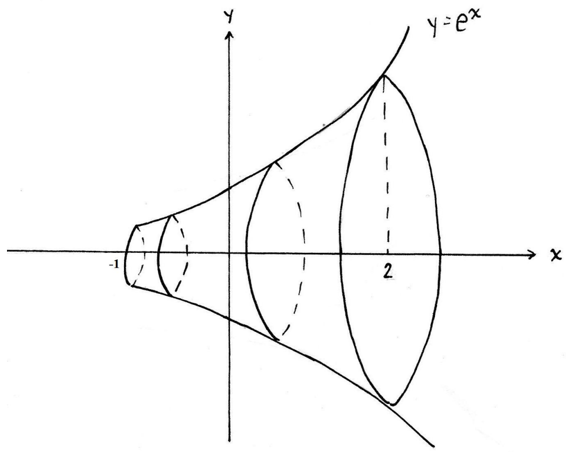


Example 23. Find the volume of the solid of revolution generated by rotating the region under the graph of $y = \sqrt{x}$ from x=0 to x=1 about the x-axis.



$$
\begin{aligned}
V &= \int_0^1 \pi f(x)^2 dx \\
&= \int_0^1 \pi(\sqrt{x})^2 dx \\
&= \int_0^1 \pi x dx \\
&= \pi \frac{x^2}{2}\Big|_0^1 \\
&= \frac{\pi}{2} = 1.57 \; units^3
\end{aligned}
$$

Example 24. Find the volume of the solid of revolution generated by rotating the region under $y = e^x$ from x=-1 to x=2 about the x-axis.

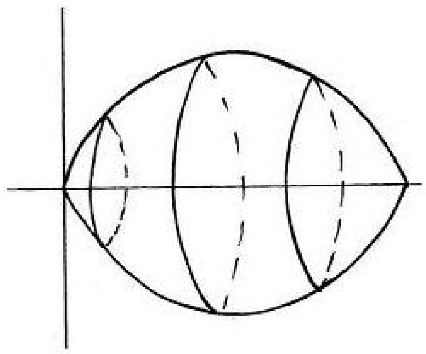


$$
\begin{aligned}
V &= \int_{-1}^2 \pi(e^x)^2 dx \\
&= \int_{-1}^2 \pi e^{2x} dx \\
&= \frac{\pi e^{2x}}{2}\Big|_{-1}^2 \\
&= \frac{\pi}{2}(e^4 - e^{-2}) \\
&= 85.55 \; units^3
\end{aligned}
$$

Example 25. Find the volume of the solid of revolution generated by rotating $y = \sin x$ from x=0 to x=π about the x-axis.

$$Recall\ \cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x)$$

$$and\ \sin(2x) = 2\sin(x)\cos(x)$$



$$
\begin{aligned}
V &= \int_0^{\pi} \pi(\sin x)^2\ dx \\
&= \pi \int_0^{\pi} \sin^2(x)\ dx \\
&= \pi \int_0^{\pi} \frac{1-\cos(2x)}{2} dx \\
&= \frac{\pi}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} \\
&= \frac{\pi}{2}\left[(\pi - 0) - (0 - 0)\right] \\
&= \frac{\pi^2}{2} = 4.93\ units^3
\end{aligned}
$$

Exercise: Find the volume of the solid generated by rotating the positive part of $2x - x^2$ around the x-axis.
Answer: $\frac{16\pi}{15} = 3.35$

6 Improper Integrals

Proper integrals have the form $\int_a^b f(x)dx, \quad b < \infty$
Improper integrals have the form $\int_a^{\infty} f(x)dx$.
Why would we use this? Because infinity is a useful abstraction of "very big" or"very far."

Example 26. The rate of sales of a new product is $\frac{dS}{dt} = \frac{400}{(t+2)^2}$ where t is the time in weeks and S is amount in dollars. If the product were on sale forever, how much would it make?

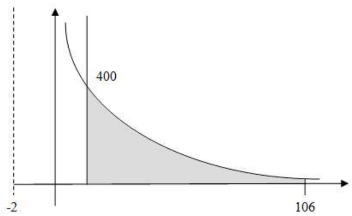
$$
\begin{aligned}
S_\infty &= \int_0^\infty \frac{400}{(t+2)^2}dt & u &= t+2 \\
&= \int_{t=0}^{t=\infty} 400u^{-2}du & \frac{du}{dt} &= 1 \\
&= \left[-400u^{-1}\right]_{t=0}^{\infty} \\
&= \left[-400(t+2)^{-1}\right]_0^\infty \\
&= \left[\frac{-400}{\infty+2}\right] - \left[\frac{-400}{0+2}\right] \Leftarrow \mathit{WARNING!} \\
(&= 0 + 200) \\
(&= \$200 \quad) & &\Leftarrow [\mathit{Secret\ Knowledge}]
\end{aligned}
$$

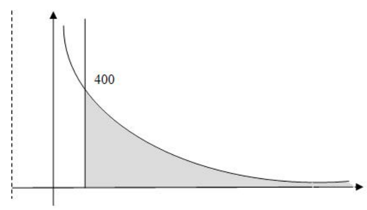
But we didn't really substitute ∞ into an equation. Instead, we take the limit so this is what we really do have:

$$
\begin{aligned}
S_\infty &= \lim_{T\to\infty} \int_0^T 400(t+2)^{-2}dt \\
&= \lim_{T\to\infty} \int_{t=0}^{t=T} 400u^{-2}du \\
&= \lim_{T\to\infty} \left[-400u^{-1}\right]_{t=0}^{t=T} \\
&= \lim_{T\to\infty} \left[-400(t+2)^{-1}\right]_0^T \\
&= \lim_{T\to\infty} \left[-\frac{400}{T+2} + \frac{400}{0+2}\right] \\
&= \lim_{T\to\infty} \left[-\frac{400}{T+2} + 200\right] \\
&= \$200
\end{aligned}
$$

But of course we never really wait forever. What if we waited a really long time, like two years (104 weeks)?

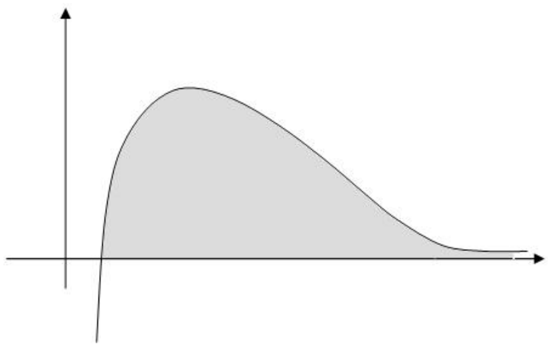
$$
\begin{aligned}
S(2) &= \int_0^{104} 400(t+2)^{-2} dt \\
&= \left[-400(t+2)^{-1}\right]_0^{104} \\
&= -\frac{400}{104+2} + 200 \\
&= -3.77 + 200 \\
&= \$196.23
\end{aligned}
$$





Example 27. $\int_1^\infty \frac{\ln t}{t} dt$
Try substitution:

$$\begin{aligned}
u &= \ln t \\
\frac{du}{dt} &= \frac{1}{t} \\
dt &= t du \\
\int_{t=1}^{t=\infty} \frac{u}{t} t du &= \left.\frac{u^2}{2}\right|_{t=1}^{t=\infty} \\
&= \left.\frac{(\ln t)^2}{2}\right|_1^\infty \\
&= \lim_{T\to\infty} \left.\frac{(\ln t)^2}{2}\right|_1^T \\
&= \infty - 0
\end{aligned}$$



$$= \infty$$

Try integration by parts:

$$\begin{aligned}
u &= \ln t & v' &= \frac{1}{t} \\
u' &= \frac{1}{t} & v &= \ln t
\end{aligned}$$

$$\begin{aligned}
I &= (\ln t)^2\Big|_1^\infty - \int_1^\infty \frac{1}{t} \ln t dt \\
&= (\ln t)^2\Big|_1^\infty - I \\
2I &= (\ln t)^2\Big|_1^\infty \\
I &= \left.\frac{(\ln t)^2}{2}\right|_1^\infty \\
&= \lim_{T\to\infty} \left.\frac{(\ln t)^2}{2}\right|_1^T \\
&= \lim_{T\to\infty} \left[\frac{(\ln T)^2}{2} - \frac{\ln(1)}{2}\right] \\
&= \infty - 0 \\
&= \infty
\end{aligned}$$

If an integral is infinite, we say it <u>diverges</u>.
If an integral is finite, we say it <u>converges</u>.

Example 28. Does $\int_0^\infty e^{-2t}dt$ converge or diverge?

$$\begin{aligned}
\int_0^\infty e^{-2t}dt &= \lim_{T\to\infty}\int_0^T e^{-2t}dt \\
&= \lim_{T\to\infty} \left.\frac{e^{-2t}}{-2}\right|_0^T \\
&= \lim_{T\to\infty}\left[\frac{e^{-2T}}{-2} - \frac{e^0}{-2}\right] \\
&= \lim_{T\to\infty}\left[\frac{-1}{2e^{2T}} + \frac{1}{2}\right] \\
&= \frac{1}{2} \therefore \textit{ converges.}
\end{aligned}$$

Example 29. Does $\int_0^\infty \frac{x}{\sqrt{x+2}}dx$ converge or diverge?

$$\begin{aligned}
&\textit{Let } u = x+2 \\
&du = dx
\end{aligned}$$

$$\begin{aligned}
\int_0^\infty \frac{x}{\sqrt{x+2}}dx &= \lim_{T\to\infty}\int_{x=0}^{x=T} \frac{x}{\sqrt{u}}du \\
&= \lim_{T\to\infty}\int_{x=0}^{x=T} \frac{u-2}{\sqrt{u}}du \qquad (\textit{since } u = x+2, x = u-2) \\
&= \lim_{T\to\infty}\int_{x=0}^{x=T} \left(\frac{u}{\sqrt{u}} - \frac{2}{\sqrt{u}}\right)du \\
&= \lim_{T\to\infty}\int_{x=0}^{x=T} \frac{u}{u^{\frac{1}{2}}} - \frac{2}{u^{\frac{1}{2}}}du \\
&= \lim_{T\to\infty}\int_{x=0}^{x=T} u^{\frac{1}{2}} - 2u^{-\frac{1}{2}}du \\
&= \lim_{T\to\infty}\left[\frac{2u^{\frac{3}{2}}}{3} - 4u^{\frac{1}{2}}\right]_{x=0}^{x=T} \\
&= \lim_{T\to\infty}\left[\frac{2(x+2)^{\frac{3}{2}}}{3} - 4(x+2)^{\frac{1}{2}}\right]_{x=0}^{x=T} \\
&= \lim_{T\to\infty}\left[\frac{2(T+2)^{\frac{3}{2}}}{3} - 4(T+2)^{\frac{1}{2}}\right] - \left[\frac{2(2)^{\frac{3}{2}}}{3} - 4(2)^{\frac{1}{2}}\right] \\
&= \lim_{T\to\infty}(T+2)^{\frac{1}{2}}\left[\frac{2}{3}(T+2) - 4\right] - \left[\frac{2(2)^{\frac{3}{2}}}{3} - 4(2)^{\frac{1}{2}}\right] \\
&= \infty(\infty - 4) - \left[\frac{2(2)^{\frac{3}{2}}}{3} - 4(2)^{\frac{1}{2}}\right] \\
&= \infty \qquad \therefore \textit{ diverges}
\end{aligned}$$

Example 30. $\int_0^\infty \frac{1}{2e^{2x}}dx$

$$
\begin{aligned}
\int_0^\infty \frac{1}{2e^{2x}}dx &= \frac{1}{2}\lim_{T\to\infty}\int_0^T e^{-2x}dx \\
&= \frac{1}{2}\lim_{T\to\infty} -\frac{1}{2}e^{-2x}\Big|_0^T \\
&= -\frac{1}{4}\lim_{T\to\infty}\left[e^{-2T}-1\right] \\
&= -\frac{1}{4}(0-1) \\
&= \frac{1}{4}
\end{aligned}
$$

Example 31. The rate at which you learn is $\frac{dC}{dt} = 50t^2e^{-t}$. How much will you learn if you study forever?

$C_\infty = \int_0^\infty 50t^2e^{-t}dt$

Try taking the constant out - this does not simplify.

Try substitution - nothing cancels.

Try integration by parts:

$$
\begin{aligned}
&u = 50t^2 \qquad\qquad v' = e^{-t} \\
&u' = 100t \qquad\qquad v = -e^{-t} \\
C_\infty &= -50t^2e^{-t} + 100\int_0^\infty te^{-t}dt
\end{aligned}
$$

Use integration by parts again

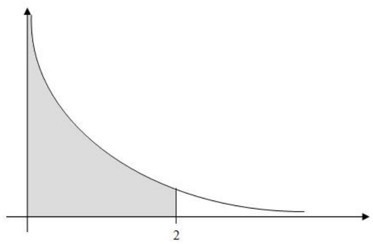
$$
\begin{aligned}
&u = t \qquad\qquad v' = e^{-t} \\
&u' = 1 \qquad\qquad v = -e^{-t} \\
C_\infty &= -50t^2e^{-t} + 100\left[-te^{-t} + \int_0^\infty e^{-t}dt\right] \\
&= \left[-50t^2e^{-t} - 100te^{-t} - 100e^{-t}\right]_0^\infty \\
&= \lim_{T\to\infty}\left[(-50T^2e^{-T} - 100Te^{-T} - 100e^{-T}) - (0-0-100)\right] \\
&= \lim_{T\to\infty}\left[-\frac{50T^2}{e^T} - \frac{100T}{e^T} - \frac{100}{e^T} + 100\right] \\
&= \frac{\infty}{\infty} - \frac{\infty}{\infty} - 0 + 100 \therefore \text{ use l'Hôpital's rule on the first two terms} \\
&= \lim_{T\to\infty}\left[-\frac{100T}{e^T} - \frac{100}{e^T} - \frac{100}{e^T}\right] + 100 \\
&= \frac{\infty}{\infty} - 0 - 0 + 100 \therefore \text{use l'Hôpital's rule again} \\
&= \lim_{T\to\infty}\left[-\frac{100}{e^T} - \frac{100}{e^T} - \frac{100}{e^T}\right] + 100 \\
&= -0 - 0 - 0 + 100 \\
&= 100\%
\end{aligned}
$$

Therefore if you study forever you will learn 100 percent.

