MAT 1341 F (Fall 2016)

INTRODUCTION TO LINEAR ALGEBRA

PRELIMINARIES

Write name on board!

Let students come in and show "lecture-01.pdf"!

- Emmy Noether:
  - As child interested in Music and Dancing.
  - Degree / License for teaching English and French.
  - Higher education in Mathematics.
  - PhD in 1907 (Erlangen).
  - Then, moved to famous University of Göttingen.

  No nerd!

- Albert Einstein admired her work.
- ✉ Albert Einstein → David Hilbert
- Discussion about her "Habilitation":
  - 1915, David Hilbert says "Eine Fakultät ist doch keine Badeanstalt." / "But a department not a community swimming pool (where men and women are separated when changing their clothes)."
  - Unsuccessful!
  - Until 1919, when WWI was over...

  Being a mathematician!!

  Lost license to teach!

- Another terrible change of rules in 1933. Being jewish, Emmy Noether emigrated to the United States.
- Passed away in 1935.
- ✉ Albert Einstein → NY Times

  Letter to the Editor, instead of obituary!

  2 weeks after her death

- "Poetry of logical ideas." Keep this statement in mind!

Syllabus

My first lecture at uOttawa!

- Introduce yourself!
- Emphasize the most important aspects from the syllabus...
- Have a first look into "Vector Spaces First"

Interrupt me!

Show them the optional textbook!

Tell students about the title!

Questions?

§1 COMPLEX NUMBERS

Both "Noether" and "Cardano" are also lunar craters ☺

Gerolamo Cardano (1501 - 1576)

Question | Find two numbers whose product is 40 and whose sum is 10.

Approach |

$$x \cdot (10 - x) = 40$$

$$\Leftrightarrow -x^2 + 10x - 40 = 0$$

Talk about equivalence arrows!

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-10 \pm \sqrt{10^2 - 4 \cdot 40}}{-2}$$

$$= 5 \pm \frac{\sqrt{-60}}{\sqrt{4}}$$

Not good!

$$= 5 \pm \sqrt{-15}$$

Let's assume, we could...

$$= 5 \pm \sqrt{15} \cdot \sqrt{-1}$$

$$= 5 \pm \sqrt{15} \cdot i$$

"If there was a solution, it would have to look like this."

§ 1.1 Extensions and $\mathbb{C}$

It is always about getting used to things!

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

That's already non-trivial for children!

Remember Hippasus of Metapont!

This is in the spirit of Cardano!

But now, we do it more rigorously:

$$\mathbb{C} := \{ a + bi \mid a, b \in \mathbb{R} \}$$

real part

imaginary part

Note: Two complex numbers are equal if and only if ("iff") their real and their imaginary parts agree!

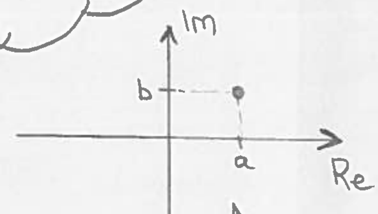
i is a formal symbol!

Notation:

For $z = a + bi \in \mathbb{C}$, we write:

$$\mathrm{Re}(z) = a$$
$$\mathrm{Im}(z) = b$$

Geometry:



helpful!

We may think of every real number $a \in \mathbb{R}$ as a complex number $z = a + 0 \cdot i$ with $\mathrm{Re}(z) = a$ and $\mathrm{Im}(z) = 0$. Hence, $\mathbb{R} \subseteq \mathbb{C}$.

Definition:

Let $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$ $(a_1, a_2, b_1, b_2 \in \mathbb{R})$ be two arbitrary complex numbers. Then:

Special Ex 1.5

$$z_1 + z_2 := (a_1 + a_2) + (b_1 + b_2)\, i$$
$$z_1 \cdot z_2 := (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)\, i$$

Note: $i^2 = i \cdot i = -1$

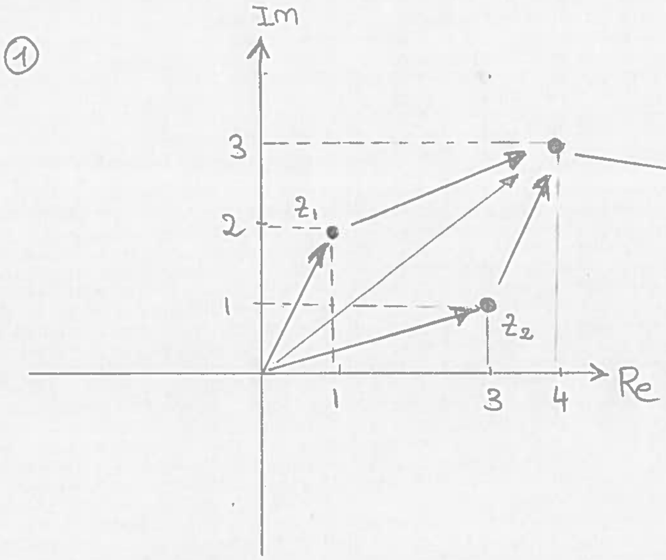
Examples:

$z_1 = 1 + 2i \quad , \quad z_2 = 3 + i$

4i would look bad!

Maybe involve students!

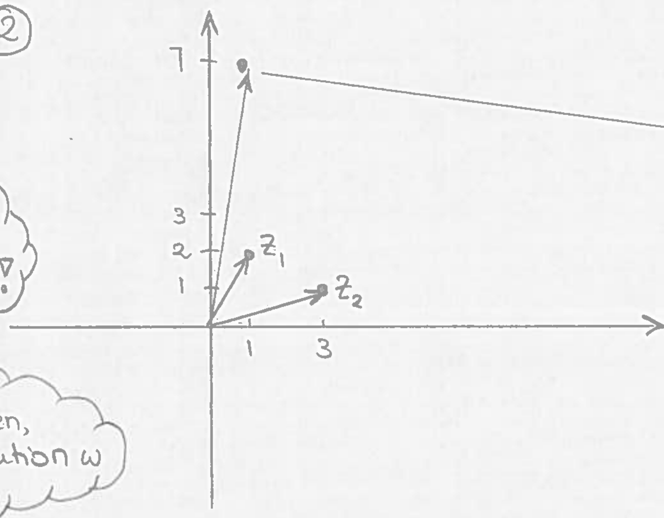
①



$z_1 + z_2 = 4 + 3i$

The addition of complex numbers corresponds to the addition of vectors in the complex plane.

②



$z_1 \cdot z_2 = 1 + 7i$

For those who are interested!

$z = a + ib$. Then, $-z$ is a solution w to $z + w = 0$.

We are going to see how to interpret the multiplication of complex numbers geometrically.

Say words about inverses!

③ Substraction:

$$z_1 - z_2 = (1 + 2i) + (-3 - i) = 1 + 2i - 3 - i = -2 + i$$

$-\quad +\quad +$

④ Division:

Ex 1.1

First Lemma 1.1.a!

$$\frac{z_1}{z_2} = \frac{1+2i}{3+i} = \frac{(1+2i)\cdot(3-i)}{(3+i)\cdot(3-i)} = \frac{5+5i}{10} = \frac{1}{2} + \frac{1}{2}i$$

We omit "+0i"!

LECTURE 2

$\mathbb{C} = \{ a+bi \mid a,b \in \mathbb{R} \}$

Keep in mind: (1) $i^2 = -1$.

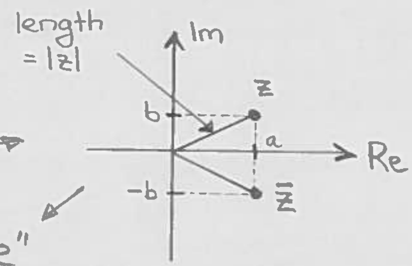
(2) To divide by $a+bi$, expand with $a-ib$.

Definition:

Let $z = a+bi$. Then, Ex 1.3

$\bar{z} := a - bi$ "complex conjugate"

$|z| := \sqrt{a^2+b^2}$ "modulus"/"absolute value"



Observation: $z\bar{z} = (a+bi)(a-bi) = a^2+b^2$, $\Rightarrow z\bar{z} = |z|^2$.

We may therefore obtain for $z \neq 0$:

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$$ a formula for the inverse $\frac{1}{z}$

Other useful properties for all $z, w \in \mathbb{C}$: Ex 1.4 Nice !

(1) $\overline{z+w} = \bar{z} + \bar{w}$ ("first add then conjugate = first conjugate then add")

Proof Let $z = a+bi$, $w = \alpha + \beta i$. Then,

$\overline{z+w} = (a+\alpha) - (b+\beta)i = (a+\alpha) + (-b-\beta)i$,

$\bar{z} + \bar{w} = (a-bi) + (\alpha - \beta i) = (a+\alpha) + (-b-\beta)i$ $\square$

(2) $\overline{zw} = \bar{z} \cdot \bar{w}$ (3) $\bar{\bar{z}} = z$ (4) $\bar{z} = z \Leftrightarrow \mathrm{Im}(z) = 0 \Leftrightarrow z \in \mathbb{R}$

(5) $\bar{z} = -z \Leftrightarrow \mathrm{Re}(z) = 0 \Leftrightarrow:$ z is purely imaginary

(6) $|z| \in \mathbb{R}$ with $|z| \geq 0$. Moreover, $|z| = 0 \Leftrightarrow z = 0$.