7 Infinite Integrands

We want to find $\int_0^2 \frac{1}{\sqrt{x}}dx$. Why is this a problem? Because $\frac{1}{\sqrt{x}}$ is not defined at 0.
But let's try to do what we did with improper integrals:

$$
\begin{aligned}
\int_0^2 \frac{1}{\sqrt{x}}dx &= \lim_{\epsilon \to 0^+} \int_\epsilon^2 \frac{1}{\sqrt{x}}dx \\
&= \lim_{\epsilon \to 0^+} \int_\epsilon^2 x^{-\frac{1}{2}}dx \\
&= \lim_{\epsilon \to 0^+} \left[2x^{\frac{1}{2}}\right]_\epsilon^2 \\
&= \lim_{\epsilon \to 0^+} (2\sqrt{2} - 2\sqrt{\epsilon}) \\
&= 2\sqrt{2} - 0 \\
&= 2\sqrt{2} = 2.828 \text{ where } 0^+ \text{means the limit from the right.}
\end{aligned}
$$



Therefore, even though the function goes to ∞ at 0, the area is well-defined. (It's like the improper integral, only turned sideways.)

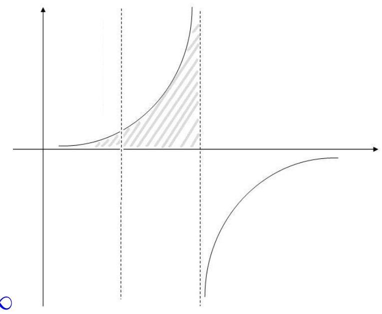
Example 32. $\int_0^3 (-4x^{-2} - 3x^{-1} + 1)dx$

$$
\begin{aligned}
\int_0^3 (-4x^{-2} - 3x^{-1} + 1)dx &= \lim_{\epsilon \to 0^+} \left[4x^{-1} - 3\ln x + x\right]_\epsilon^3 \\
&= \lim_{\epsilon \to 0^+} \left[\frac{4}{3} - 3\ln 3 + 3\right] - \left[\frac{4}{\epsilon} - 3\ln \epsilon + \epsilon\right] \\
&= \lim_{\epsilon \to 0^+} \left[\frac{13}{3} - 3\ln 3 - \frac{4}{\epsilon} + 3\ln \epsilon - \epsilon\right] \\
&= \frac{13}{3} - 3\ln 3 - \infty + 3(-\infty) - 0 \\
&= -\infty
\end{aligned}
$$

Example 33. $\int_0^1 \frac{1}{1-x}dx$

$$\int_0^1 \frac{1}{1-x}dx = \lim_{\epsilon \to 1^-} \int_0^{\epsilon} \frac{1}{1-x}dx$$

$$\begin{aligned}
&\textit{Substitute } u = 1-x \\
&\qquad \frac{du}{dx} = -1 \\
&= \lim_{\epsilon \to 1^-} \int_{x=0}^{x=\epsilon} -\frac{1}{u}du \\
&= \lim_{\epsilon \to 1^-} (-\ln u)\Big|_{x=0}^{x=\epsilon} \\
&= \lim_{\epsilon \to 1^-} -\ln(1-x)\Big|_0^{\epsilon} \\
&= \lim_{\epsilon \to 1^-} (-\ln(1-\epsilon) + \ln(1))
\end{aligned}$$



$$= -\ln 0 + 0 = +\infty$$

Example 34. $\int_{-1}^{3} -\frac{1}{x^2}dx$

The function $-\frac{1}{x^2}$ has domain $\{x \in \mathbb{R} : x \neq 0\}$, which means we need to split the integral at 0.

$$\begin{aligned}
\int_{-1}^{3} -\frac{1}{x^2}dx &= \lim_{\epsilon \to 0^-} \int_{-1}^{\epsilon} -\frac{1}{x^2}dx + \lim_{\epsilon \to 0^+} \int_{\epsilon}^{3} -\frac{1}{x^2}dx \\
&= \lim_{\epsilon \to 0^-} \frac{1}{x}\Big|_{-1}^{\epsilon} + \lim_{\epsilon \to 0^+} \frac{1}{x}\Big|_{\epsilon}^{3} \\
&= \lim_{\epsilon \to 0^-} \left(\frac{1}{\epsilon} + 1\right) + \lim_{\epsilon \to 0^+} \left(\frac{1}{3} - \frac{1}{\epsilon}\right) \\
&= (-\infty + 1) + \left(\frac{1}{3} - \infty\right) \\
&= -\infty
\end{aligned}$$

8 Partial Fractions

First introductory example

We don't know how to integrate the fraction $\int \frac{x}{x+2}dx$, but we can write the fraction in a simpler way and use known rules to find the integral as follows:

$$\int \frac{x}{x+2}dx = \int \frac{(x+2)-2}{x+2}dx = \int \left[1 - \frac{2}{x+2}\right] dx = x - 2\ln|x+2| + C.$$

Second introductory example

We don't know integrate the fraction $\int \frac{3x-2}{x(x-2)}dx$. But if we simplify the fraction as

$$\frac{3x-2}{x(x-2)} = \frac{1}{x} + \frac{2}{x-2}$$

(check this!) then we can integrate as follows:

$$\int \frac{3x-2}{x(x-2)}dx = \int \left(\frac{1}{x} + \frac{2}{x-2}\right) dx = \ln|x| + 2\ln|x-2| + C.$$

General Idea

Rational functions are fractions of polynomials, i.e., if $P(x)$ and $Q(x)$ are polynomials, then $P(x)/Q(x)$ is called a rational function. We already know how to integrate some of them, namely the following building blocks (you need to know these!)

$$\begin{aligned}
\int \frac{1}{x+a}dx &= \ln|x+a| + C, \\
\int \frac{1}{x^2+1}dx &= \arctan(x) + C = \tan^{-1}(x) + C, \\
\int \frac{x}{x^2+1}dx &= \frac{1}{2}\ln(x^2+1) + C.
\end{aligned}$$

(You don't have to memorize the last one; you could use substitution to solve it.)

If we can split a rational function into sums of these building blocks, then we can integrate easily. The goal of this section is to find a technique to integrate (find antiderivatives of) all rational functions. We only consider cases where $\deg(Q) \leq 2$, i.e., the highest power of x in the denominator is no more than 2. The idea is to decompose a rational function into a sum of simpler rational functions, namely the three examples above, which we know how to integrate.

Recipe for partial fractions

To find the integral of a rational function $P(x)/Q(x)$, follow these steps.

1. If $\deg(P) \geq \deg(Q)$ then use long division to split the rational function into several parts. Now assume that $\deg(P) < \deg(Q)$.

2. If $Q(x) = ax^2 + bx + c = a(x-x_1)(x-x_2)$ has two distinct real roots, the one can find numbers A, B such that
$$\frac{P(x)}{Q(x)} = \frac{1}{a}\left[\frac{A}{x-x_1} + \frac{B}{x-x_2}\right].$$
Then use the natural logarithm to integrate the two terms.

3. If $Q(x) = ax^2 + bx + c = a(x-x_1)^2$ has only one real root, the one can find numbers A, B such that
$$\frac{P(x)}{Q(x)} = \frac{1}{a}\left[\frac{A}{x-x_1} + \frac{B}{(x-x_1)^2}\right].$$
Then one can integrate using substitution, the logarithm, and direct integration.

4. If $Q(x) = ax^2 + bx + c$ has no real roots, then complete the square to get
$$Q(x) = a\left[\left(x - \frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2\right] = a[(x-A)^2 + B].$$
Then use the natural logarithm and the arctan to integrate the two terms (potentially substitute first).

We illustrate each of these cases with examples.

Example 35. $P(x) = x^2 + 1, Q(x) = x - 1.$

We have $\deg(P) = 2 > 1 = \deg(Q)$, so we need to do long division. We find

$$x^2 + 1 = (x - 1)(x + 1) + 2.$$

Therefore

$$\int \frac{x^2 + 1}{x - 1} dx = \int \left[x + 1 + \frac{2}{x - 1} \right] dx = \frac{x^2}{2} + x + 2 \ln |x - 1| + C.$$

Example 36. $P(x) = 2x^3 + 3x^2 + 2x + 4, Q(x) = x^2 + 1.$

Again, since $\deg(P) = 3 > 2 = \deg(Q)$, we need to do long division. We find

$$2x^3 + 3x^2 + 2x + 4 = (x^2 + 1)(2x + 3) + 1.$$

Therefore

$$\int \frac{2x^3 + 3x^2 + 2x + 4}{x^2 + 1} dx = \int \left[2x + 3 + \frac{1}{x^2 + 1} \right] dx = x^2 + 3x + \arctan(x) + C.$$

Example 37. $P(x) = 2x + 1, Q(x) = x^2 + x - 2.$

This time, $\deg(P) = 1 < 2 = \deg(Q)$, so no long division is necessary. Instead, we factor Q as $Q(x) = (x - 1)(x + 2)$, so that

$$\frac{2x + 1}{x^2 + x - 2} = \frac{2x + 1}{(x - 1)(x + 2)}.$$

On the other hand, for two numbers, A, B, we find

$$\frac{A}{x - 1} + \frac{B}{x + 2} = \frac{(A + B)x + 2A - B}{(x - 1)(x + 2)}.$$

Comparing with the expression above, we find that $A + B = 2$ and $2A - B = 1$. Hence, $A = B = 1$. Then we integrate

$$\int \left[\frac{2x + 1}{x^2 + x - 2} \right] dx = \int \left[\frac{1}{x - 1} + \frac{1}{x + 2} \right] dx = \ln |x - 1| + \ln |x + 2| + C.$$

Example 38. $P(x) = x + 5, Q(x) = x^2 - 4x + 4.$

Again, $\deg(P) = 1 < 2 = \deg(Q)$, so no long division is necessary. But $Q(x) = (x - 2)^2$, has only a single root, i.e.,

$$\frac{x + 5}{x^2 - 4x + 4} = \frac{x + 5}{(x - 2)^2}.$$

On the other hand, for two numbers, A, B, we find

$$\frac{A}{x - 2} + \frac{B}{(x - 2)^2} = \frac{Ax - 2A + B}{(x - 2)^2}.$$

Comparing with the expression above, we find that $A = 1$ and $-2A + B = 5$. Hence, $A = 1, B = 7$. Then we integrate

$$\int \left[\frac{x + 5}{x^2 - 4x + 4} \right] dx = \int \left[\frac{1}{x - 2} + \frac{7}{(x - 2)^2} \right] dx = \ln |x - 2| - \frac{7}{x - 2} + C.$$

Example 39. $P(x) = 3x + 2, Q(x) = x^2 - 2x + 5.$

No long division necessary. However, Q has no real roots. We complete the square

$$Q(x) = x^2 - 2x + 5 = x^2 - 2x + 1 - 1 + 5 = (x-1)^2 + 4.$$

Now we write

$$\int \frac{3x+2}{x^2-2x+5}dx = \int \frac{3x+2}{(x-1)^2+4}dx = \frac{1}{4}\int \frac{3x+2}{(\frac{x-1}{2})^2+1}dx.$$

This is a case for substitution. We choose $u = \frac{x-1}{2}$ so that $x = 2u+1$ and $dx = 2du$. Then we get

$$\frac{1}{4}\int \frac{3x+2}{(\frac{x-1}{2})^2+1}dx = \frac{1}{2}\int \frac{6u}{u^2+1}du + \frac{1}{2}\int \frac{5}{u^2+1}du.$$

The first of these integrals requires another substitution, $w = u^2+1$, the second is again an arctan. With this we find

$$\frac{1}{2}\int \frac{6u}{u^2+1}du + \frac{1}{2}\int \frac{5}{u^2+1}du = \frac{1}{2}\int \frac{3}{w}dw + \frac{1}{2}\int \frac{5}{u^2+1}du = \frac{3}{2}\ln|w| + \frac{5}{2}\arctan(u) + C.$$

After back-substituting, we find that the integral with respect to x is given by

$$\frac{3}{2}\ln\left|\frac{x^2}{4} - \frac{x}{2} + \frac{5}{4}\right| + \frac{5}{2}\arctan\left(\frac{x-1}{2}\right) + C.$$

Example 40. $P(x) = x^2 - 2, Q(x) = x^2 - 3x + 2.$

Long division first, or the simpler way

$$\frac{x^2-2}{x^2-3x+2} = \frac{x^2-3x+2+3x-4}{x^2-3x+2} = 1 + \frac{3x-4}{x^2-3x+2}.$$

Now, the denominator is $Q(x) = (x-1)(x-2)$, hence we set the partial fractions as

$$\frac{A}{x-1} + \frac{B}{x-2} = \frac{(A+B)x - (2A+B)}{x^2-3x+2}.$$

Hence, we need $A+B = 3$ and $2A+B = 4$, which is given by $A = 1, B = 2$. Now we can integrate

$$\int \frac{x^2-2}{x^2-3x+2}dx = \int \left(1 + \frac{1}{x-1} + \frac{2}{x-2}\right) dx = x + \ln|x-1| + 2\ln|x-2| + C.$$

9 Differential Equations

Your town is suffering an epidemic. Your chances of catching the disease are proportional to the probability you meet a carrier of the disease. That is: (probability that you are in a place at a given time) $\times$ (probability that a carrier is also in given place at given time).

Outline of the process:

- Word problem
- Translate into equations
- Sharpen up equations
- Initial conditions
- One-variable problem

- Separation of variables
- Solve
- Find constant of integration
- Sketch solution
- Biological interpretation

S - susceptible individuals
I - infected individuals

$$\frac{dS}{dt} \propto -SI$$
$$\frac{dI}{dt} \propto SI$$

I increases due to encounters (more people get sick).
S decreases by the same amount (susceptible people become infected).

Sharper: replace "$\propto$" with "$= \beta$". Let's suppose that 10% of people are infected initially.

$$S' = -\beta SI$$
$$I' = \beta SI$$
$$S' + I' = 0$$
$$S + I = N \ (constant)$$
$$\therefore S = N - I$$
$$I' = \beta(N - I)I$$ which is a single autonomous differential equation.

How do we solve this? Using separation of variables.

Steps:
1. Put state variable on one side and time variable (including dt) on the other.
2. Integrate both sides.
3. Set integrals equal to each other.
4. Combine two integrating constants into one.
5. Solve for the state variable (may rewrite the constant in a more convenient form.)
6. Solve for the constant using the intial condition.

$$\frac{dI}{dt} = \beta(N-I)I$$

$$\frac{dI}{(N-I)I} = \beta dt$$

$$\int \frac{dI}{(N-I)I} = \beta \int dt$$

$$\frac{1}{(N-I)I} = \frac{A}{N-I} + \frac{B}{I}$$

$$1 = AI + B(N-I)$$

$$I = 0 \qquad 1 = BN \qquad B = \frac{1}{N}$$

$$I = N \qquad 1 = AN \qquad A = \frac{1}{N}$$

$$\int \left(\frac{1}{N}\frac{1}{N-I} + \frac{1}{N}\frac{1}{I}\right) dI = \beta \int dt$$

$$\frac{1}{N}(-\ln|N-I|) + \frac{1}{N}\ln|I| = \beta t + c \leftarrow \textit{ only one constant.}$$

$$\frac{1}{N}\ln\frac{I}{N-I} = \beta t + c$$

$$\ln\frac{I}{N-I} = \beta N t + cN$$

$$\frac{I}{N-I} = e^{\beta N t}e^{cN} = ke^{\beta N t}$$

$$I = (N-I)ke^{\beta N t}$$

$$I = Nke^{\beta N t} - Ike^{\beta N t}$$

$$I(1 + ke^{\beta N t}) = Nke^{\beta N t}$$

$$I = \frac{Nke^{\beta N t}}{1 + ke^{\beta N t}}$$

$$I(0) = \frac{Nk}{1+k} = \frac{N}{10} \qquad (10\% \textit{ of population infected})$$

$$10k = 1 + k$$

$$9k = 1$$

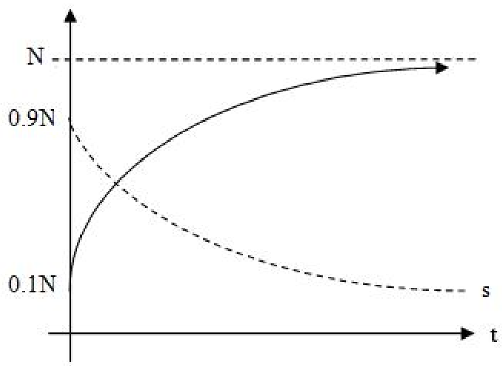
$$k = \frac{1}{9}$$

$$I = \frac{\frac{N}{9}e^{\beta N t}}{1 + \frac{e^{\beta N t}}{9}} = \frac{Ne^{\beta N t}}{9 + e^{\beta N t}}$$

$$\lim_{t\to\infty} I(t) = \lim_{t\to\infty}\frac{Ne^{\beta N t}}{9 + e^{\beta N t}}$$

$$= \lim_{t\to\infty}\frac{\beta N^2 e^{\beta N t}}{\beta N e^{\beta N t}} \quad \text{using l'Hopital's rule}$$

$$= N$$



Therefore, eventually everyone gets infected.

Exercise: Check $I(0) = 0.1N$ and I satisfies $I' = \beta(N-I)I$.

Example 41. Suppose $\frac{dx}{dt} = x + x^2$.

a) Set $y = \frac{1}{x}$ and find a differential equation for y.

b) Solve for x if x(0)=1.

Solution: a)

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2} \\ \frac{dy}{dt} &= \frac{dy}{dx}\frac{dx}{dt} \\ &= -\frac{1}{x^2}(x+x^2) \\ &= -\frac{1}{x} - 1 \\ &= -y - 1\end{aligned}$$

Therefore we have transformed a nonlinear equation into a linear equation.

b)

$$\begin{aligned}\frac{dy}{y+1} &= -dt \\ \int \frac{1}{y+1}dy &= -\int dt \\ \ln|y+1| &= -t + c \\ y + 1 &= e^{-t+c} = Ae^{-t} \\ y &= Ae^{-t} - 1 \\ \frac{1}{x} &= Ae^{-t} - 1 \\ x &= \frac{1}{Ae^{-t} - 1} \\ x(0) &= \frac{1}{A-1} = 1 \\ A - 1 = 1 &\to A = 2 \\ \therefore x &= \frac{1}{2e^{-t} - 1}\end{aligned}$$

What if we solve directly?

$$\frac{dx}{x+x^2} = dt$$
$$\int \frac{dx}{x+x^2} = \int dt$$
$$\frac{1}{x(1+x)} = \frac{A}{x} + \frac{B}{1+x}$$
$$1 = A(1+x) + Bx$$
$$x = 0 \qquad 1 = A$$
$$x = -1 \qquad 1 = -B$$
$$\int \left(\frac{1}{x} - \frac{1}{1+x}\right) dx = \int dt$$
$$\ln|x| - \ln|1+x| = t + c$$
$$\ln\left|\frac{x}{1+x}\right| = t + c$$
$$\frac{x}{1+x} = e^{t+c} = ke^t$$
$$x = ke^t(1+x)$$
$$x(1-ke^t) = ke^t$$
$$x = \frac{ke^t}{1-ke^t}$$
$$x(0) = \frac{k}{1-k} = 1 \to k = 1-k \to 2k = 1 \to k = \frac{1}{2}$$
$$x = \frac{\frac{1}{2}e^t}{1-\frac{1}{2}e^t} = \frac{e^t}{2-e^t}$$

Is this the same answer? Yes since $x = \frac{e^t}{2-e^t}\frac{e^{-t}}{e^{-t}} = \frac{1}{2e^{-t}-1}$.

Example 42. $\frac{dx}{dt} = \frac{x}{2x-1} \qquad x(0) = 1$

$$\frac{2x-1}{x}dx = dt$$
$$\int \frac{2x-1}{x}dx = \int dt$$
$$\int 2 - \frac{1}{x}dx = t + c$$
$$2x - \ln|x| = t + c$$

In this case we can't find the solution explicitly, because we can't isolate x. But we can still find c.

$$x(0) = 1 \Rightarrow 2(1) - \ln(1) = 0 + c \Rightarrow 2 = c$$
$$\therefore 2x - \ln|x| = t + 2 \text{ is the implicit solution.}$$

Exercise: Solve $x' = \frac{x^3-3x}{t}$ with $x(1) = 2$

10 Equilibria

What is a derivative? It is a rate of change. The system is at equilibrium if there is no change. That is, the derivative is zero.

Example 43. $\frac{dx}{dt} = x + x^2$
We know that $x = \frac{1}{Ae^{-t}-1} = \frac{e^t}{A-e^t}$

$$\lim_{t\to\infty} x = -1 \qquad\qquad \lim_{t\to-\infty} x = 0 \text{ with } A \neq 0$$

Finding the equilibria:

$$\begin{aligned}
&x + x^2 = 0\\
&x(1+x) = 0\\
&x = 0, \quad x = -1\\
&A = 0 \Rightarrow x = -1 \text{ always} \therefore \text{ equilibrium.}\\
&A = \infty \Rightarrow x = 0 \text{ always} \therefore \text{ equilibrium.}
\end{aligned}$$

Example 44. Disease example: $I' = \beta(N-I)I$

$$I' = 0 \Rightarrow \beta(N-I)I = 0$$

$$\beta = 0 \qquad N - I = 0 \qquad I = 0$$

$$\beta = 0 \qquad I = N \qquad I = 0$$

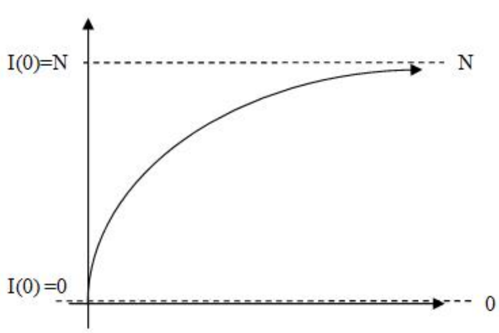
$$\nearrow \qquad \uparrow \qquad \nwarrow$$

no transmission   everyone infected   nobody infected

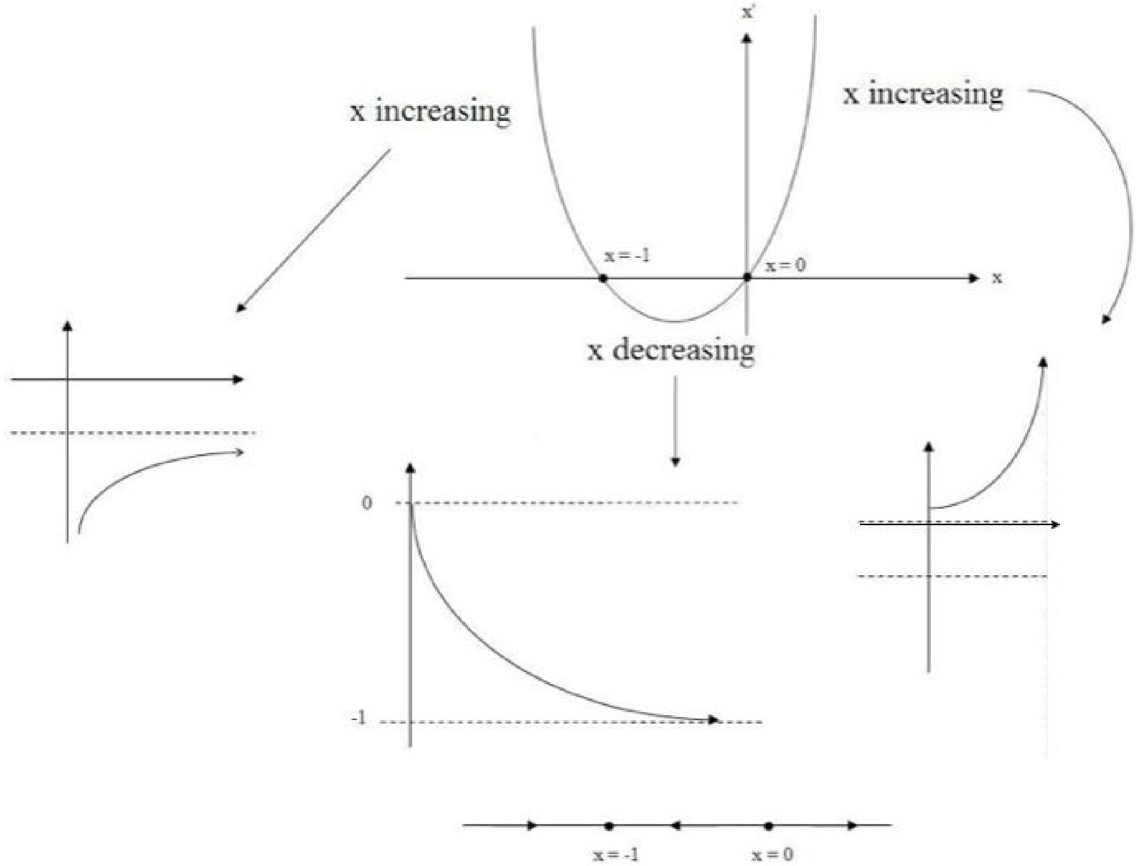
$$\text{Solution}: I = \frac{Nke^{\beta Nt}}{1+ke^{\beta Nt}}$$

$$\beta = 0 \Rightarrow I = \frac{Nk}{1+k} \qquad \text{Recall that } \lim_{t\to\infty} I = N$$

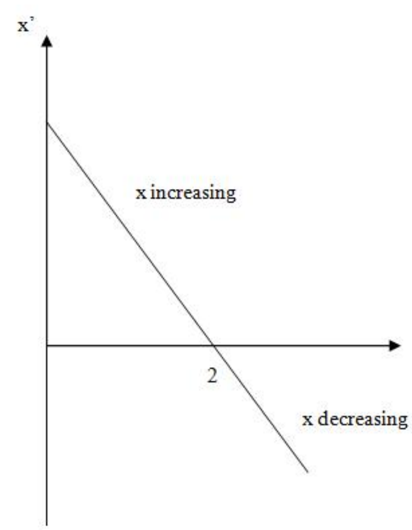
Thus there is no change over time.



Example 45. $x' = x + x^2$



Example 46. $\frac{dx}{dt} = 2 - x$



$$Equilibrium: \; 2 - x = 0 \; \Rightarrow \; x = 2.$$

11 Stability

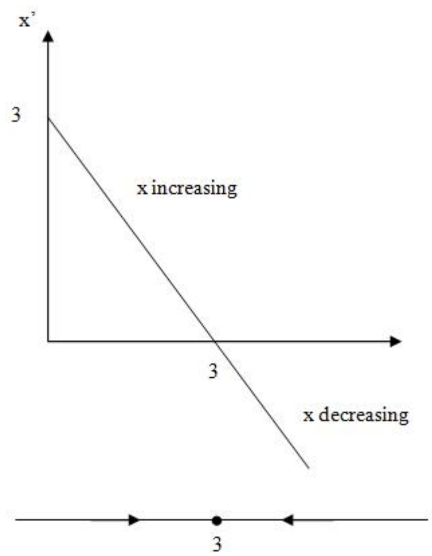
Definition 11.1. *An equilibrum is* <u>*stable*</u> *if solutions that begin near the equilibrium approach the equilibrium. An equilibrium is* <u>*unstable*</u> *if solutions that begin near the equilibrium move away from the equilibrium.*

$\therefore$ x=2 is a stable equilibrium in the previous example.

Example 47. $\frac{dx}{dt} = x + x^2$

$$x = -1 \text{ is stable}$$
$$x = 0 \text{ is unstable.}$$

Example 48. $\frac{dx}{dt} = 3 - x$
Equilibrium: $3 - x = 0 \Rightarrow x = 3$
It looks like $x = 3$ is stable. How can we know for sure?

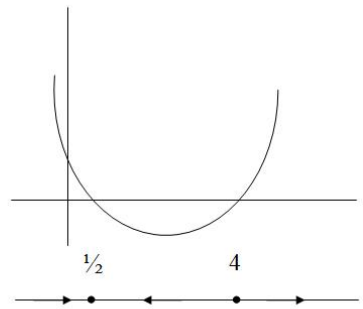


$$\begin{aligned}
\frac{dx}{3-x} &= dt \\
\int \frac{dx}{3-x} &= \int dt \\
-\ln|3-x| &= t + c \\
\ln|3-x| &= -t - c \\
3 - x &= e^{-t-c} \\
x &= 3 - Ae^{-t} \\
\lim_{t\to\infty} x &= 3 - A \lim_{t\to\infty} e^{-t} \\
&= 3 - 0 \\
&= 3 \quad \text{regardless of what A is.}
\end{aligned}$$

Example 49. $\frac{dx}{dt} = 2x^2 - 9x + 4$. Determine equilibria and their stability.

Equilibria :

$$2x^2 - 9x + 4 = 0$$
$$(2x - 1)(x - 4) = 0$$



$$x = \frac{1}{2}, \ x = 4$$

$x = \frac{1}{2}$ is stable and $x = 4$ is unstable.

What if we can't sketch the graph?

Stability Theorem Suppose y^* is an equilibrium of $\frac{dy}{dt} = f(y)$. Then

$$y^* \text{ is stable if } f'(y^*) < 0$$
$$y^* \text{ is unstable if } f'(y^*) > 0$$

Example 50. $\frac{dx}{dt} = 2x^2 - 9x + 4$. Determine all equilibria and their stability.

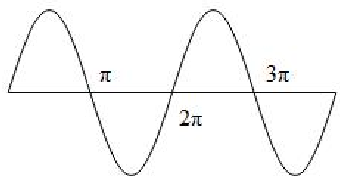
We know that $x^* = \frac{1}{2}, 4$.

$$f(x) = 2x^2 - 9x + 4$$
$$f'(x) = 4x - 9$$
$$f'\left(\frac{1}{2}\right) = 2 - 9 = -7 < 0 \qquad \therefore \frac{1}{2} \textit{ is stable.}$$
$$f'(4) = 16 - 9 = 7 > 0 \qquad \therefore 4 \textit{ is unstable.}$$

Example 51. $\frac{dx}{dt} = \sin x, \quad x \geqslant 0$. Determine all equilibria and their stability.