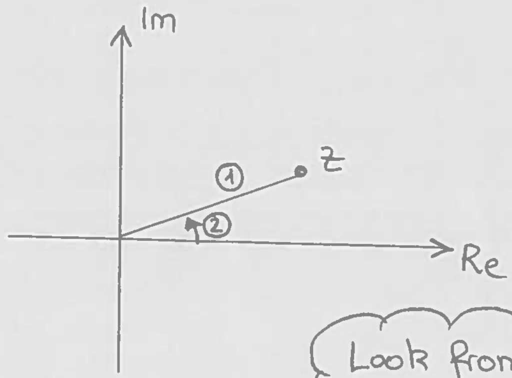
(7) $|z| = |\bar{z}|$

Proof Let $z = a+bi$. Then, $|z| = \sqrt{a^2+b^2}$, $|\bar{z}| = \sqrt{a^2+(-b)^2} = \sqrt{a^2+b^2}$. $\square$

(8) $|zw| = |z| \cdot |w|$ ← Important for polar form!

(9) $|z+w| \leq |z| + |w|$ ← Δ inequality!

§ 1.2 Polar Form



Describe $z \in \mathbb{C}$ in terms of

1. modulus $|z| \geq 0$
2. argument $\operatorname{Arg}(z) =: \vartheta$

with $-\pi < \vartheta \leq \pi$

Look from 0 into the positive real axis and measure the angle in mathematically positive direction, that is, counterclockwise!

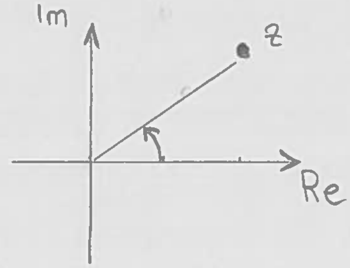
uniquely determined unless $z=0$

Of course, $\vartheta' = \vartheta + n \cdot 2\pi$, $n \in \mathbb{Z}$ also works!

$-\pi < \vartheta \leq \pi$ is called the principal argument of z!

Example:

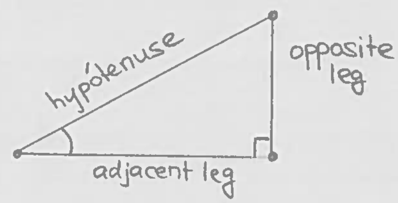
$z = 4 + 3i$



$|z| = \sqrt{4^2 + 3^2} = \sqrt{16+9} = \sqrt{25} = 5$

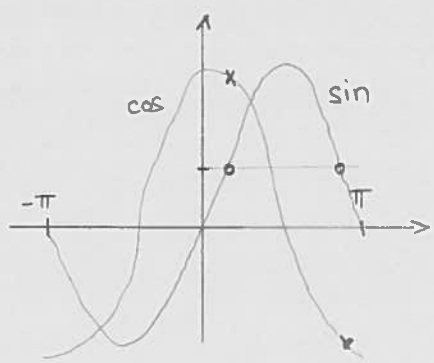
$\mathrm{Arg}(z) = ?$

In general:



$\sin(\vartheta) = \dfrac{\text{opposite leg}}{\text{hypotenuse}} = \dfrac{\mathrm{Im}(z)}{|z|}$

$\cos(\vartheta) = \dfrac{\text{adjacent leg}}{\text{hypotenuse}} = \dfrac{\mathrm{Re}(z)}{|z|}$



This allows us to reconstruct ϑ.

In our case: $\sin(\vartheta) = \frac{3}{5}$, $\cos(\vartheta) = \frac{4}{5}$

$\Rightarrow \vartheta \approx 36.87°$

Observe:

$\sin(\vartheta) = \dfrac{\mathrm{Im}(z)}{|z|}$ and $\cos(\vartheta) = \dfrac{\mathrm{Re}(z)}{|z|}$. Hence, we may rewrite:

$$z = \mathrm{Re}(z) + i \cdot \mathrm{Im}(z) = |z| \cdot \cos(\vartheta) + i \cdot |z| \cdot \sin(\vartheta)$$
$$= |z| \cdot \left(\cos(\vartheta) + i \cdot \sin(\vartheta)\right)$$

Sometimes just denoted by "r"

"Polar form of z"
(Version 1)

Ex 1.2

Taylor's magic

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

This yields $e^{ix} = \cos(x) + i \cdot \sin(x)$. In fact, one defines the complex exponential function using Taylor series.

Hence:

$$z = |z| \cdot \big(\cos(\vartheta) + i\sin(\vartheta)\big) = |z| \cdot e^{i\vartheta}$$

"Polar form of z"
(version 2)

Observations:

(a) If $r > 0$, then

$$r e^{i\vartheta} = r' e^{i\vartheta'} \iff r = r' \text{ and } \vartheta = \vartheta' + 2\pi n \text{ for some } n \in \mathbb{Z}.$$

(b) $\overline{r \cdot e^{i\vartheta}} = r \cdot e^{-i\vartheta}$

(c) $|e^{i\vartheta}| \stackrel{!}{=} 1$

Consider examples!

$2i$ $\quad -i \quad -1 \quad 1+i$

Multiplication of complex numbers in polar form

$$z = r \cdot e^{i\vartheta} \quad , \quad w = s \cdot e^{i\varphi}$$

$$\begin{aligned}
z \cdot w &= r \cdot (\cos(\vartheta) + i \sin(\vartheta)) \cdot s \cdot (\cos(\varphi) + i \sin(\varphi)) \\
&= rs \cdot \left[\cos(\vartheta)\cos(\varphi) - \sin(\vartheta)\sin(\varphi) + i \cdot (\cos(\vartheta)\sin(\varphi) + \cos(\varphi)\sin(\vartheta)) \right] \\
&= rs \cdot \left[\cos(\vartheta + \varphi) + i \sin(\vartheta + \varphi) \right] \\
&= rs \cdot e^{i(\vartheta + \varphi)}
\end{aligned}$$

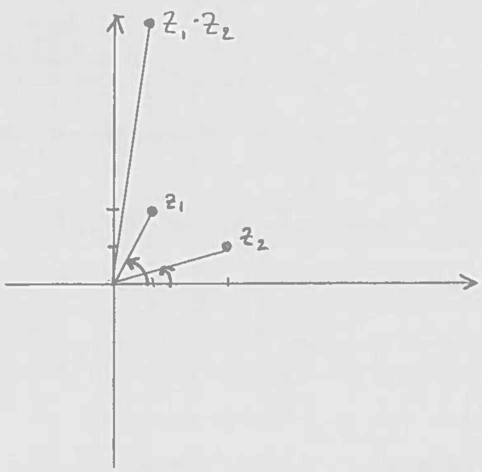
formulae

IMPORTANT RULE: Multiply the moduli (absolute values)!
Add the arguments!

Like usual multiplication with exponents!

also division!

Consider our example from before:



$z_1 = 1 + 2i$
$z_2 = 3 + i$

Example: $\frac{i}{1+i} = \frac{e^{i\frac{\pi}{2}}}{\sqrt{2}\, e^{i\frac{\pi}{4}}} = \frac{1}{\sqrt{2}}\, e^{i\left(\frac{\pi}{2} - \frac{\pi}{4}\right)} = \frac{\sqrt{2}}{2}\, e^{i\frac{\pi}{4}}$

LECTURE 3

§1.3 Fundamental Theorem of Algebra

Warm up: W15-T1 Slides # 187 - 230

Paul Erdős

Within $\mathbb{C}$, we can find solutions for Cardano's problem:

$$-x^2 + 10x - 40 = 0$$

Namely, $x_1 := 5 + \sqrt{15}\, i$ and $x_2 := 5 - \sqrt{15}\, i = \overline{x_1}$. We may decompose:

$$-x^2 + 10x - 40 = (-1)\cdot(x - x_1)\cdot(x - x_2)$$

This works in general:

Theorem (Gauß 1797):

Every polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ with complex coefficients can be completely factored using complex numbers:

$$p(z) = a_n \cdot (z - \omega_1) \cdot \ldots \cdot (z - \omega_n)$$

with $a_n, w_k \in \mathbb{C}$.

roots

Application (optional).