Equilibia: $\sin x = 0 \rightarrow \ x = 0, \pi, 2\pi, 3\pi, ...$

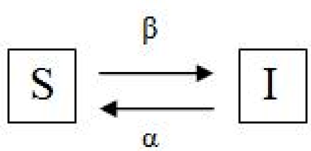
$$
\begin{aligned}
f'(x) &= \cos x \\
f'(0) &= \cos 0 = 1 \\
f'(\pi) &= \cos \pi = -1 \\
f'(2\pi) &= \cos 2\pi = 1 \\
f'(3\pi) &= \cos 3\pi = -1 \\
&\vdots
\end{aligned}
$$



$\therefore$ the even multiples of π are unstable equilibria and the odd multiples of π are stable equilibria.

Example 52. SIS epidemic. In this disease people get sick as before, but they also recover after a while.

$$
\begin{aligned}
S' &= \alpha I - \beta SI \\
I' &= \beta SI - \alpha I \\
S' + I' &= 0 \rightarrow S + I = N \\
I' &= \beta(N - I)I - \alpha I
\end{aligned}
$$



In this case we can't solve as we did before, but we can still discover pertinent information.

Equilibria:

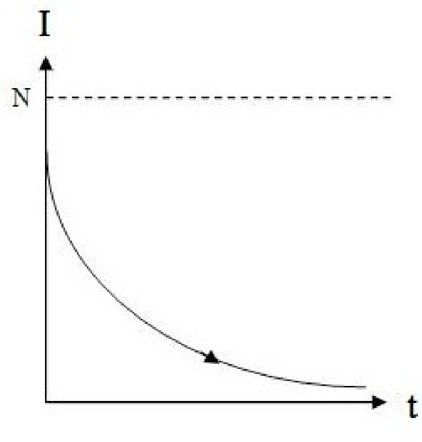
$$
\begin{aligned}
&[\beta(N - I) - \alpha]I = 0 \\
&I = 0 \qquad \beta(N - I) - \alpha = 0 \\
&\qquad\qquad N - I = \frac{\alpha}{\beta} \\
&\qquad\qquad I = N - \frac{\alpha}{\beta} = \frac{\beta N - \alpha}{\beta}
\end{aligned}
$$

Two cases:

$$
\begin{aligned}
&(i)\beta N - \alpha < 0 \\
&(ii)\beta N - \alpha > 0 \\
&f(I) = \beta(N - I)I - \alpha I \\
&\quad\;\; = \beta NI - \beta I^2 - \alpha I \\
&f'(I) = \beta N - 2\beta I - \alpha
\end{aligned}
$$

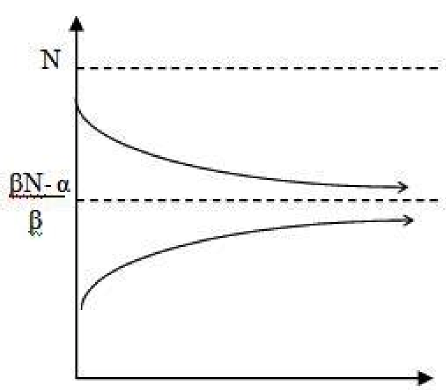
Case (i) I=0 is the only realistic equilibrium.

$$f'(0) = \beta N - \alpha < 0 \text{ in this case only!}$$
$$\therefore\ I = 0 \text{ is stable. That is, the disease dies out on its own.}$$



Case(ii) Two equilibria: I=0 and $I = \frac{\beta N - \alpha}{\beta}$

$$f'(0) = \beta N - \alpha > 0 \therefore\ I = 0 \textit{ is unstable in this case.}$$
$$\begin{aligned} f'\left(\frac{\beta N - \alpha}{\beta}\right) &= \beta N - 2\beta\left[\frac{\beta N - \alpha}{\beta}\right] - \alpha \\ &= \beta N - 2(\beta N - \alpha) - \alpha \\ &= \beta N - 2\beta N + 2\alpha - \alpha \\ &= -\beta N + \alpha \\ &= -(\beta N - \alpha) < 0 \end{aligned}$$
$$\therefore \frac{\beta N - \alpha}{\beta} \text{ is stable in this case.}$$

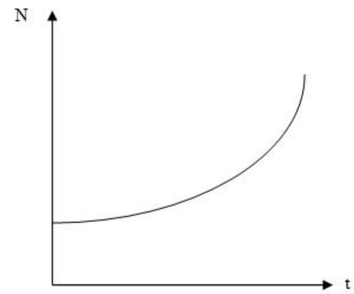


Therefore, if $\beta N - \alpha < 0$, ie recovery rate is high or transmission probability is low, then the disease dies out.

If $\beta N - \alpha > 0$, ie recovery rate is low or transmission probability is high, then the disease becomes established in the population. It doesn't infect everyone, but a certain proportion of people are always infected at any given time. Such a disease is called endemic.

12 The Logistic Equation

Many populations grow quickly at first but slow their growth when they run out of space. For example, rabbits on an island.
Unlimited growth can be written as $\frac{dN}{dt} = rN \Rightarrow N = Ae^{rt}$.



Slight limitation on growth $\frac{dN}{dt} = rN - bN^2$

$$= rN(1 - \frac{b}{r}N)$$

$$= rN(1 - \frac{N}{K}) \qquad K = \frac{r}{b} \text{ (we'll see why later)}$$

Separation of variables:

$$\frac{dN}{N(1-\frac{N}{K})} = rdt$$

$$\frac{1}{N(1-\frac{N}{K})} = \frac{A}{N} + \frac{B}{1-\frac{N}{K}}$$

$$1 = A(1-\frac{N}{K}) + BN$$

$$N = 0 \qquad 1 = A$$

$$N = K \qquad 1 = BK \qquad B = \frac{1}{K}$$

$$\int \left(\frac{1}{N} + \frac{1}{K(1-\frac{N}{K})} \right) dN = \int rdt$$

$$\int \left(\frac{1}{N} + \frac{1}{K-N} \right) dN = \int rdt$$

$$\ln N - \ln{(K-N)} = rt + c$$

$$\ln \frac{N}{K-N} = rt + c$$

$$\frac{N}{K-N} = De^{rt}$$

$$N = (K-N)De^{rt}$$

$$N(1+De^{rt}) = KDe^{rt}$$

$$N = \frac{KDe^{rt}}{1+De^{rt}}$$

$$N(0) = N_0 \Rightarrow N_0 = \frac{KD}{1+D}$$

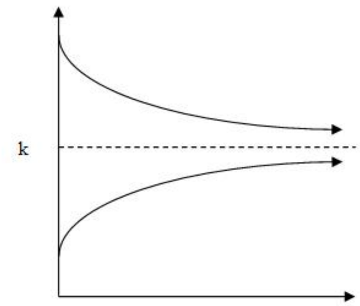
$$N_0(1+D) = KD$$

$$N_0 + N_0 D = KD$$

$$N_0 = (K-N_0)D$$

$$D = \frac{N_0}{K-N_0}$$

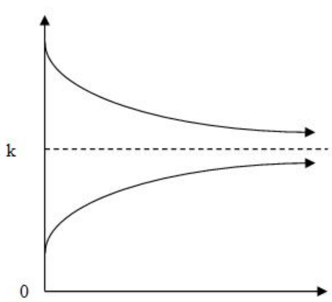
$$
\begin{aligned}
N &= \frac{\frac{KN_0e^{rt}}{K-N_0}}{1+\frac{N_0e^{rt}}{K-N_0}} \\
&= \frac{KN_0e^{rt}}{K-N_0+N_0e^{rt}} \\
&= \frac{Ke^{rt}}{\frac{K}{N_0}-1+e^{rt}} \\
&= \frac{K}{\left(\frac{K}{N_0}-1\right)e^{-rt}+1}
\end{aligned}
$$



$\lim_{t\to\infty} N(t) = K$ K is called the carrying capacity.

Or we have the equilibria:

$$
\begin{aligned}
&N = 0, 1-\frac{N}{K} = 0 \Rightarrow N = K \\
&f'(N) = r\left(1-\frac{N}{K}\right) - \frac{rN}{K} \\
&f'(0) = r > 0 \therefore \textit{ unstable} \\
&f'(K) = -r < 0 \therefore \textit{ stable} \\
&(\textit{much easier})
\end{aligned}
$$



13 Newton's Law of Cooling

Newton says that the rate at which heat is lost from an object is proportional to the difference between the temperature of that object and the ambient temperature.
How can we turn this into a differential equation?
Let H=heat
A=ambient temperature

$$\frac{dH}{dt} \quad \underbrace{= \alpha} \quad (A - H)$$

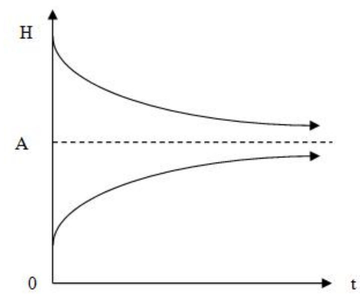
↑ Rate at which heat changes

↑ is proportional to

↖ difference between temperature and ambient temperature

The equilibrium is $H = A$ (assuming $\alpha \neq 0$ otherwise objects never lose heat)

$$f(H) = \alpha(A - H)$$
$$f'(H) = -\alpha$$



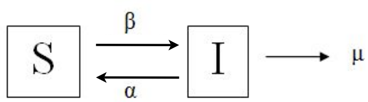
$\therefore\ H = A \textit{ is stable.}$

i.e. eventually objects will warm up to, or cool down to the ambient temperature in the room.

14 Two-Dimensional Differential Equations

SIS epidemic with death:

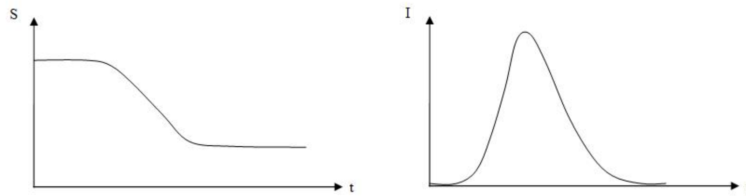
$$S' = \alpha I - \beta SI$$
$$I' = \beta SI - \alpha I - \mu I$$



What can we do with this?
Adding the equations together, we have $S' + I' = -\mu I \neq 0$
Therefore the population is not constant.
The equilibria are:

$$\alpha I - \beta SI = 0 \qquad \Rightarrow I = 0 \text{ or } S = \frac{\alpha}{\beta}$$
$$\beta SI - \alpha I - \mu I = 0 \qquad \Rightarrow I = 0 \text{ or } S = \frac{\alpha + \mu}{\beta}$$

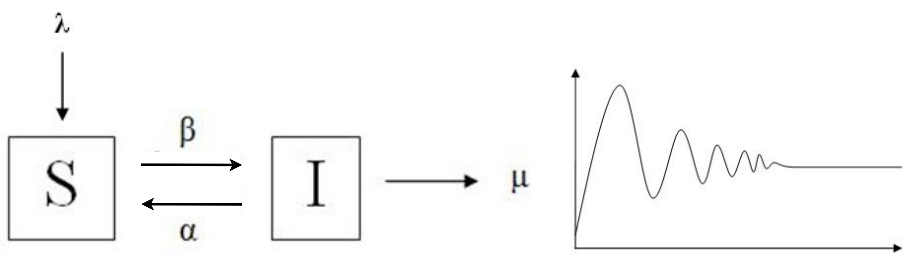
This suggests that eventually $I = 0$. Is this good?
What else? Answer (for now): Numerics

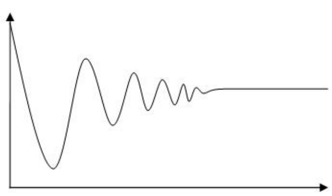


Eventually the disease dies out but it kills a portion of the population along the way.

Example 53. SIS epidemic, with migration

$$S' = \lambda + \alpha I - \beta SI$$
$$I' = \beta SI - \alpha I - \mu I$$

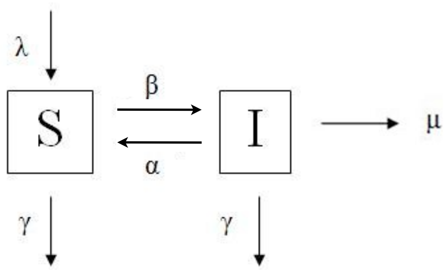


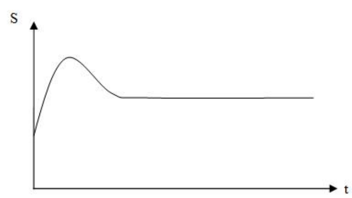


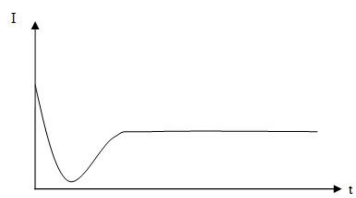
The disease oscillates but settles down.

Example 54. SIS epidemic with migration and natural death

$$
\begin{aligned}
S' &= \lambda + \alpha I - \beta SI - \gamma S \\
I' &= \beta SI - \alpha I - \mu I - \gamma I \\
&\text{Adding together, if } \mu = 0, \text{we have} \\
S' + I' &= \lambda - \gamma(S + I) \\
N' &= \lambda - \gamma N \\
N &\to \frac{\lambda}{\gamma}
\end{aligned}
$$

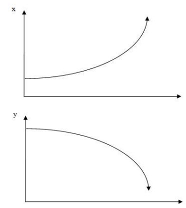






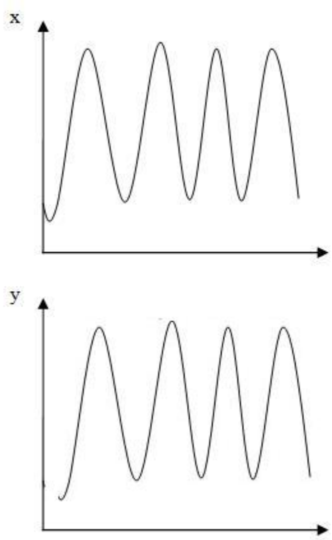
Example 55. Foxes and Rabbits
In the absence of foxes, rabbits grow without bound. In the absence of rabbits, foxes die as there is nothing to eat.
No eating:

$$x' = \lambda x \qquad \text{rabbits grow}$$
$$y' = -\delta y \qquad \text{foxes die}$$



When foxes eat rabbits, rabbits die and foxes increase.

$$x' = \lambda x - \epsilon xy$$
$$y' = \epsilon xy - \delta y$$



15 Complex Numbers

15.0.1 Introductory consideration

We can easily solve the equation $x^2 - 4 = 0$. The answer is $x = \pm 2$; in particular, x is a rational number, even an integer. The equation $x^2 - 2 = 0$ is a bit more tricky. The solution $x = \pm\sqrt{2}$ is not a rational number. Instead, we have defined the square root of a positive number as the real number that gives the original number back when multiplied by itself. But what should we do with the equation $x^2 + 1 = 0$? The answer cannot be a real number. (Why?) Can we do the same as above and define a number whose square equals -1? Indeed, this is what mathematicians did in the eighteenth century (it was a daring act and caused a lot of controversy), and they called that number 'i' or *imaginary*. (We will see that complex numbers are hardly more imaginary than $\sqrt{2}$.)

15.1 Definition

A *complex number* z is a number of the form

$$z = a + b\mathrm{i}$$

with real numbers a, b and the symbol i that satisfies $\mathrm{i}^2 = -1$. We call $a = \mathrm{Re}(z)$ the *real part* of z and $b = \mathrm{Im}(z)$ the *imaginary part* of z. The real number a can be considered the complex number $a + 0\mathrm{i}$. A complex number of the form $z = b\mathrm{i}$ is called *purely imaginary*.

15.2 Addition, subtraction, and multiplication of complex numbers

Complex numbers are easily added, subtracted and multiplied, if we keep the rule $\mathrm{i}^2 = -1$ in mind and use the distributive laws.

$$(a + b\mathrm{i}) \pm (c + d\mathrm{i}) = (a \pm c) + (b \pm d)\mathrm{i}$$

$$(a + b\mathrm{i}) \times (c + d\mathrm{i}) = ac + bc\mathrm{i} + ad\mathrm{i} + bd\mathrm{i}^2 = (ac - bd) + (ad + bc)\mathrm{i}\,,$$

since $\mathrm{i}^2 = -1$.

15.2.1 Examples

1. $(3 + 5\mathrm{i}) + (2 - 7\mathrm{i}) = 5 - 2\mathrm{i}$
2. $(0.5 + 1.7\mathrm{i}) - (0.8 - 2.6\mathrm{i}) = -0.3 + 4.3\mathrm{i}$
3. $(-3 + 2\mathrm{i}) \times (4 - 5\mathrm{i}) = (-12 - (-10)) + (15 + 8)\mathrm{i} = -2 + 23\mathrm{i}$
4. $(2 - 0.5\mathrm{i}) \times (3 + 4\mathrm{i}) = (6 - (-2)) + (-1.5 + 8)\mathrm{i} = 8 + 6.5\mathrm{i}$
5. $(9 + 2\mathrm{i}) + 5 = (9 + 2\mathrm{i}) + (5 + 0\mathrm{i}) = 14 + 2\mathrm{i}$
6. $-3\mathrm{i} + (2 + 3\mathrm{i}) = (0 - 3\mathrm{i}) + (2 + 3\mathrm{i}) = 2 + 0\mathrm{i} = 2$
7. $2 \times (3 - 5\mathrm{i}) = 6 - 10\mathrm{i}$
8. $3\mathrm{i} \times (-1 + 4\mathrm{i}) = -12 - 3\mathrm{i}$

Before we look at inverses and division of complex numbers, we introduce the *complex conjugate* of a complex number.

15.3 The complex conjugate

The *complex conjugate* of $z = a + b\mathrm{i}$ is $\bar{z} = a - b\mathrm{i}$, i.e., we simply change the sign of the imaginary part. Since the multiplication

$$z\bar{z} = (a + b\mathrm{i})(a - b\mathrm{i}) = a^2 + b^2$$

always produces a non-negative real number, we can take the square root. We define the *modulus* or *absolute value* of $z = a + b\mathrm{i}$ as

$$|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}.$$

From the identity $z\bar{z} = |z|^2$, we find the inverse of z to be

$$\frac{1}{z} = z^{-1} = \bar{z}/|z|^2.$$

Example 56. Start with $z = 3+4i$ and $w = 2-i$. The complex conjugates are $\bar{z} = 3-4i$ and $\bar{w} = 2+i$. The absolute values are $|z| = 5$ and $|w| = \sqrt{5}$. The inverses are

$$z^{-1} = \frac{1}{25}(3-4i)$$
$$w^{-1} = \frac{1}{5}(2+i).$$

Finally, we can divide

$$\frac{z}{w} = z\frac{\bar{w}}{|w|^2} = \frac{1}{5}(2+11i)$$
$$\frac{w}{z} = w\frac{\bar{z}}{|z|^2} = \frac{1}{25}(2-11i).$$

Another way to think about this: make the denominator real (similar to the way you'd rationalize the denominator) by multiplying top and bottom by the conjugate of the denominator (i.e., "real-ize" the denominator). Thus,

$$\begin{aligned}\frac{1}{z} &= \frac{1}{3+4i} \\ &= \frac{1}{3+4i}\cdot\frac{3-4i}{3-4i} \\ &= \frac{3-4i}{3^2-(4i)^2} \\ &= \frac{3-4i}{3^2+4^2} \\ &= \frac{3-4i}{25}\end{aligned}$$

Similarly for $\frac{1}{w}$:

$$\begin{aligned}\frac{1}{w} &= \frac{1}{2+i}\cdot\frac{2-i}{2-i} \\ &= \frac{2-i}{2^2-i^2} \\ &= \frac{2-i}{5} \\ \frac{z}{w} &= \frac{3+4i}{2-i}\cdot\frac{2+i}{2+i} \\ &= \frac{6+3i+8i+4i^2}{2^2+1^2} \\ &= \frac{2+11i}{5} \\ \frac{w}{z} &= \frac{2-i}{3+4i}\cdot\frac{3-4i}{3-4i} \\ &= \frac{6-8i-3i+4i^2}{9+16} \\ &= \frac{2-11i}{25}\end{aligned}$$

Example 57. Start with $z = 1-4i$ and $w = 0.5+3i$. The complex conjugates are $\bar{z} = 1+4i$ and $\bar{w} = 0.5-3i$. The absolute values are $|z| = \sqrt{17}$ and $|w| = \sqrt{37/4}$. The inverses are

$$z^{-1} = \frac{1}{17}(1+4i)$$
$$w^{-1} = \frac{4}{37}(0.5-3i).$$

Division gives

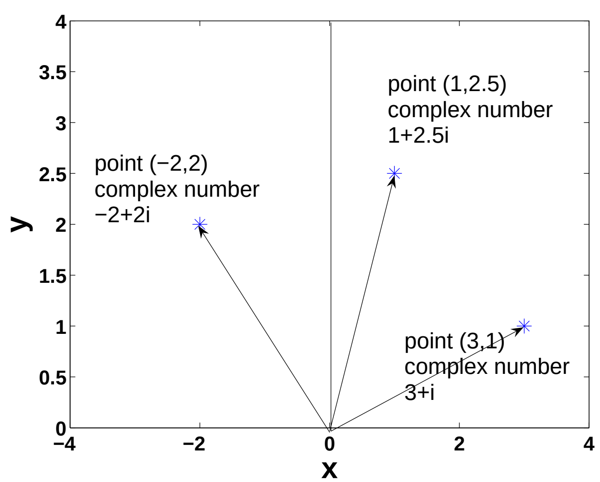
$$\frac{z}{w} = z\frac{\bar{w}}{|w|^2} = \frac{4}{37}(-11.5 - 5\text{i})$$
$$\frac{w}{z} = w\frac{\bar{z}}{|z|^2} = \frac{1}{17}(-11.5 + 5\text{i}).$$

Alternatively, we can "real-ize" the denominator as before. Thus $\frac{z}{w}$ and $\frac{w}{z}$ are

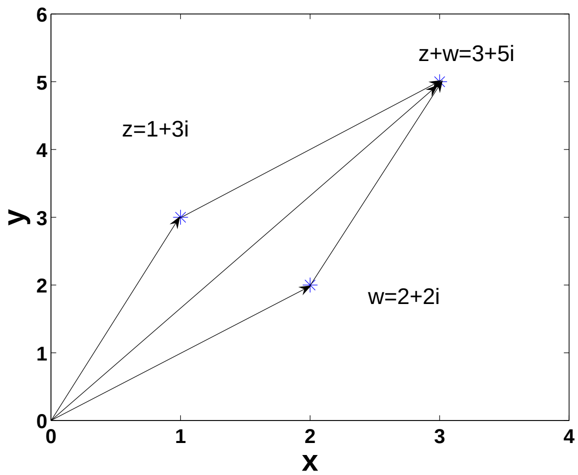
$$\begin{aligned}
\frac{z}{w} &= \frac{1-4\text{i}}{0.5+3\text{i}} \cdot \frac{0.5-3\text{i}}{0.5-3\text{i}} \\
&= \frac{0.5-3\text{i}-2\text{i}+12\text{i}^2}{0.25+9} \\
&= \frac{-11.5-5\text{i}}{9.25} \\
\frac{w}{z} &= \frac{0.5+3\text{i}}{1-4\text{i}} \cdot \frac{1+4\text{i}}{1+4\text{i}} \\
&= \frac{0.5+2\text{i}+3\text{i}-12}{1+16} \\
&= \frac{-11.5+5\text{i}}{17}
\end{aligned}$$

15.4 Geometric interpretation

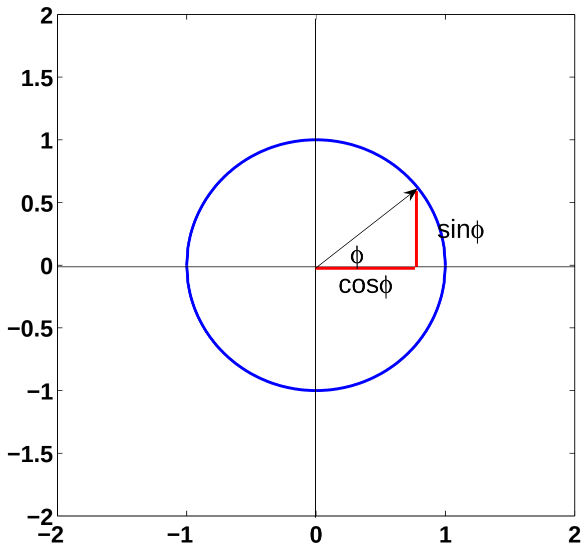
It is very helpful to think of a complex number as a point in the plane with the real part as the x-value and the imaginary part as the y-value. Hence, we identify the complex number $z = a + b\text{i}$ with the point (a, b) or with the vector (arrow) from the origin to the point (a, b). (We will talk about vectors in more detail shortly). Then the absolute value of the complex number is simply the distance of the corresponding point from the origin or the length of the vector (arrow).



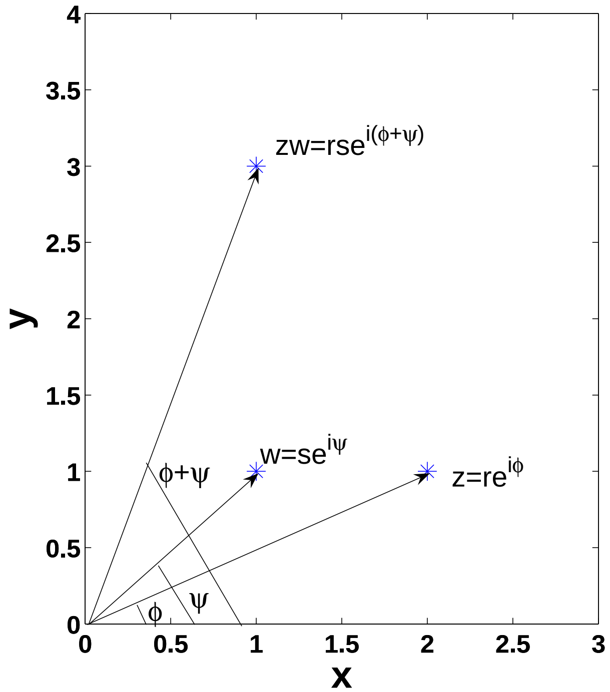
With this correspondence, the addition of complex numbers become the addition of vectors as it is known from the physics of forces.



To interpret multiplication, we take a slightly different point of view. Instead of giving the coordinates of the vector as the endpoint (a, b), we consider its length $r \geq 0$ and the angle ϕ it makes with the x-axis (counterclockwise) as $(r\cos\phi, r\sin\phi)$. This representation is called *polar coordinates*.



Then multiplication of two numbers is simply multiplication of the lengths and addition of the angles.



We write

$$z = r(\cos\phi + \mathrm{i}\sin\phi) \quad \text{and} \quad w = s(\cos\psi + \mathrm{i}\sin\psi).$$

Then we multiply, using the trigonometric identities

$$\begin{aligned} zw &= r(\cos\phi + \mathrm{i}\sin\phi) \times s(\cos\psi + \mathrm{i}\sin\psi) \\ &= rs[\cos\phi\cos\psi - \sin\phi\sin\psi + \mathrm{i}(\cos\phi\sin\psi + \cos\psi\sin\phi)] \\ &= rs[\cos(\phi+\psi) + \mathrm{i}\sin(\phi+\psi)]. \end{aligned}$$

15.5 Observation and definition

Every complex number of the form $z = \cos\phi + \mathrm{i}\sin\phi$ has absolute value one. We introduce the exponential notation (known as Euler's formula)

$$\exp(\mathrm{i}\phi) = e^{\mathrm{i}\phi} = \cos\phi + \mathrm{i}\sin\phi.$$

It might look strange at first, but the same rules as for the real exponential function apply. In fact, if we denote

$$f(\phi) = \cos\phi + \mathrm{i}\sin\phi$$

then

$$\begin{aligned} f'(\phi) &= -\sin\phi + \mathrm{i}\cos\phi \\ \frac{f'(\phi)}{f(\phi)} &= \frac{-\sin\phi + \mathrm{i}\cos\phi}{\cos\phi + \mathrm{i}\sin\phi} \\ &= \frac{-\sin\phi + \mathrm{i}\cos\phi}{\cos\phi + \mathrm{i}\sin\phi} \cdot \frac{\cos\phi - \mathrm{i}\sin\phi}{\cos\phi - \mathrm{i}\sin\phi} \\ &= \frac{-\sin\phi\cos\phi + \mathrm{i}\sin^2\phi + \mathrm{i}\cos^2\phi + \cos\phi\sin\phi}{\cos^2\phi + \sin^2\phi} \\ &= \mathrm{i} \end{aligned}$$

since $\cos^2\phi + \sin^2\phi = 1$. Now integrate:

$$\int \frac{f'(\phi)}{f(\phi)} d\phi = \int \mathrm{i} d\phi$$
$$\ln f(\phi) = \mathrm{i}\phi$$
$$f(\phi) = e^{\mathrm{i}\phi}$$

and thus $\cos\phi + \mathrm{i}\sin\phi = e^{\mathrm{i}\phi}$.

This has many advantages. First of all, we can write any complex number in polar coordinates as $z = re^{\mathrm{i}\phi}$. And we can easily multiply complex numbers in this form. For example, the calculation above becomes a single step (no need to look up the trig identities)

$$re^{\mathrm{i}\phi} \times se^{\mathrm{i}\psi} = rse^{\mathrm{i}(\phi+\psi)}.$$

Example 58.