If $p(z) = a_n z^n + \dots + a_1 z + a_0$ with real coefficients, then $p(\omega) = 0 \Rightarrow p(\overline{\omega}) = 0$. So, whenever a root of $p(z)$ is not real, we know that it comes with its conjugate. Multiplying the two factors $(z - \omega)$ and $(z - \overline{\omega})$, we obtain a new factor:

$$\Big(z^2 - \underbrace{(\omega + \overline{\omega})}_{\substack{= 2\,\mathrm{Re}(z) \\ \in \mathbb{R}}} z + \underbrace{\omega\overline{\omega}}_{\substack{= |\omega|^2 \\ \in \mathbb{R}}}\Big)$$

Hence, every real polynomial can be factored into real polynomials of degree at most two!

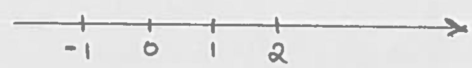
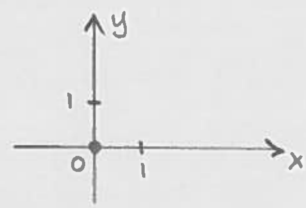
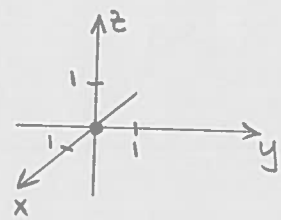
Time admitting, show slides about the Mandelbrot set ...

Gives students time to relax!

already done at the end of lecture 2

§ 2 VECTOR GEOMETRY

§ 2.1 Introduction

algebraic point of view	geometric point of view
$\mathbb{R}$ ("scalars")	
$\mathbb{R}^2 = \{(x,y) \mid x,y \in \mathbb{R}\}$	 (model for $\mathbb{C}$)
$\mathbb{R}^3 = \{(x,y,z) \mid x,y,z \in \mathbb{R}\}$	
$\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4) \mid x_i \in \mathbb{R}\}$	space-time (model for $\mathbb{H}$) Hamiltonians

$\exists$ "exists"

"for all"

$\vdots$

$\forall n \in \mathbb{N}$:

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R}\}$$

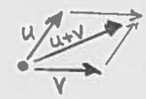
Just think about statistics ...

The algebraic structure of $\mathbb{R}^n$

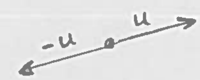
(1) Equality $\quad (x_1, \dots, x_n) = (y_1, \dots, y_n) \Leftrightarrow \forall i \in \{1, \dots, n\}: x_i = y_i$

(2) Addition $\quad (x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$ "componentwise"

(3) Zero vector $\quad (0, \dots, 0) \in \mathbb{R}^n$ Ex 2.1



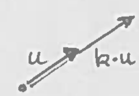
(4) Negative vector $\quad -(x_1, \dots, x_n) = (-x_1, \dots, -x_n)$



Remark (3) is the vector whose addition doesn't change anything. For each vector $(x_1, \dots, x_n)$, (4) is the vector whose addition results in $(0, \dots, 0)$!

(5) Multiplication BY A SCALAR !!!

For $k \in \mathbb{R}$, $k \cdot (x_1, \dots, x_n) = (kx_1, \dots, kx_n)$



Definition: Let $k_1, \dots, k_m \in \mathbb{R}$ and $u_1, \dots, u_m \in \mathbb{R}^n$. Then, the vector $k_1 u_1 + \dots + k_m u_m \in \mathbb{R}^n$ is called a linear combination of $u_1, \dots, u_m$.

Example:

$(0,1)$ is a linear combination of $(1,1)$, $(2,0)$.

Indeed:

$$(0,1) = 1 \cdot (1,1) - \tfrac{1}{2} \cdot (2,0)$$

Formally:

I $\quad 0 = a + 2b$

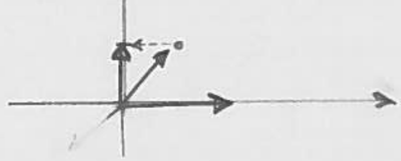
II $\quad 1 = a$

$0 = 1 + 2b$

$1 = a$

$-1/2 = b$

$1 = a$



Example (p. 17):

$(0,1,0)$ is not a linear combination of $(1,2,3)$ and $(1,0,0)$. Indeed,

$$\begin{pmatrix}0\\1\\0\end{pmatrix} = a\cdot\begin{pmatrix}1\\2\\3\end{pmatrix} + b\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} \iff \begin{array}{ll} \mathrm{I} & 0 = a+b \\ \mathrm{II} & 1 = 2a \\ \mathrm{III} & 0 = 3a \\ \hline & 0 = a+b \\ & \left.\begin{array}{l}\tfrac{1}{2} = a \\ 0 = a\end{array}\right\} \lightning \end{array}$$

Sometimes written in columns

Properties

V1 Associativity
$u+(v+w) = (u+v)+w$

V2 $\exists$ additive identity ($0 \in \mathbb{R}^n$)
that is: $v+0 = v$

V3 $\forall v \in \mathbb{R}^n$ $\exists$ inverse $-v \in \mathbb{R}^n$
that is: $v+(-v) = 0$

V4 Commutativity
$u+v = v+u$

properties of vectors

S1 Distributivity I
$\alpha(u+v) = \alpha u + \alpha v$

S2 Distributivity II
$(\alpha+\beta)v = \alpha v + \beta v$

S3 Associativity
$(\alpha\beta)v = \alpha(\beta v)$

S4 $1 \in \mathbb{R}$ does nothing
that is: $1v = v$

properties of scalar multiplication

Ex 2.2

We will soon generalize and start in a situation where these properties hold, without necessarily working with $\mathbb{R}^n$.

§ 2.2 The dot product

NB
$\|u\|=0 \Leftrightarrow u=0$

$n=3$

$u=(x_1,x_2,x_3)$, $v=(y_1,y_2,y_3)$

$u\cdot v = x_1y_1+x_2y_2+x_3y_3$

This way

- $\|u\| = \sqrt{u\cdot u} = \sqrt{x_1^2+x_2^2+x_3^2}$ "length/norm of u"
- $\|u-v\|$ = distance between u and v
- $u\cdot v=0 \Leftrightarrow u \perp v$ "orthogonal/perpendicular"

arbitrary $n\in\mathbb{N}$

$u=(x_1,\dots,x_n)$, $v=(y_1,\dots,y_n)$

$u\cdot v := x_1y_1+\dots+x_ny_n$

This way

- $\|u\| := \sqrt{u\cdot u} = \sqrt{x_1^2+\dots+x_n^2}$
  NB $\|u\|=0 \Leftrightarrow u=0$
  "length / norm of u"
- $\|u-v\|:=$ distance between u and v
- $u\cdot v=0 \Leftrightarrow: u\perp v$ "orthogonal / perpendicular"

"Euclidean space"

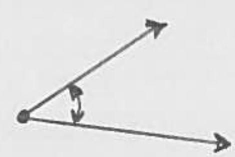
Ex 2.3

Sometimes, other dot products are considered. Each of them gives rise to another notion of "distance" and "orthogonality".

The dot product enables us to define angles between vectors as well...

$v \neq 0$

First, some work...



Theorem (Cauchy-Schwarz)

Let $u, v \in \mathbb{R}^n$. Then $|u\cdot v| \leq \|u\|\cdot\|v\|$.

Proof Without loss of generality, $u\neq 0$ and $v\neq 0$. Then,

PREFACE TO LECTURE 4

Properties of the dot product

Let $u, v, w \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$. Then:

- $(u+v) \cdot w = u \cdot w + v \cdot w$
- $u \cdot (v+w) = u \cdot v + u \cdot w$
- $(\lambda u) \cdot w = \lambda (u \cdot w) = u \cdot (\lambda w)$
- $\| \lambda u \| = |\lambda| \cdot \|u\|$

LECTURE 4

Present the Cauchy-Schwarz inequality (with its proof) on the slides...

Blackboard:

Cauchy-Schwarz inequality : $|u \cdot v| \le \|u\| \cdot \|v\|$

One consequence:

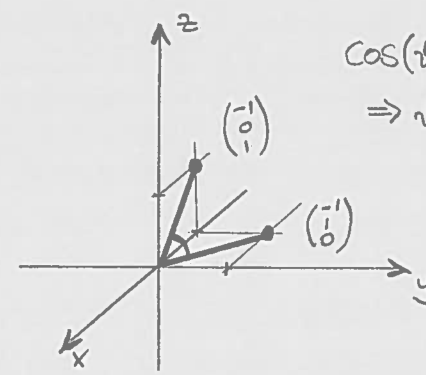
$$\begin{aligned} \|u+v\|^2 &= (u+v)\cdot(u+v) = u\cdot u + 2\,u\cdot v + v\cdot v \\ &= \|u\|^2 + 2\,u\cdot v + \|v\|^2 \\ &\le \|u\|^2 + 2\cdot\|u\|\cdot\|v\| + \|v\|^2 \\ &= (\|u\| + \|v\|)^2 \end{aligned}$$

Therefore:

$$\|u+v\| \le \|u\| + \|v\| \qquad \text{"triangle inequality"}$$

§2.3 Angles between vectors

Generalizing the notion of an angle to higher dims...