1. The complex number $z = 1 + \mathrm{i}$ has modulus $|z| = \sqrt{2}$ and angle $\phi = \pi/4$. Hence $z = 1 + \mathrm{i} = \sqrt{2}e^{\mathrm{i}\pi/4}$.
2. The complex number $w = \sqrt{3} + \mathrm{i}$ has modulus $|w| = 2$ and angle $\phi = \pi/6$.
   Hence $w = \sqrt{3} + \mathrm{i} = 2e^{\mathrm{i}\pi/6}$.
3. Their product is $zw = (\sqrt{3} - 1) + (\sqrt{3} + 1)\mathrm{i} = 2\sqrt{2}e^{\mathrm{i}5\pi/12}$.
4. In general, if $z = a + b\mathrm{i}$ then $r = |z| = \sqrt{a^2 + b^2}$. The argument ϕ is not uniquely defined. If we restrict it between $-\pi$ and π the we get

$$\begin{cases} \phi = \arctan(b/a) & \text{if} \quad a > 0 \\ \phi = \arctan(b/a) + \pi & \text{if} \quad a < 0, b > 0 \\ \phi = \arctan(b/a) - \pi & \text{if} \quad a < 0, b < 0 \end{cases}$$

16 Linear Algebra

Example 59. Find equilibria of

$$x' = 2x + y$$
$$y' = 3x + 4y$$

Solution :

$2x + y = 0 \qquad R_1$

$3x + 4y = 0 \qquad R_2$

$8x + 4y = 0 \qquad R_1 \to R_1$ x 4

$3x + 4y = 0$

$8x + 4y = 0$

$-5x \quad = 0 \qquad R_2 \to R_2 - R_1$

$\Rightarrow \; x = 0 \qquad 8(0) + 4y = 0 \Rightarrow \; y = 0$

There is one solution.

Example 60. Find equilibria of

$$\begin{aligned} x' &= 6x - 5y + 4z - 5 \\ y' &= 4x - 4y + 3z - 2 \\ z' &= 6x - 7y + 5z - 3 \end{aligned}$$

Solution :

$$\begin{aligned} 6x - 5y + 4z &= 5 \qquad R_1 \\ 4x - 4y + 3z &= 2 \qquad R_2 \\ 6x - 7y + 5z &= 3 \qquad R_3 \end{aligned}$$

$$\begin{aligned} 6x - 5y + 4z &= 5 \\ 4x - 4y + 3z &= 2 \\ -2y + z &= -2 \qquad R_3 \to R_3 - R_1 \end{aligned}$$

$$\begin{aligned} 6x + 3y \quad &= 13 \qquad R_1 \to R_1 - 4R_3 \\ 4x + 2y \quad &= 8 \qquad R_2 \to R_2 - 3R_3 \\ -2y + 2 &= -2 \end{aligned}$$

$$\begin{aligned} 6x + 3y \quad &= 13 \\ 2x + y \quad &= 4 \qquad R_2 \to R_2/2 \\ -2y + 2 &= -2 \end{aligned}$$

$$\begin{aligned} 0 &= 1 \qquad R_1 \to R_1 - 3R_2 \\ 2x + y \quad &= 4 \\ -2y + z &= -2 \end{aligned}$$

$\therefore$ no solution.

Example 61. Solve:

$$\begin{aligned} -x + 3y - z &= -4 \qquad R_1 \\ 3x - 8y + 5z &= 14 \qquad R_2 \\ 2x - 6y + 2z &= 8 \qquad R_3 \end{aligned}$$

$$\begin{aligned} -x + 3y - z &= -4 \\ y + 2z &= 2 \qquad R_2 \to R_2 + 3R_1 \\ 0 &= 0 \qquad R_3 \to R_3 + 2R_1 \end{aligned}$$

Since we have more variables than equations , we can let z be any number, ie $z = t, t \in \mathbb{R}$

$$\begin{aligned} y + 2t = 2 &\Rightarrow y = 2 - 2t \\ -x + 3(2 - 2t) - t &= -4 \\ x &= 4 + 6 - 6t - t = 10 - 7t \end{aligned}$$

Therefore, there are infinitely many solutions.

16.1 Shorthand notation: Matrices

A matrix is a rectangular array of numbers. We have m rows and n columns, yielding an m x n matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

We can rewrite a system of equations in matrix form:

$$\begin{matrix} -x & +3y & -z & = -4 \\ 3x & -8y & +5z & = 14 \\ 2x & -6y & +2z & = 8 \end{matrix} \quad \Rightarrow \quad \underset{\substack{\uparrow \\ \text{square matrix}}}{\begin{bmatrix} -1 & 3 & -1 \\ 3 & -8 & 5 \\ 2 & -6 & 2 \end{bmatrix}} \underset{\substack{\uparrow}}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{\substack{\uparrow \\ \text{column vectors}}}{\begin{bmatrix} -4 \\ 14 \\ 8 \end{bmatrix}} \quad \Rightarrow \quad \underset{\substack{\uparrow \\ \text{augmented matrix}}}{\left[\begin{array}{ccc|c} -1 & 3 & -1 & -4 \\ 3 & -8 & 5 & 14 \\ 2 & -6 & 2 & 8 \end{array}\right]}$$

Example 62. Solve

$$\begin{aligned} 5x + 6y &= 13 \\ x + 2y &= 2 \end{aligned}$$

$$\left[\begin{array}{cc|c} 5 & 6 & 13 \\ 1 & 2 & 2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 0 & -4 & 3 \\ 1 & 2 & 2 \end{array}\right] \quad R_1 \to R_1 - 5R_2$$

$$\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & -4 & 3 \end{array}\right] \quad R_1 \Leftrightarrow R_2$$

This matrix is upper triangular.

$$-4y = 3 \Rightarrow y = -\frac{3}{4}$$

$$x + 2y = 2$$

$$x + 2\left(-\frac{3}{4}\right) = 2$$

$$x - \frac{3}{2} = 2$$

$$x = 2 + \frac{3}{2} = \frac{7}{2}$$

Example 63. Solve

$$\begin{aligned} x + y + z &= 6 \\ 2x + y - z &= -6 \\ 3x + 2y - 5z &= -35 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 1 & -1 & -6 \\ 3 & 2 & -5 & -35 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -1 & -3 & -18 \\ 0 & -1 & -8 & -53 \end{array}\right] \qquad \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -1 & -3 & -18 \\ 0 & 0 & -5 & -35 \end{array}\right] \qquad \begin{array}{l} \\ \\ R_3 - R_2 \end{array}$$

$$\begin{aligned} &-5z = -35 && \Rightarrow z = 7 \\ &-y - 3z = -18 \\ &-y - 3(7) = -18 \end{aligned}$$

$$\begin{aligned} &-y - 21 = -18 \\ &-y = 3 && \Rightarrow y = -3 \\ &x + y + z = 6 \\ &x - 3 + 7 = 6 \\ &x = 6 - 4 && \Rightarrow x = 2 \end{aligned}$$

Example 64. Solve

$$\begin{aligned} x - y - 2z &= 3 \\ -2x - y + z &= 4 \\ 3x \quad\;\; - 3z &= -1 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 3 \\ -2 & -1 & 1 & 4 \\ 3 & 0 & -3 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 3 \\ 0 & -3 & -3 & 10 \\ 0 & 3 & 3 & -10 \end{array}\right] \quad \begin{array}{c} \\ R_2 + 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 3 \\ 0 & -3 & -3 & 10 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{c} \\ \\ R_3 + R_2 \end{array}$$

$$\begin{aligned} & z = t && \Rightarrow z = t \\ & -3y - 3z = 10 && \\ & -3y - 3t = 10 && \\ & -3y = 10 + 3t && \Rightarrow y = -\frac{10}{3} - t \\ & x - y - 2z = 3 && \\ & x - \left(-\frac{10}{3} - t\right) - 2t = 3 && \Rightarrow x = -\frac{1}{3} + t \end{aligned}$$

17 Matrices

Two matrices A and B are equal if and only if all their entries are equal.

17.1 Matrix addition

Matrix addition is performed by adding individual entries.

Example 65. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $\quad B = \begin{bmatrix} 10 & 19 & -8 \\ 3 & -1 & -15 \\ 2 & -2 & 0 \end{bmatrix}$

$$A + B = \begin{bmatrix} 11 & 21 & -5 \\ 7 & 4 & -9 \\ 9 & 6 & 9 \end{bmatrix}$$

17.2 Multiplication of matrices by a constant

Multiplying a matrix A by a constant c multiplies each entry.

Example 66. $-3B = \begin{bmatrix} -30 & -57 & 24 \\ -9 & 3 & 45 \\ -6 & 6 & 0 \end{bmatrix}$

Example 67. $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad N = \begin{bmatrix} -5 & -6 \\ -7 & -8 \end{bmatrix}$ $\quad P = \begin{bmatrix} 2 & 0 \\ -1 & 9 \end{bmatrix}$

Find M+2N-3P.

$$\begin{aligned} M+2N-3P &= \begin{bmatrix}1&2\\3&4\end{bmatrix} + 2\begin{bmatrix}-5&-6\\-7&-8\end{bmatrix} - 3\begin{bmatrix}2&0\\-1&9\end{bmatrix} \\ &= \begin{bmatrix}1&2\\3&4\end{bmatrix} + \begin{bmatrix}-10&-12\\-14&-16\end{bmatrix} + \begin{bmatrix}-6&0\\3&-27\end{bmatrix} \\ &= \begin{bmatrix}1-10-6&2-12\\3-14+3&4-16-27\end{bmatrix} \\ &= \begin{bmatrix}-15&-10\\-8&-39\end{bmatrix} \end{aligned}$$

17.3 Matrix Multiplication

Matrix multiplication: $\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}\begin{bmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{bmatrix} = \begin{bmatrix}a_{11}b_{11}+a_{12}b_{21}&a_{11}b_{12}+a_{12}b_{22}\\a_{21}b_{11}+a_{22}b_{21}&a_{21}b_{12}+a_{22}b_{22}\end{bmatrix}$

Example 68.

$$\begin{aligned} \begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\begin{bmatrix}5&6\\-7&-1\\-2&0\end{bmatrix} &= \begin{bmatrix}1(5)+2(-7)+3(-2)&1(6)+2(-1)+3(0)\\4(5)+5(-7)+6(-2)&4(6)+5(-1)+6(0)\end{bmatrix} \\ &= \begin{bmatrix}5-14-6&6-2\\20-35-12&24-5\end{bmatrix} \\ &= \begin{bmatrix}-15&4\\-27&19\end{bmatrix} \end{aligned}$$

Important: The number of <u>columns</u> of the first matrix must match the number of <u>rows</u> of the second matrix.

Example 69.

$$NP = \begin{bmatrix}-5&-6\\-7&-8\end{bmatrix}\begin{bmatrix}2&0\\-1&9\end{bmatrix} = \begin{bmatrix}-10+6&-54\\-14+8&-72\end{bmatrix} = \begin{bmatrix}-4&-54\\-6&-72\end{bmatrix}$$

$$PN = \begin{bmatrix}2&0\\-1&9\end{bmatrix}\begin{bmatrix}-5&-6\\-7&-8\end{bmatrix} = \begin{bmatrix}-10&-12\\5-63&6-72\end{bmatrix} = \begin{bmatrix}-10&-12\\-58&-66\end{bmatrix}$$

$\therefore NP \neq PN \qquad \therefore$ order is important.

Example 70.

$$\begin{aligned} AB &= \begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\begin{bmatrix}10&19&-8\\3&-1&-15\\2&-2&0\end{bmatrix} \\ &= \begin{bmatrix}10+6+6&19-2-6&-8-30\\40+15+12&76-5-12&-32-75\\70+24+18&133-8-18&-56-120\end{bmatrix} \\ &= \begin{bmatrix}22&11&-38\\67&59&-107\\112&107&-176\end{bmatrix} \end{aligned}$$

Exercise: Find BA. Show that $AA^2 = A^2A = A^3$.

17.4 The Identity Matrix

The identity matrix I is an $n \times n$ matrix with 1's down the diagonal and 0's elsewhere.

$$I = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

For any matrix A, $AI = IA = A$.

17.5 Determinants

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then the determinant of A is $\det(A) = a_{11}a_{22} - a_{12}a_{21}$.

Example 71. $\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1(4) - 2(3) = 4 - 6 = -2.$

Example 72. Find $\det(P - N)$

$$\begin{aligned} \det(P-N) &= \det\left(\begin{bmatrix} 2 & 0 \\ -1 & 9 \end{bmatrix} - \begin{bmatrix} -5 & -6 \\ -7 & -8 \end{bmatrix}\right) \\ &= \det \begin{bmatrix} 7 & 6 \\ 6 & 17 \end{bmatrix} \\ &= 7(17) - 6(6) \\ &= 83 \end{aligned}$$

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then $\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$.

(Look at the pattern, don't memorise the formula.)

Example 73. Find $\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

$$\begin{aligned} \det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} &= 1(5)(9) + 2(6)(7) + 3(4)(8) - 3(5)(7) - 1(6)(8) - 2(4)(9) \\ &= 45 + 84 + 96 - 105 - 48 - 72 \\ &= 0 \end{aligned}$$

Example 74. Find $\det \begin{bmatrix} 8 & 2 & 0 \\ 0 & 3 & 5 \\ -4 & 1 & 0 \end{bmatrix}$.

$$\begin{aligned} \det \begin{bmatrix} 8 & 2 & 0 \\ 0 & 3 & 5 \\ -4 & 1 & 0 \end{bmatrix} &= 8(3)(0) + 2(5)(-4) + 0(1)(0) - 0(3)(-4) - 8(1)(5) - 0(2)(0) \\ &= -40 - 40 \\ &= -80 \end{aligned}$$

17.6 Inverse of a matrix

Suppose A is an $n \times n$ square matrix. If there exists an $n \times n$ matrix B such that $AB = BA = I$, then B is called the inverse of A and is denoted A^{-1}.

Theorem 17.1. *A^{-1} exists if and only if* $\det A \neq 0$.

Therefore $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is not invertible, but $\begin{bmatrix} 8 & 2 & 0 \\ 0 & 3 & 5 \\ -4 & 1 & 0 \end{bmatrix}$ is.

To find the inverse:
1. Write the augmented matrix $[A|I]$
2. Row reduce until the matrix is $[I|B]$
3. The matrix B is the inverse of A

Example 75. Find the inverse of $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 0 & -1 & 1 & -2 \\ 1 & 3 & 0 & 1 \end{array}\right] \qquad R_1 - 2R_2$$

$$\left[\begin{array}{cc|cc} 0 & -1 & 1 & -2 \\ 1 & 0 & 3 & -5 \end{array}\right] \qquad R_2 + 3R_1$$

$$\left[\begin{array}{cc|cc} 0 & 1 & -1 & 2 \\ 1 & 0 & 3 & -5 \end{array}\right] \qquad R_1 \times (-1)$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 3 & -5 \\ 0 & 1 & -1 & 2 \end{array}\right] \qquad R_1 \Leftrightarrow R_2$$

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

Example 76. Find the inverse of $\begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} 2 & 6 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 0 & 0 & 1 & -2 \\ 1 & 3 & 0 & 1 \end{array}\right] \qquad R_1 - 2R_2$$

Since the first row consists of only 0s, we can never obtain the identity matrix on the left hand side. Therefore we can't find the inverse.