$\cos(\vartheta) = \frac{1}{2}$
$\Rightarrow \vartheta = \frac{\pi}{3}$

Def For $u, v \in \mathbb{R}^n$, $u \neq 0$, $v \neq 0$, the angle between u and v is defined as ϑ with:

- $\cos(\vartheta) = \dfrac{u \cdot v}{\|u\|\cdot\|v\|}$
- $0 \le \vartheta \le \pi$

Ex 2.4

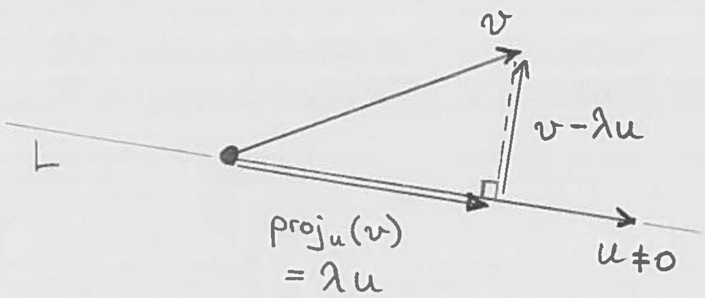
NB $u \perp v \iff u \cdot v = 0 \iff \vartheta = \frac{\pi}{2}$

$u \parallel v \iff \vartheta = 0$ or $\vartheta = \pi$

in which case: $u \cdot v = \pm \|u\| \cdot \|v\|$

↑ not hard!

§2.4 Orthogonal projections onto lines in $\mathbb{R}^n$



What is a suitable value for $\lambda \in \mathbb{R}$?

$$v - \lambda u \perp u$$
$$\Leftrightarrow \quad (v - \lambda u) \cdot u = 0$$
$$\Leftrightarrow \quad v \cdot u - \lambda (u \cdot u) = 0$$
$$\Leftrightarrow \quad \lambda = \frac{v \cdot u}{u \cdot u}$$

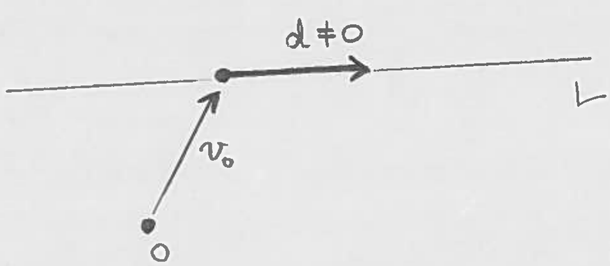
Hence,
$$\operatorname{proj}_u(v) = \frac{v \cdot u}{u \cdot u}\, u.$$

Ex 2.5

NB $\operatorname{proj}_u(v)$ is the unique point on L whose distance to v is minimal.

§3 LINES AND PLANES

§3.1 Describing lines



$$L = \{ v_0 + t d \mid t \in \mathbb{R} \}$$

support vector

direction vector

"vector form"

"parametric form"

$v_0 = (a, b, c)$, $d = (d_1, d_2, d_3)$

$$\begin{aligned} x &= a + t d_1 \\ y &= b + t d_2 \\ z &= c + t d_3 \end{aligned}$$

is an alternative notation!

Example:

$$L_1 = \left\{ \begin{pmatrix}1\\2\end{pmatrix} + t\begin{pmatrix}3\\4\end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\} \qquad L_2 = \left\{ \begin{pmatrix}5\\6\end{pmatrix} + t\begin{pmatrix}7\\8\end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

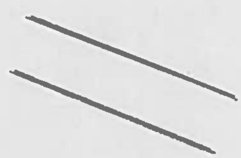
$L_1 \cap L_2 = ?$ Let's use the algebraic point of view!

$$\begin{pmatrix}1\\2\end{pmatrix} + t\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}5\\6\end{pmatrix} + s\begin{pmatrix}7\\8\end{pmatrix} \Leftrightarrow \begin{array}{ll} \text{I} & 1+3t = 5+7s \\ \text{II} & 2+4t = 6+8s \end{array}$$

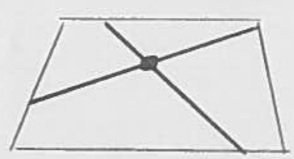
$$\Leftrightarrow \dots \Leftrightarrow s = -1 \text{ and } t = -1$$

Hence, $L_1 \cap L_2 = \left\{ \begin{pmatrix}-2\\-2\end{pmatrix} \right\}$.

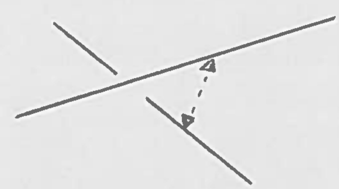
Lines in $\mathbb{R}^3$



"parallel"
(or even identical)

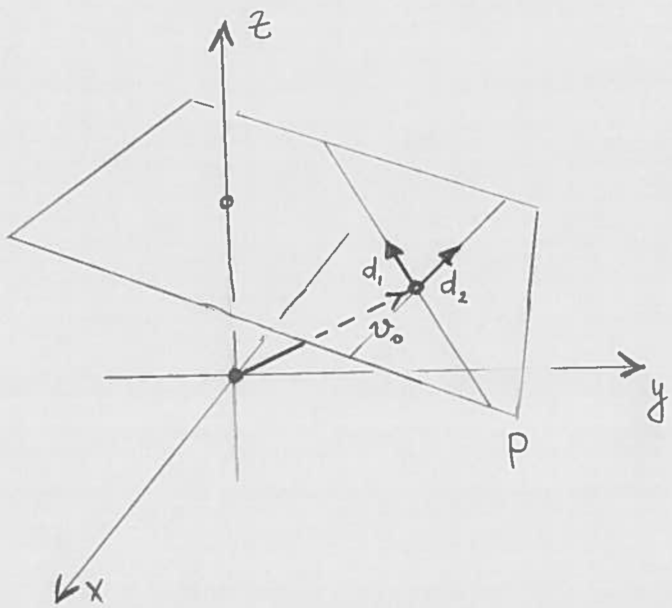


"intersecting in 1 pt"



"skew"

§ 3.2 Describing planes in $\mathbb{R}^3$

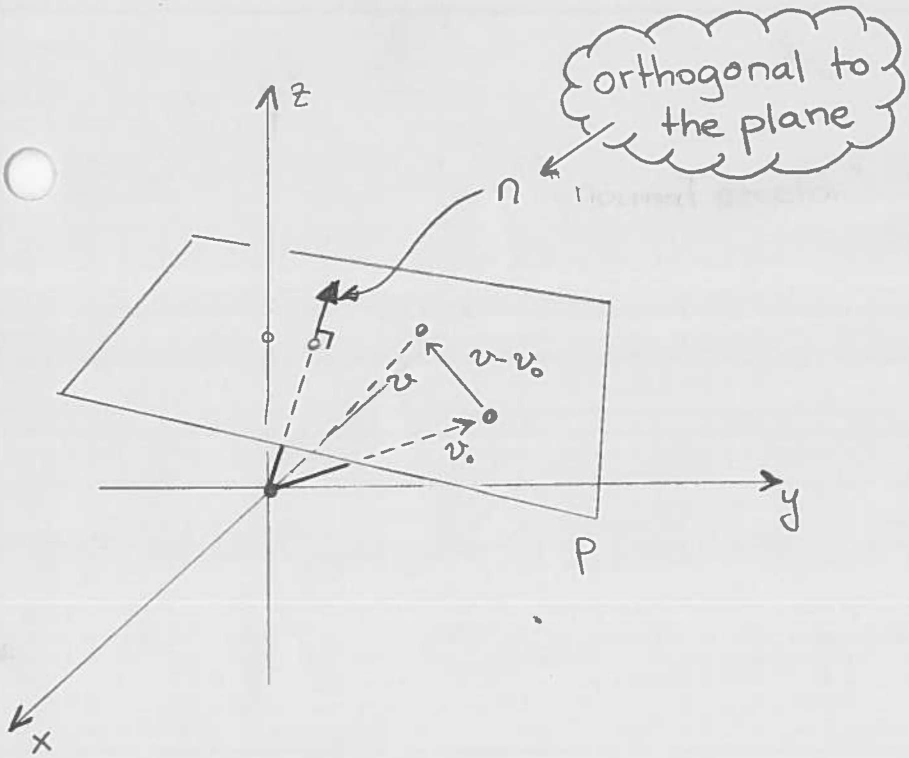


support vector (v_0), direction vectors (d_1, d_2)

$$P = \{ v_0 + t d_1 + s d_2 \mid t, s \in \mathbb{R} \}$$

"vector form"

This will be the way to generalize to higher dimensions!



Alternative way in $\mathbb{R}^3$:

$$P = \{ v \in \mathbb{R}^3 \mid (v - \underbrace{v_0}_{\text{support vector}}) \cdot \underbrace{n}_{\text{normal vector}} = 0 \}$$

"point-normal form"

$$= \{ v \in \mathbb{R}^3 \mid v \cdot n = \underbrace{v_0 \cdot n}_{\text{fixed number } \in \mathbb{R}} \}$$

<u>Example:</u> $n = (2, -1, 1)$, $v_0 = (0, 0, 3)$

$$P = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 3 \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; 2x - y + z = 3 \right\}$$

"Cartesian equation"

<u>Remark:</u> The coefficients in the cartesian equation give us a normal vector for the plane!

The coefficients and the normal vector are not unique!

Plane with $n = (1, 2, 3)$ and $v_0 = (4, 5, 6)$:

$$\left. \begin{aligned} x + 2y + 3z &= 32 \\ -2x - 4y - 6z &= -64 \end{aligned} \right\} \text{ same plane}$$

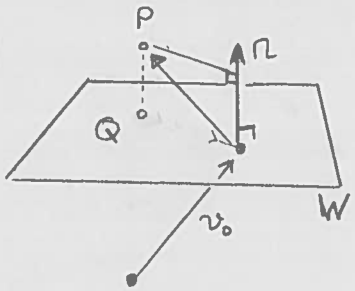
both are correct solutions

LECTURE 5

My numbers differ from textbook!

§ 3.3 Working with normal vectors

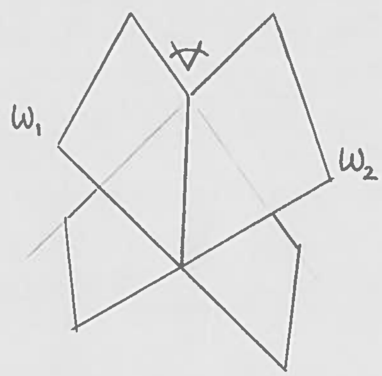
a)



Distance between P and W

$= \| P - Q \| = \| \operatorname{proj}_n (P - v_0) \|$

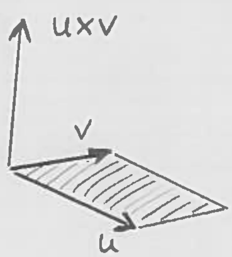
b) Def^n The angle between two planes in $\mathbb{R}^3$ is the angle between their normal vectors.



In particular,

$W_1 \perp W_2 \Leftrightarrow$ their normal vectors are orthogonal.

§ 3.4 Cross product in $\mathbb{R}^3$



Computation:

e.g. $u = (1,2,3)$ $v = (4,5,6)$

$$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = \begin{pmatrix} -3 \\ 6 \\ -3 \end{pmatrix}$$

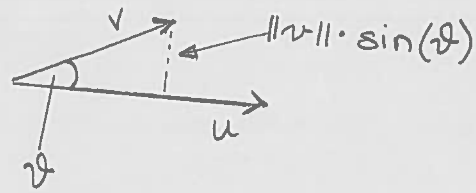
$$\begin{array}{ccccc} \hat{i} & \hat{j} & \hat{k} & \hat{i} & \hat{j} \\ 1 & 2 & 3 & 1 & 2 \\ 4 & 5 & 6 & 4 & 5 \end{array}$$

add ——

substract - - - -

Properties $\forall\, u, v, w \in \mathbb{R}^3$

- $u \times v = -\, v \times u$
- $(u \times v) \cdot u = (u \times v) \cdot v = 0$ ← orthogonal on u and v
- $(u + v) \times w = u \times w + v \times w$
- $\|u \times v\| = \|u\| \cdot \|v\| \cdot \sin(\vartheta)$ ← area of the parallelogram spanned by u and v



Warning! $u \times (v \times w) \neq (u \times v) \times w$

§ 3.5 Applications of the cross product

Slides!

How to prepare for the test? Solve (and understand) the past ones ...

Start distributing checklist!

§ 4 VECTOR SPACES

§ 4.1 An example different from $\mathbb{R}^n$

$$
\begin{aligned}
E_1 &: \quad x - y - z = -1 \\
E_2 &: \quad 2x - y + z = 1 \\
E_3 &: \quad -x + 2y + 4z = 4
\end{aligned}
$$

Using these equations we can construct new ones:

$$
\left.
\begin{aligned}
E_2 - 2E_1 &: \quad y + 3z = 3 \\
E_1 + E_3 &: \quad y + 3z = 3
\end{aligned}
\right\} \Rightarrow E_2 - 2E_1 = E_1 + E_3
$$

We may add equations and we may multiply them by a scalar $c \in \mathbb{R}$. In fact, the properties S1 - S4 and V1 - V4 are all satisfied.

Slides!

Checklist!

§ 4.2 Definition

Definition Any set V with two operations ("+" and "·") that satisfies the axioms on our checklist is called a vector space.

with "+" and "·" as in §4.1

Examples: $\mathbb{R}^n$, {equations in x, y, z}, ...

§ 4.3 Coming back to our example from § 4.1 ...

$$\mathcal{E} = \{ k_1 E_1 + k_2 E_2 + k_3 E_3 \mid k_i \in \mathbb{R} \} \subseteq \{\text{equations in } x, y, z\}$$

$\mathcal{E}$ is the set of all equations obtainable from E_1, E_2, E_3.

$\mathcal{E}$ is a vector space, too.

with "+" and "·" as in § 4.1

Subspace

This allows us to translate questions:

- Can we solve E_1, E_2, E_3 for x?
- Are there $k_1, k_2, k_3, x_0 \in \mathbb{R}$ such that "$x = x_0$" is $k_1 E_1 + k_2 E_2 + k_3 E_3$? Gauß
- Is there $x_0 \in \mathbb{R}$ such that "$x = x_0$" is a linear combination of E_1, E_2, E_3?

- Is there $x_0 \in \mathbb{R}$ such that "$x = x_0$" $\in \mathcal{E}$?

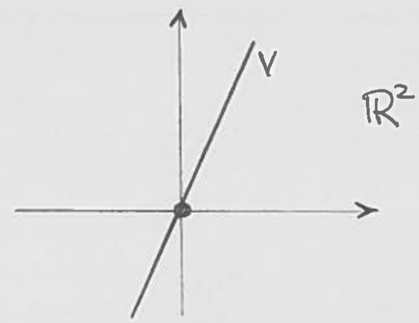
Similar to: Is some point in some plane ?

§ 4.4 Further examples

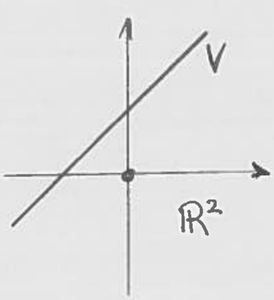
a) $V = \{0\}$ with $0 + 0 = 0$ and $c \cdot 0 = 0$ for all $c \in \mathbb{R}$.

b) $V = \{(x, 2x) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^2$ with "+" and "·" inherited from $\mathbb{R}^2$ (standard operations)

Check all axioms, as is done in Example 15 (p. 42) ...



c) $V = \{(x, x+2) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^2$ with "+" and "·" inherited from $\mathbb{R}^2$



Is NOT A VECTOR SPACE !

Showing that one axiom fails is enough.

③ $(0,0) \notin V$ — done

Remark We will discuss a "subspace test" that will help us to check whether a subset of a vector space with the inherited operations is actually a vector space.

LECTURE 6

Repeat p.22 of these notes in detail using the slides !!!

a) $V = \{0\}$

<u>with</u> $0 + 0 = 0$ and $\forall c \in \mathbb{R} : c \cdot 0 = 0$

b) $V = \{(x, 2x) \mid x \in \mathbb{R}\}$

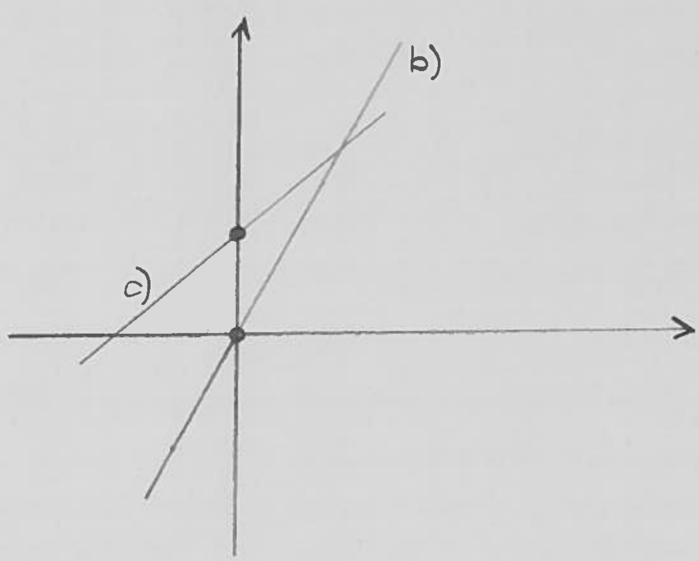
<u>with</u> "+" and "·" inherited from $\mathbb{R}^2$ (standard operations)