Check: $\det \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} = 2(3) - 6(1) = 0 \quad \therefore$ the inverse doesn't exist.

Theorem 17.2. *If* $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ *and* $a_{11}a_{22} - a_{12}a_{21} \neq 0$ *then A is invertible and*

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

Note: This only works for 2 x 2 matrices. The denominator is the determinant.

Exercise: Check this for $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$.

Example 77. Find the inverse of $\begin{bmatrix} 1 & -1 & -1 \\ 2 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 & 0 & 1 \end{array}\right] \qquad \begin{array}{c} R_2 - 2R_1 \\ R_3 + R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 1 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 & 0 & 1 \end{array}\right] \qquad R_1 + R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 1 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} \end{array}\right] \qquad R_3/(-2)$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -\frac{1}{2} & 1 & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} \end{array}\right] \qquad \begin{array}{c} R_1 - 2R_3 \\ R_2 - 3R_3 \end{array}$$

$$\therefore \text{ the inverse is } \begin{bmatrix} 0 & 1 & 1 \\ -\frac{1}{2} & 1 & \frac{3}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

17.7 Transpose of a matrix

The transpose of a matrix swaps its rows and its columns.

Example 78. Find the transpose of $A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & -1 & 6 \\ -1 & 1 & -1 \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -1 & 1 \\ 3 & 6 & -1 \end{bmatrix}$$

Example 79. Find the transpose of $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

$$B^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

17.8 Solving linear equations

Example 80. Solve

$$\begin{aligned} 2x + 5y &= 4 \\ x + 3y &= -1 \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 4 \\ -1 \end{bmatrix} \\ \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 3(4) - 5(-1) \\ -1(4) + 2(-1) \end{bmatrix} \\ &= \begin{bmatrix} 17 \\ -6 \end{bmatrix} \\ &\therefore \; x = 17, \; y = -6 \end{aligned}$$

Therefore if A^{-1} exists then the solution to $A\vec{x} = \vec{b}$ is $\vec{x} = A^{-1}\vec{b}$.

Example 81. Solve

$$\begin{aligned} x - y - z &= -1 \\ 2x - y + z &= 2 \\ -x + y - z &= 1 \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 1 & -1 & -1 \\ 2 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 1 \\ -\frac{1}{2} & 1 & \frac{3}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 + 1 \\ \frac{1}{2} + 2 + \frac{3}{2} \\ \frac{1}{2} - \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \\ &\therefore \; x = 3, \; y = 4, \; z = 0 \end{aligned}$$

17.9 Eigenvalues and Eigenvectors

If A is a square matrix and $\vec{x}$ is a vector, then $A\vec{x}$ is also a vector. Could it be possible that $A\vec{x}$ is a scalar multiple of $\vec{x}$, say $\lambda\vec{x}$?
If so $\quad A\vec{x} = \lambda\vec{x}$

$A\vec{x} - \lambda\vec{x} = \vec{0}$ (the zero vector)
We can't factor out $\vec{x}$ because $(A-\lambda)\vec{x}$ makes no sense. (How do you substract a number from a matrix?)
But I comes to the rescue! Recall, $I\vec{x} = \vec{x}$.
$\therefore$ $\lambda\vec{x} = \lambda I\vec{x}$ if we want.

$$\begin{aligned} A\vec{x} - \lambda\vec{x} &= \vec{0} \\ A\vec{x} - \lambda I\vec{x} &= \vec{0} \\ (A-\lambda I)\vec{x} &= \vec{0} \qquad \text{this is fine since } \lambda I \text{ is a square matrix.} \end{aligned}$$

If $\vec{x} = \vec{0}$ then this is trivial. So let's suppose that $\vec{x} \neq \vec{0}$.
If $A - \lambda I$ is invertible, then we could take the inverse $\vec{x} = (A-\lambda I)^{-1}\vec{0} = \vec{0}$ (anything multiplied by $\vec{0}$ is $\vec{0}$.)
But this is no good, as we don't want $\vec{x} = \vec{0}$.
$\therefore$ $A - \lambda I$ is not invertible. ie. $\det(A-\lambda I) = 0$. This gives us a formula for λ that's independent of $\vec{x}$.
The values λ are called eigenvalues and the vectors $\vec{x}$ are called eigenvectors.

Example 82. Find all eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$.

$$\begin{aligned} \det(A-\lambda I) &= \det\left(\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{bmatrix}\right) \\ &= (1-\lambda)(2-\lambda) - (2)(3) \\ &= \lambda^2 - \lambda - 2\lambda + 2 - 6 \\ &= \lambda^2 - 3\lambda - 4 = 0 \\ (\lambda-4)(\lambda+1) &= 0 \\ \lambda = 4,\ \lambda &= -1 \text{ are the eigenvalues.} \end{aligned}$$

To find eigenvectors, put into the equation $(A-\lambda I)\vec{x} = \vec{0}$.

$\lambda = 4:$

$$\begin{aligned} (A-4I)\vec{x} &= \vec{0} \\ \left(\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ -3x_1 + 2x_2 &= 0 \\ 3x_1 - 2x_2 &= 0 \end{aligned}$$

But these two equations are fundamentally the same.

Note: this will always be true.
We are left with one equation and two variables: $3x_1 - 2x_2 = 0$

$$\begin{aligned} \therefore\ \textit{let } x_2 &= t \\ x_1 &= \frac{2}{3}t \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} t, \qquad t \in \mathbb{R},\ t \neq 0. \end{aligned}$$

$$\lambda = -1: \qquad (A - (-1)I)\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix} 1-(-1) & 2 \\ 3 & 2-(-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} 2x_1 + 2x_2 &= 0 \\ 3x_1 + 3x_2 &= 0 \end{aligned} \Rightarrow x_1 + x_2 = 0$$

$$let\ x_2 = s,\ x_1 = -s$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} s, \qquad s \in \mathbb{R}, s \neq 0.$$

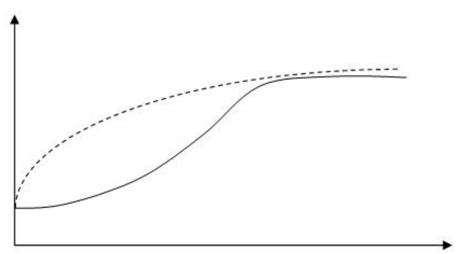
Note: It's crucial to specify that $t \neq 0$ and $s \neq 0$, because the cases $t = 0$ and $s = 0$ correspond to the zero vector, which cannot be an eigenvector.

17.9.1 Algorithm for eigenvalues and eigenvectors

- Use $\det(A - \lambda I) = 0$ to find a polynomial of degree n
- Solve over the complex plane for n eigenvalues
- Use $(A - \lambda I)\vec{x} = \vec{0}$ to solve for $\vec{x}$
- You must get a row of zeroes
- The free variable cannot equal zero.

17.9.2 Biological interpretation of eigenvalues and eigenvectors

- If all $\lambda_i < 0$ then the disease is eliminated.
- If any $\lambda_i > 0$ then the disease propagates.
- $\vec{x}$ is a measure of the speed of propagation.
- Control measures aim to reduce λ_{max} below 0.



Similar λ's, different $\vec{x}$'s

Example 83. Find the eigenvalues and eigenvectors of $B = \begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}$

Eigenvalues:

$$\begin{aligned} \det(B - \lambda I) &= \det \begin{bmatrix} 1-\lambda & 4 \\ 1 & -2-\lambda \end{bmatrix} \\ &= (1-\lambda)(-2-\lambda) - 4 \\ &= \lambda^2 - \lambda + 2\lambda - 2 - 4 \\ &= \lambda^2 + \lambda - 6 \\ &= (\lambda - 2)(\lambda + 3) = 0 \\ \lambda &= 2, -3 \end{aligned}$$

Eigenvectors:

$\lambda = 2:$

$$\begin{bmatrix} 1-2 & 4 \\ 1 & -2-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$x_1 - 4x_2 = 0$$
$$x_2 = t$$
$$x_1 = 4t$$
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} t, \qquad t \in \mathbb{R},\ t \neq 0$$

$\lambda = -3:$

$$\begin{bmatrix} 1-(-3) & 4 \\ 1 & -2-(-3) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$x_1 + x_2 = 0$$
$$x_2 = s$$
$$x_1 = -s$$
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} s, \qquad s \in \mathbb{R},\ s \neq 0$$

Example 84. Find the eigenvalues and eigenvectors of $C = \begin{bmatrix} -1 & 1 & 8 \\ 5 & 3 & -1 \\ 0 & 0 & 9 \end{bmatrix}$

Eigenvalues:

$$\begin{aligned}
\det(C - \lambda I) &= \det \begin{bmatrix} -1-\lambda & 1 & 8 \\ 5 & 3-\lambda & -1 \\ 0 & 0 & 9-\lambda \end{bmatrix} \\
&= (-1-\lambda)(3-\lambda)(9-\lambda) + 0 + 0 - 0 - 0 - 5(1)(9-\lambda) \\
&= (9-\lambda)[(-1-\lambda)(3-\lambda) - 5] \\
&= (9-\lambda)[\lambda^2 + \lambda - 3\lambda - 3 - 5] \\
&= (9-\lambda)(\lambda^2 - 2\lambda - 8) \\
&= (9-\lambda)(\lambda - 4)(\lambda + 2) = 0 \\
\lambda &= 9,\ 4,\ -2
\end{aligned}$$

Eigenvectors:

$$\lambda = 9: \quad \begin{bmatrix} -1-9 & 1 & 8 \\ 5 & 3-9 & -1 \\ 0 & 0 & 9-9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -10 & 1 & 8 \\ 5 & -6 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -11 & 6 \\ 5 & -6 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad R_1 + 2R_2$$

$$-11x_2 + 6x_3 = 0$$

$$5x_1 - 6x_2 - x_3 = 0$$

$$x_3 = t$$

$$x_2 = \frac{6}{11}t$$

$$5x_1 - 6\left(\frac{6}{11}t\right) - t = 0$$

$$x_1 = \frac{47}{55}t$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{47}{55} \\ \frac{6}{11} \\ 1 \end{bmatrix} t, \qquad t \in \mathbb{R},\ t \neq 0$$

$$\lambda = 4: \quad \begin{bmatrix} -1-4 & 1 & 8 \\ 5 & 3-4 & -1 \\ 0 & 0 & 9-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -5 & 1 & 8 & 0 \\ 5 & -1 & -1 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -5 & 1 & 8 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right] \qquad R_2 + R_1$$

$$x_3 = 0$$

$$-5x_1 + x_2 + 8x_3 = 0$$

$$x_2 = s$$

$$-5x_1 + s + 8(0) = 0$$

$$x_1 = \frac{1}{5}s$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ 1 \\ 0 \end{bmatrix} s, \qquad s \in \mathbb{R},\ s \neq 0$$

$$\lambda = -2: \qquad \begin{bmatrix} -1-(-2) & 1 & 8 \\ 5 & 3-(-2) & -1 \\ 0 & 0 & 9-(-2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 8 & 0 \\ 5 & 5 & -1 & 0 \\ 0 & 0 & 11 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 8 & 0 \\ 0 & 0 & -41 & 0 \\ 0 & 0 & 11 & 0 \end{array}\right] \qquad R_2 - 5R_1$$

$$x_3 = 0$$

$$x_1 + x_2 + 8x_3 = 0$$

$$x_2 = r$$

$$x_1 + r = 0 \to x_1 = -r$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} r, \qquad r \in \mathbb{R},\ r \neq 0$$

Exercise. Find the eigenvalues and eigenvectors of

$$D = \begin{bmatrix} 0 & -8 & 6 \\ 2 & 0 & 4 \\ 0 & 0 & -3 \end{bmatrix} \qquad\qquad E = \begin{bmatrix} 2 & 3 & 5 \\ 4 & 6 & 10 \\ 0 & 0 & 0 \end{bmatrix}$$

17.10 Finding general solutions of linear systems

To find the general solution of $\begin{bmatrix} x' \\ y' \end{bmatrix} = J \begin{bmatrix} x \\ y \end{bmatrix}$, let's start with a solution of the following form:

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^{\lambda t} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \lambda e^{\lambda t} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = J \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \lambda e^{\lambda t} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = J e^{\lambda t} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\Rightarrow \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = J \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

This implies λ is an eigenvalue of J with corresponding eigenvector $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

The general form for 2 x 2 matrices is: $\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{\lambda_1 t} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + c_2 e^{\lambda_2 t} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$.

Example 85. Solve the following system with initial conditions $x(0) = 5, y(0) = 1$.

$$x' = -6x + 9y$$
$$y' = 3x$$

$$J = \begin{bmatrix} -6 & 9 \\ 3 & 0 \end{bmatrix}$$

$$\begin{aligned}
\det(J - \lambda I) &= \det \begin{bmatrix} -6-\lambda & 9 \\ 3 & -\lambda \end{bmatrix} \\
&= (6+\lambda)\lambda - 27 \\
&= \lambda^2 + 6\lambda - 27 \\
&= (\lambda + 9)(\lambda - 3) \\
&\to \lambda = -9, 3
\end{aligned}$$

$$\lambda_1 = -9 : \begin{bmatrix} -6+9 & 9 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & 9 & 0 \\ 3 & 9 & 0 \end{array}\right]$$

$$v_1 + 3v_2 = 0$$
$$v_2 = r$$
$$v_1 = -3r$$

$\therefore$ *the eigenvectors are given by* $\begin{bmatrix} -3 \\ 1 \end{bmatrix} r, r \in \mathbb{R}, r \neq 0.$

$$\lambda_2 = 3 : \begin{bmatrix} -6-3 & 9 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -9 & 9 & 0 \\ -3 & 3 & 0 \end{array}\right]$$

$$w_1 - w_2 = 0$$
$$w_2 = s$$
$$w_1 = s$$

$\therefore$ *the eigenvectors are given by* $\begin{bmatrix} 1 \\ 1 \end{bmatrix} s, s \in \mathbb{R}, s \neq 0.$

So we have $\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{-9t} \begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
Now we use the initial conditions to find the values of c_1 and c_2.

$$-3c_1 + c_2 = 5$$
$$c_1 + c_2 = 1$$
$$\Rightarrow c_1 = -1, c_2 = 2$$

The solution is given by $\begin{bmatrix} x \\ y \end{bmatrix} = -e^{-9t} \begin{bmatrix} -3 \\ 1 \end{bmatrix} + 2e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Exercise: Show that this solution satisfies both the ODEs and the initial condition.