<u>That is:</u> $(x, 2x) + (y, 2y) = (x + y, 2x + 2y)$

$c \cdot (x, 2x) = (c \cdot x, c \cdot 2x)$

c) $V = \{(x, x+2) \mid x \in \mathbb{R}\}$

<u>with</u> "+" and "·" inherited from $\mathbb{R}^2$ (standard operations)



d) <u>Definition:</u> A matrix is a table of numbers. If it has m rows and n columns, we say its size is $m \times n$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ is of size } 2 \times 3$$

$M_{mn}(\mathbb{R}) :=$ {all matrices of size $m \times n$ with entries from $\mathbb{R}$}

$V = M_{22}(\mathbb{R})$ <u>with</u> componentwise addition and componentwise scalar multiplication

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

$$2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

Check {all} axioms, as is done in Example 17 (p. 44) ...

e) Let $[a,b] = \{x \in \mathbb{R} \mid a \le x \le b\}$.

$$V = F[a,b] = \{ f \mid f : [a,b] \to \mathbb{R} \}$$

<u>with</u> $(f+g)(x) := f(x) + g(x)$

$(cf)(x) := c(f(x))$

Mention Fourier series ...

f) $V = F(\mathbb{R}) = \{ f \mid f : \mathbb{R} \to \mathbb{R} \}$

§5 SUBSPACES

§5.1 Definition and Subspace Test

V vector space

$W \subseteq V$ a subset

Definition W is called a subspace of V, $W \leq V$, if, given it is endowed with the addition and scalar multiplication of V, it is a vector space itself.

Note

$\{(x, 2x) \mid x \in \mathbb{R}\} \leq \mathbb{R}^2$

with standard operations

Theorem (Subspace Test)

Let V be a vector space. A subset $W \subseteq V$ is a subspace of V iff the following hold:

(i) $0 \in W$

(ii) $\forall u, v \in W: \ u + v \in W$ ("closed under addition")

(iii) $\forall c \in \mathbb{R}, \forall u \in W: cu \in W$ ("closed under scalar multiplication")

Idea of the proof

(i), (ii), (iii) ensure axioms 3, 1, 2. Axioms 5 - 10 are inherited from V, as seen at the beginning of this lecture. And this yields 4 as discussed in Exercise 4.14 b.

(□)

$\underbrace{(-1)\cdot v}_{=: -v} + v = (-1+1)\cdot v = 0 \cdot v = 0$

because:

$0 \cdot v = (0+0)\cdot v = 0\cdot v + 0 \cdot v$

$\Rightarrow 0 = 0 \cdot v$

§5.2 Examples

a) Planes through the origin in $\mathbb{R}^3$ ← choose $u_0 = 0$ as support vector

$$T = \{ u \in \mathbb{R}^3 \mid u \cdot n = 0 \} \leq \mathbb{R}^3$$

fixed normal vector $n \neq 0$

Proof Apply the subspace test.

(i) $0 \in T$ because $0 \cdot n = 0$.

(ii) Let $u, v \in T$. Then, $(u+v) \cdot n = u \cdot n + v \cdot n = 0$.
So, $u + v \in T$.

(iii) Let $c \in \mathbb{R}$, $u \in T$. Then, $(cu) \cdot n = c\,(u \cdot n) = c \cdot 0 = 0$.
So, $cu \in T$.

□

Note If a plane does not go through the origin, (i) is not satisfied and it is therefore not a subspace.

LECTURE 7

Recall the subspace test:

V vector space, $W \subseteq V$ subset. W is a subspace of V iff:

(i) $0 \in W$ (ii) $\forall u, v \in W: u + v \in W$ (iii) $\forall c \in \mathbb{R}, u \in W: cu \in W$.

We have seen:

In $\mathbb{R}^3$: any plane through $(0,0,0)$ — subspace

any plane not through $(0,0,0)$ — no subspace

Now:

- In $\mathbb{R}^n$: any line through $(0,\dots,0)$ — subspace

  Proof Let $d \in \mathbb{R}$, $d \neq 0$, be a direction vector.

  $L = \{ t \cdot d \mid t \in \mathbb{R} \}$.

  (i) $(0,\dots,0) = 0 \cdot d \in L$.

  (ii) $t_1 \cdot d + t_2 \cdot d = (t_1 + t_2) d \in L$.

  (iii) $c \cdot (td) = (ct) d \in L$. $\square$

  any line not through $(0,\dots,0)$ — no subspace

- In $F(\mathbb{R})$:

c) $V = F(\mathbb{R})$ with "+" and "·" as before

$\mathbb{P} = \{\text{polynomial functions } \mathbb{R} \to \mathbb{R}\} \leq F(\mathbb{R})$

can be written as
$p(x) = a_n x^n + \dots + a_1 x + a_0$

d) Definition

The transpose of an $m \times n$ matrix A is the $n \times m$ matrix A^T whose rows are the columns of A.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$

Definition

$A \in M_{22}(\mathbb{R})$ symmetric $:\Longleftrightarrow A^T = A$

$V = M_{22}(\mathbb{R})$ with "+" and "·" as before

$S = \{A \in V \mid A \text{ symmetric}\} = \{A \in V \mid A^T = A\} \leq V$

Proof on p. 56 !

§6 THE SPAN OF VECTORS IN A VECTOR SPACE

Two ways of describing subspaces:

(1) $W = \{\text{ things } \mid \text{ conditions on things }\}$

e.g. $\{(x, y, z) \in \mathbb{R}^3 \mid x - 2y + z = 0\}$

(2) $U = \{$ things with parameters $\mid$ parameters are real $\}$

e.g. $\left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \;\middle|\; a,b,d \in \mathbb{R} \right\}$

$$= a \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

3 basic ingredients

§6.1 Converting from (1) to (2)

Example:

$$\begin{aligned} W &= \{(x,y,z) \in \mathbb{R}^3 \mid x - 2y + z = 0\} \\ &= \{(x,y,z) \mid x,y \in \mathbb{R},\ z = -x + 2y\} \\ &= \{(x, y, -x+2y) \mid x,y \in \mathbb{R}\} \end{aligned}$$

$$= x \cdot (1,0,-1) + y \cdot (0,1,2)$$

2 basic ingredients

§6.2 Definition of span

$v_1, \dots, v_m \in V$

Definition

(i) If $a_1, \dots, a_m \in \mathbb{R}$, then $a_1 v_1 + \dots + a_m v_m$ is called a linear combination of $v_1, \dots, v_m$.

(ii) $\underline{\text{span}}\{v_1, \dots, v_m\}$

$$= \{a_1 v_1 + \dots + a_m v_m \mid a_1, \dots, a_m \in \mathbb{R}\}$$

= set of all linear combinations of $v_1, \dots, v_m$

<u>We say:</u> $v_1, \dots, v_m$ spans $\text{span}\{v_1, \dots, v_m\}$

——— " ——— is a spanning set for $\text{span}\{v_1, \dots, v_m\}$

(iii) A vector space, or subspace, W is <u>spanned</u> by $v_1, \dots, v_m$ if $W = \text{span}\{v_1, \dots, v_m\}$.

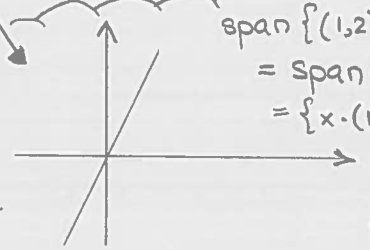
Examples:

- In § 6.1 we had $W = \text{span}\{(1,0,-1), (0,1,2)\}$.
- In the preliminaries to §6 we had $U = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

<u>Note</u> $\text{span}\{(1,0), (0,1)\} = \mathbb{R}^2$

$\text{span}\{(1,2), (3,6)\} \neq \mathbb{R}^2$

$(1,0) \notin$



$$\begin{aligned} &\text{span}\{(1,2),(3,6)\} \\ &= \text{span}\{(1,2)\} \\ &= \{x \cdot (1,2) \mid x \in \mathbb{R}\} \\ &= \{(x, 2x) \mid x \in \mathbb{R}\} \end{aligned}$$

We know this space from Lecture 6!

$$\begin{aligned} &\{a \cdot (1,2) + b \cdot (3,6) \mid a, b \in \mathbb{R}\} \\ &= \{a \cdot (1,2) + b \cdot 3 \cdot (1,2) \mid a, b \in \mathbb{R}\} \\ &= \{(a+3b) \cdot (1,2) \mid a, b \in \mathbb{R}\} \\ &= \{x \cdot (1,2) \mid x \in \mathbb{R}\} \end{aligned}$$

§ 6.3 Spanned sets are subspaces

Theorem Let V be a vector space, $\{v_1, \dots, v_m\} \subseteq V$, $U := \text{span}\{v_1, \dots, v_m\}$. Then,

(i) U is a subspace of V

(ii) If W is a subspace of V with $\{v_1, \dots, v_m\} \subseteq W$, then $U \subseteq W$. Hence, U is the "smallest" subspace containing $\{v_1, \dots, v_m\}$.

Proof

(i) Apply the subspace test:

1. $0 = 0 \cdot v_1 + \dots + 0 \cdot v_m \in U$
2. $u = a_1 \cdot v_1 + \dots + a_m \cdot v_m$, $v = b_1 v_1 + \dots + b_m v_m$. Then, $u + v = (a_1 + b_1) \cdot v_1 + \dots + (a_m + b_m) v_m \in U$.
3. $c \in \mathbb{R}$, $u = a_1 v_1 + \dots + a_m v_m$. Then, $cu = (ca_1) v_1 + \dots + (ca_m) v_m \in U$.

(ii) clear!

$\square$

Example

$$W = \{(x, y, x-y) \mid x, y \in \mathbb{R}\}$$
$$= \{x \cdot (1,0,1) + y \cdot (0,1,-1) \mid x, y \in \mathbb{R}\}$$
$$= \text{span}\{(1,0,1), (0,1,-1)\}$$

$\therefore$ Subspace of $\mathbb{R}^3$.

LECTURE 8

At the beginning, show slides to illustrate and recapitulate the notion of a span.

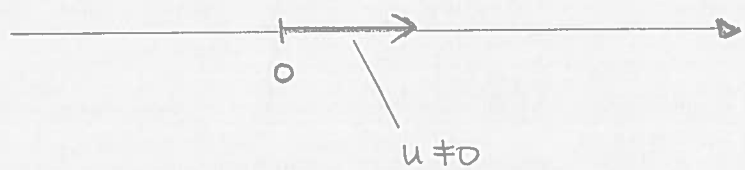
Then, give the proof from page 30.

(ii) Since $\{v_1, \dots, v_m\} \subseteq W$ and since W, being a vector space, is closed under addition and scalar multiplication, it must contain $\operatorname{span}\{v_1, \dots, v_m\} = U$.

$\square$

§6.4 Subspaces of $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$

a) $\mathbb{R}$: $\{0\}$, $\mathbb{R}$.



If W is a subspace of $\mathbb{R}$ with $0 \neq u \in W$, then every $x \in \mathbb{R}$ is in W:

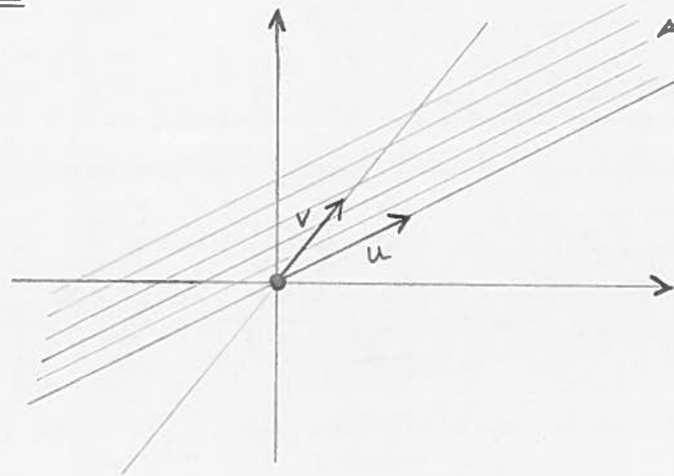
$$x = \left(\frac{x}{u}\right) \cdot u \in \operatorname{span}(\{u\}) \subseteq W$$

Hence, $W = \mathbb{R}$.

b) $\mathbb{R}^2$: $\{0\}$, lines through 0, $\mathbb{R}^2$.

<u>Idea:</u>

See Slides!



Either $W=\{0\}$, or $0 \neq u \in W$

<u>in which case</u> $\operatorname{span}\{u\} \subseteq W$.

Then, either W is line through 0, or $\operatorname{span}\{u\} \not\ni v \in W$

<u>in which case</u> $\operatorname{span}\{u,v\} \subseteq W$.

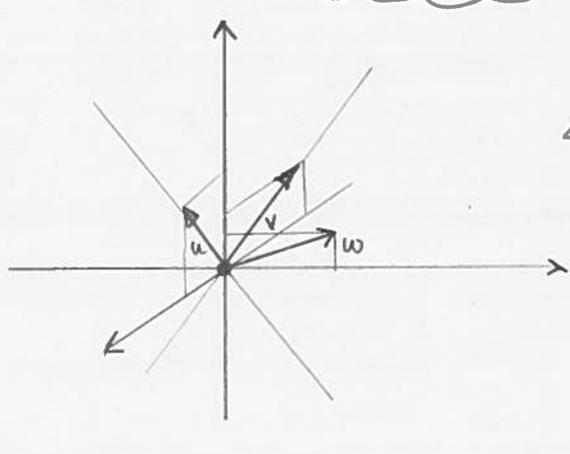
Then, $W = \mathbb{R}^2$.

c) $\mathbb{R}^3$: $\{0\}$, lines through 0, planes through 0, $\mathbb{R}^3$.

Analogously.

Explain verbally!

... and draw!



§6.5 Final Remark

The theorem from §6.3 does also allow us to check if two spans are equal:

$$\operatorname{span}\{u_1,\dots,u_m\} = \operatorname{span}\{v_1,\dots,v_n\}$$

$$\Leftrightarrow \quad \{u_1,\dots,u_m\} \subseteq \operatorname{span}\{v_1,\dots,v_n\}$$

$$\text{and } \{v_1,\dots,v_n\} \subseteq \operatorname{span}\{u_1,\dots,u_m\}.$$

Example

$$\operatorname{span}\{(1,0),(0,1)\} = \operatorname{span}\{(1,1),(1,-1)\}$$

because:

$$\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}, \quad \begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}, \quad \text{so "}\subseteq\text{".}$$

$$\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}, \quad \begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}, \quad \text{so "}\supseteq\text{".}$$

Hence, "=". □

§7 LINEAR (IN-)DEPENDENCE

§7.1 Difficulties with spans

Recall two phenomena from §6:

a) $\operatorname{span}\{(1,0),(0,1)\} = \operatorname{span}\{(1,1),(1,-1)\}$

It is not obvious if two subspaces are equal just by looking at their spanning sets.

b) As seen on the slides for Lecture 8:

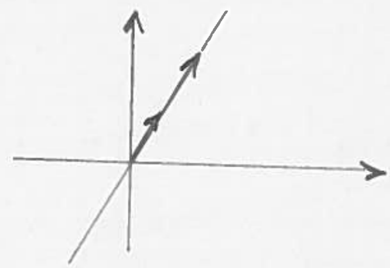
$$\text{Span}\{(1,2),(2,4)\} = \text{span}\{(1,2)\}$$

The number of elements in a spanning set does not necessarily tell you how "big" the subspace is.

Problem:

$(1,2)$ and $(2,4)$ are collinear.

"on one line"



maybe later!

Analogously, $(1,0,0), (0,1,0), (1,1,0) \in \mathbb{R}^3$ are coplanar, i.e. all lying in a plane.
And:

$$\text{span}\{(1,0,0),(0,1,0),(1,1,0)\} = \text{span}\{(1,0,0),(0,1,0)\}$$

" a plane through 0 "

§7.2 Collinearity in algebraic terms

1 vector in the span of the other!

u, v collinear

$\Leftrightarrow$ $u = k \cdot v$ for some $k \in \mathbb{R}$ or $v = k \cdot u$ for some $k \in \mathbb{R}$

This is not enough, imagine $v = 0$!

$\Leftrightarrow$ $\exists\, a, b \in \mathbb{R}$, not both 0, such that:

$$a u + b v = 0$$

2-in-1

Examples

a) $(1,2)$ and $(2,4)$ are collinear because:

$$2\cdot(1,2) - 1\cdot(2,4) = (0,0)$$

In particular, linear combinations are not unique:

$$2\cdot(1,2) - 1\cdot(2,4) = 0\cdot(1,2) + 0\cdot(2,4)$$

b) $(3,1)$ and $(0,0)$ are collinear because:

$$0\cdot(3,1) + 17\cdot(0,0) = (0,0)$$

§7.3 Coplanarity in algebraic terms

u, v, w coplanar

$\iff$ $u = kv + lw$ for some $k, l \in \mathbb{R}$

<u>or</u> $v = ku + lw$ for some $k, l \in \mathbb{R}$

<u>or</u> $w = ku + lv$ for some $k, l \in \mathbb{R}$

$\iff$ $\exists\, a, b, c \in \mathbb{R}$, not all 0, such that:

$$au + bv + cw = 0$$

3-in-1

Example

$(1,0,0), (0,1,0), (1,1,0)$ are coplanar because:

$$1\cdot(1,0,0) + 1\cdot(0,1,0) - 1\cdot(1,1,0) = (0,0,0)$$

§7.4 Definition

Definition

V vector space, $v_1, \dots, v_m \in V$. Then $\{v_1, \dots, v_m\}$ is <u>linearly dependent</u> (LD) if there are $a_1, \dots, a_m \in \mathbb{R}$, not all 0, such that $a_1 v_1 + \dots + a_m v_m = 0$.