Exercise: Solve

$$\begin{aligned} x' &= -8y + 10z \\ y' &= -2x + 8z \\ z' &= -2z \end{aligned}$$

with $x(0) = -45$, $y(0) = 7$ and $z(0) = 9$.

18 Multivariable Calculus

Real-valued functions
Almost all biological processes depend on more than one variable.

Example 86. The ambient temperature you need to survive is:

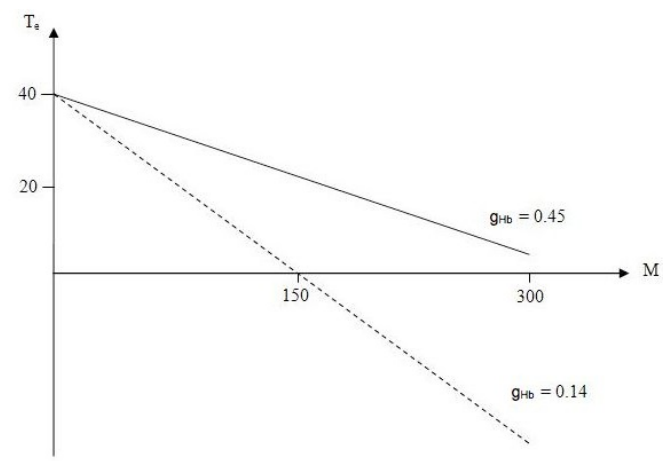
$$T_e = 36 - \frac{(0.9M - 12)(g_{Hb} + 0.95)}{27.8 g_{Hb}}$$

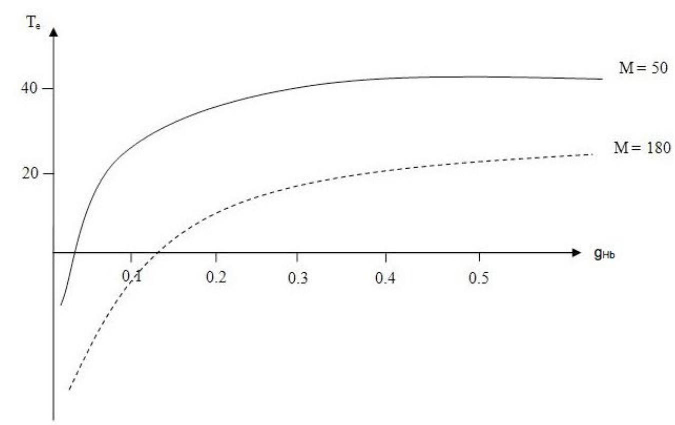
Where T_e is the temperature in degrees celcius
M is metabolic heat
g_{Hb} is thermal conduction, describing how quickly heat is lost.

$g_{Hb} = 0.45\ mol\ m^{-2}s^{-1}$	without clothing
$= 0.14$	for a wool suit
$= 0.04$	for a sleeping bag

$M = 50\ wm^{-2}$	sleeping
$= 95$	writing at a desk
$= 180$	walking at 2.5 mph
$= 350$	walking at 3.5 mph with a 40 lb pack

To meet the required temperature, we can either change M (moving when we get cold) or g_{Hb} (putting on more clothing) or both.





Definition 18.1. *A real-valued function f assigns a real number to an n-dimensional input. We write $f : \mathbb{R}^n \to \mathbb{R}$ and f has the form $f(x_1, x_2, \ldots, x_n) = w$.*

Example 87. $f(x, y) = x + y$. Evaluate f at the points $(-1, 1)$ and $(2, -5)$.

$$\begin{aligned} f(-1, 1) &= -1 + 1 = 0 \\ f(2, -5) &= 2 - 5 = -3 \end{aligned}$$

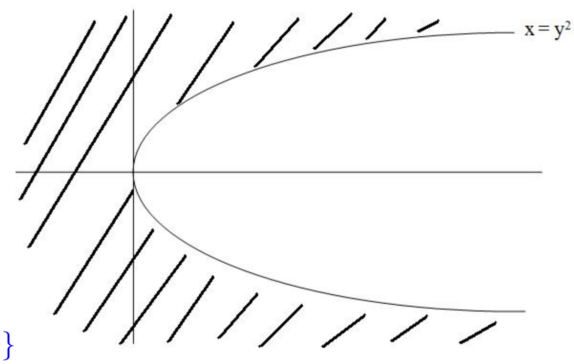
Example 88. Find the domain and range of the function $f(x, y, z) = \frac{xy}{z^2}$.

Domain: $z \neq 0$ $\therefore$ domain is $\{(x, y, z) \in \mathbb{R}^3,\ z \neq 0\}$
Range: $\mathbb{R}$ (since f can take any value).

Example 89. Find the domain and range of $f(x, y) = \sqrt{y^2 - x}$. Sketch the domain in $\mathbb{R}^2$.

$Domain: \ y^2 - x \geqslant 0 \quad \rightarrow \ y^2 \geqslant x$

$x \leqslant y^2$

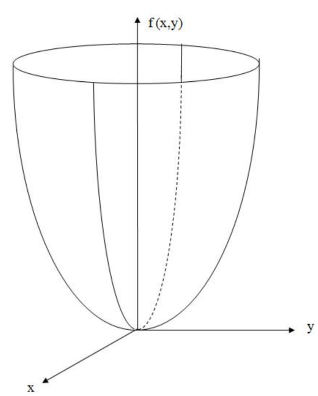


$Range: \{f \in \mathbb{R}: \ f \geqslant 0\}$

Example 90. Graph the function $f(x, y) = x^2 + y^2$

$Domain: \ \mathbb{R}^2$

$Range: \ \{f : f \geqslant 0\}$



The graph is a parabolic bowl.

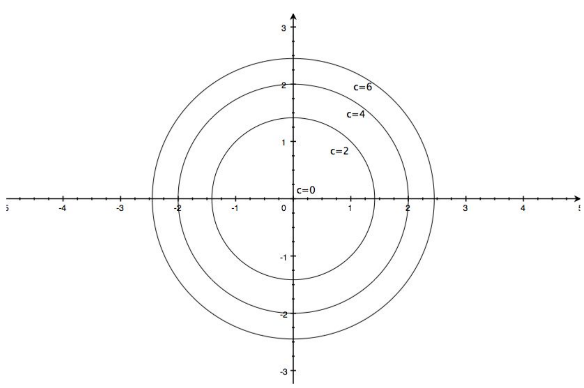
Another way to visualize functions is with <u>level curves</u>. We see these in weather maps all the time.

Definition 18.2. *Suppose $f : \mathbb{R}^2 \rightarrow \mathbb{R}$. The <u>level curves</u> of f comprise the set of (x, y) points where $f(x, y) = c$.*

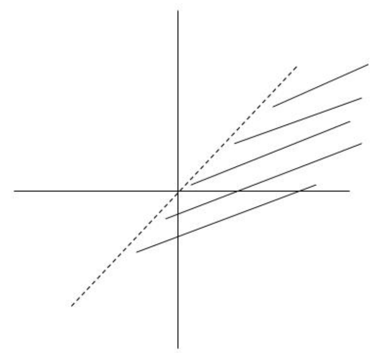
Level curves are a "snapshot" of the function, taken at constant values c. It's a way of seeing 3 dimensions but only drawing 2.

Example 91. $f(x, y) = x^2 + y^2$ Plot the level curves for $c = 0, 2, 4, 6$.

$x^2 + y^2 = c$ describes a circle with centre $(0, 0)$ and radius $\sqrt{c}$. (If $c = 0$, this is simply a point) As c increases, the circles get closer together.



Example 92. $f(x, y) = \ln(x - y)$. Find the domain and range. Sketch the domain and level curves corresponding to $c = -2, -1, 0, 1, 2$.

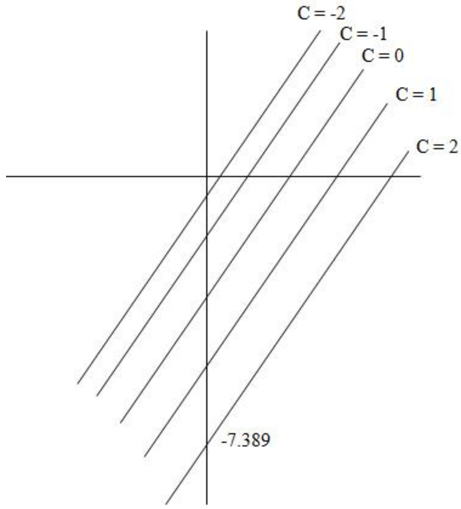


Domain: $x - y > 0 \to x > y$

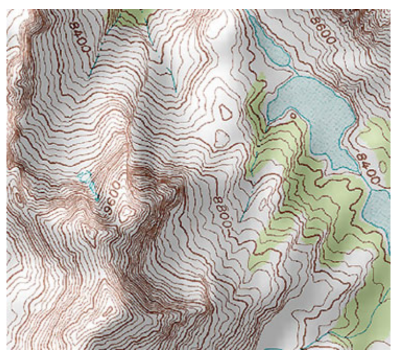
Range: $f = \mathbb{R}$

$$\begin{aligned}
\ln(x - y) &= c \\
x - y &= e^c \\
y &= x - e^c \\
y &= x - 0.135 && (c = -2) \\
y &= x - 0.367 && (c = -1) \\
y &= x - 1 && (c = 0) \\
y &= x - 2.718 && (c = 1) \\
y &= x - 7.389 && (c = 2)
\end{aligned}$$

Level curves:



Example 93. Topography



Exercise. Sketch the domain of

$$f(x, y) = \frac{4y + 2x}{x^2 + 2xy - 3}$$
$$g(x, y) = \ln(x) - \sqrt{y}$$

19 Limits

Suppose

$$\lim_{(x,y)\to(a,b)} f(x, y) = L_1 \quad \textit{and} \quad \lim_{(x,y)\to(a,b)} g(x, y) = L_2$$

Then

$$\lim_{(x,y)\to(a,b)} [f(x, y) + g(x, y)] = \lim_{(x,y)\to(a,b)} f(x, y) + \lim_{(x,y)\to(a,b)} g(x, y)$$

$$\lim_{(x,y)\to(a,b)} [cf(x, y)] = c \lim_{(x,y)\to(a,b)} f(x, y)$$

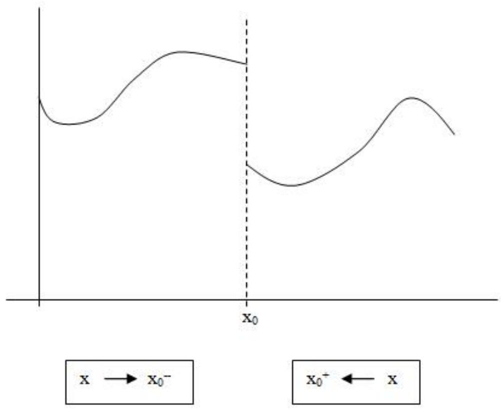
$$\lim_{(x,y)\to(a,b)} f(x, y)g(x, y) = \lim_{(x,y)\to(a,b)} f(x, y) \lim_{(x,y)\to(a,b)} g(x, y)$$

$$\lim_{(x,y)\to(a,b)} \frac{f(x, y)}{g(x, y)} = \frac{\lim_{(x,y)\to(a,b)} f(x, y)}{\lim_{(x,y)\to(a,b)} g(x, y)} \text{ if the denominator is not zero}$$

Example 94.

$$\lim_{(x,y)\to(0,0)} x^2+y^2 = 0^2+0^2 = 0$$
$$\lim_{(x,y)\to(4,-3)} x^2+y^2 = 4^2+(-3)^2 = 16+9 = 25$$
$$\lim_{(x,y)\to(2,0)} \frac{4y+2x}{x^2+2xy-3} = \frac{4(0)+2(2)}{2^2+(2)(2)(0)-3} = \frac{4}{4-3} = 4$$

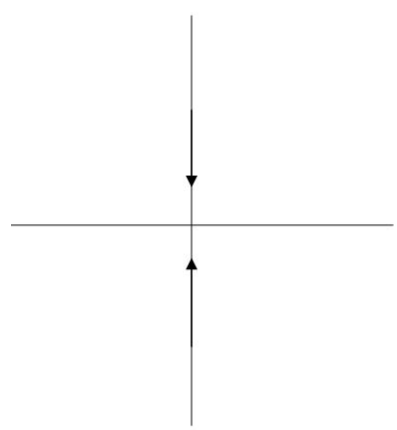
Recall: $f(x)$ is continuous at x_0 if $\lim_{x\to x_0^-} h(x) = \lim_{x\to x_0^+} h(x)$.
For one-variable functions, there are two paths: from the left and from the right. For multi-variable functions, there are many paths.



Example 95. Is $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ continuous at (0,0)?
Path 1: Along the x-axis (so y=0)

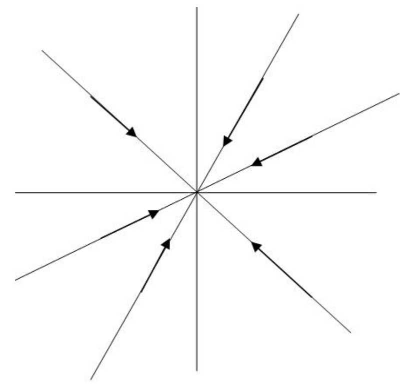
$$\lim_{(x,0)\to(0,0)} f = \lim_{x\to 0} \frac{x^2-0}{x^2+0} = \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

Path 2: Along the y-axis (so x=0)



$$\lim_{(0,y)\to(0,0)} f = \lim_{y\to 0} \frac{0-y^2}{0+y^2} = \lim_{y\to 0} \frac{-y^2}{y^2} = -1$$

Therefore the limit does not exist and f(x,y) is not continuous at (0,0).
More generally, we could approach by paths $y = mx$.



$$\lim_{(x,mx)\to(0,0)} f = \lim_{x\to 0} \frac{x^2 - m^2x^2}{x^2 + m^2x^2} = \lim_{x\to 0} \frac{1-m^2}{1+m^2}$$

The value of the limit is different for different values of m (if $m \neq 0$). Thus, the limit does not exist.

Example 96. Show that $\lim_{(x,y)\to(0,0)} \frac{4xy}{xy+y^3}$ does not exist.
Straight lines:

$$\lim_{(x,mx)\to(0,0)} \frac{4xy}{xy+y^3} = \lim_{x\to 0} \frac{4mx^2}{mx^2+(mx)^3} = \lim_{x\to 0} \frac{4}{1+m^2x} = 4$$

So it looks like the limit exists.
But let's try the path $x = y^2$.

$$\lim_{(y^2,y)\to(0,0)} \frac{4xy}{xy+y^3} = \lim_{y\to 0} \frac{4y^3}{y^3+y^3} = \frac{4}{2} = 2 \neq 4$$

Therefore, the limit does not exist.