LECTURE 9

Test 2 on
Monday, October 17
Duration: 75 mins
Content: upto and including §9

Recall from last time:

Definition

V vector space, $v_1, \dots, v_m \in V$.

(i) $\{v_1, \dots, v_m\}$ linearly dependent ("LD")

$\Leftrightarrow$ $\exists\, a_1, \dots, a_m \in \mathbb{R}$, not all 0, such that

$$a_1 v_1 + \dots + a_m v_m = 0$$

(ii) $\{v_1, \dots, v_m\}$ linearly independent ("LI")

$\Leftrightarrow$ $\{v_1, \dots, v_m\}$ not LD

$\Leftrightarrow$ The only solution to $a_1 v_1 + \dots + a_m v_m = 0$ is $a_1 = \dots = a_m = 0$.

Hence:

u, v collinear $\Leftrightarrow$ $\{u, v\}$ LD

u, v, w coplanar $\Leftrightarrow$ $\{u, v, w\}$ LD

Sometimes $\{\dots\}$ is omitted

§7.5 Examples

See pages 80 and 81 in VSF. Highlights:

a) $\{(1,0), (0,1)\}$ LI

because:

$$a \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{matrix} a = 0 \\ b = 0 \end{matrix}$$

b) $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\} \subseteq M_{22}(\mathbb{R})$ LD

because:

$$1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

c) $\{ x^2 + 4x + 4, x^2, x+1 \} \subseteq \mathbb{P}_2$ LD

because:

$$1 \cdot (x^2 + 4x + 4) - 1 \cdot (x^2) - 4 \cdot (x+1) = 0$$

d) Exercise: $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\} \subseteq \mathbb{R}^2$ ______ LI

$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\} \subseteq \mathbb{R}^3$ ______ LD

§7.6 Observations

① $\{v\}$ LI $\Leftrightarrow$ $v \neq 0$

Proof First, observe that for all $c \in \mathbb{R}$, $c \cdot 0 = 0$.
Indeed, $c \cdot 0 = c \cdot (0+0) = c \cdot 0 + c \cdot 0$, $\Rightarrow 0 = c \cdot 0$.
Now, if $v = 0$, then $1 \cdot v = 0$, whence $\{v\}$ LD.
If $v \neq 0$, then consider the equation $k \cdot v = 0$.
If there was a $k \neq 0$ solving it, then:

$$v = \frac{1}{k} \cdot k \cdot v = \frac{1}{k} \cdot 0 = 0 \quad \lightning$$

So, there can't be any such k, $\Rightarrow$ $\{v\}$ LI. $\square$

② $\{v_1, \dots, v_m\}$ LD $\Rightarrow$ any set containing $\{v_1, \dots, v_m\}$ LD

③ $\{v_1, \dots, v_m\}$ LI $\Rightarrow$ any subset of $\{v_1, \dots, v_m\}$ LI

Sketch ② and ③ !

④ $\{0\}$ LD ——— Special case of ①!

⑤ Any set containing $\{0\}$ LD ——— Special case of ②!

⑥ $\{u, v\}$ LD $\Leftrightarrow$ One vector is a multiple of the other.

Proof: "$\Rightarrow$" $au + bv = 0$ (a and b not both 0)

If $a \neq 0$, then $u = -\frac{b}{a} v$.

If $b \neq 0$, then $v = -\frac{a}{b} u$.

Maybe skip because ⑧ $\Rightarrow$ ⑥

"$\Leftarrow$" If $u = av$, then $1 \cdot u - a \cdot v = 0$, $\{u, v\}$ LD.

If $v = au$, then $a \cdot u - 1 \cdot v = 0$, $\{u, v\}$ LD. □

⑦ A set with more than 3 vectors can be LD even though no two are multiples of each other.

Consider: Example b)!

⑧ $\{v_1, \ldots, v_m\}$ LD $\Leftrightarrow$ $\exists\, k \in \{1, \ldots, m\}$ such that

$$v_k \in \text{span}\{v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_m\}$$

Proof: "$\Rightarrow$" $a_1 v_1 + \ldots + a_m v_m = 0$ (not all $a_i = 0$)

Choose k such that $a_k \neq 0$. Then,

$$v_k = -\frac{a_1}{a_k} v_1 - \ldots - \frac{a_{k-1}}{a_k} v_{k-1} - \frac{a_{k+1}}{a_k} v_{k+1} - \ldots - \frac{a_m}{a_k} v_m.$$

Hence $v_k \in \text{span}\{v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_m\}$.

"$\Leftarrow$" $v_k = a_1 v_1 + \ldots + a_{k-1} v_{k-1} + a_{k+1} v_{k+1} + \ldots + a_m v_m$

$\Rightarrow a_1 v_1 + \ldots + a_{k-1} v_{k-1} - 1 \cdot v_k + a_{k+1} v_{k+1} + \ldots + a_m v_m = 0$ (not all coeffs $= 0$) □

In §8 we will be looking for LI spanning sets!

§8 LI AND SPANNING SETS

§8.1 REDUCING LD spanning sets

Recall: $\operatorname{span}\{(1,2),(2,4)\} = \operatorname{span}\{(1,2)\}$

LD

To prove this, we observed that

$$\begin{aligned} & a\cdot(1,2) + b\cdot(2,4) \\ &= a\cdot(1,2) + b\cdot 2\cdot(1,2) \\ &= (a+2b)\cdot(1,2) \end{aligned}$$

This can be generalized:

Theorem

Consider $\operatorname{span}\{v_1,\dots,v_m\}$. If $v_1 \in \operatorname{span}\{v_2,\dots,v_m\}$, then:

$$\operatorname{span}\{v_1,\dots,v_m\} = \operatorname{span}\{v_2,\dots,v_m\}$$

Proof "$\supseteq$" is clear. We give an argument for "$\subseteq$".
$v_1 = a_2 v_2 + \dots + a_m v_m$. Now, let $w \in \operatorname{span}\{v_1,\dots,v_m\}$.
So, $w = b_1 v_1 + \dots + b_m v_m$. We can rewrite that:

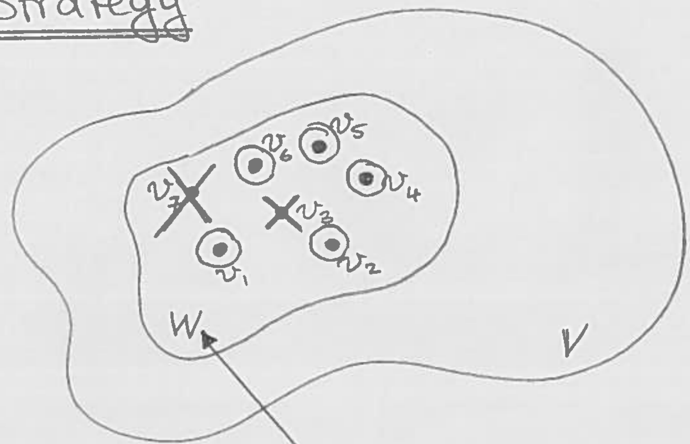
$$\begin{aligned} w &= b_1\cdot(a_2 v_2 + \dots + a_m v_m) + b_2 v_2 + \dots + b_m v_m \\ &= (b_2 + b_1 a_2)\, v_2 + \dots + (b_m + b_1 a_m)\, v_m \\ &\in \operatorname{span}\{v_2,\dots,v_m\} \end{aligned}$$

$\square$

Message: We can reduce any LD spanning set.

Use observation 8 and the above theorem!

Strategy



linear combination of the remaining ones

$$W = \operatorname{span}\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$$

$$= \operatorname{span}\{v_1, v_2, v_4, v_5, v_6, v_7\}$$

linear combination of the remaining ones

$$= \operatorname{span}\underbrace{\{v_1, v_2, v_4, v_5, v_6\}}_{\text{eventually LI}}$$

LECTURE 10

Show slides to give students an overview over the tests F14-T2, W15-T2, F15-T2 and to recapitulate §8.1 from last time.

Doing so, observe:

a) $0\,v_1 + 0\,v_2 + 0\,v_3 + 1\,v_4 + 0\,v_5 = 0$

$\Rightarrow \quad v_4 = 0\,v_1 + 0\,v_2 + 0\,v_3 + 0\,v_5 \in \operatorname{span}\{v_1, v_2, v_3, v_5\}$

b) $0\,v_1 + 2\,v_2 + 0\,v_3 - 1\,v_5 = 0$

$\Rightarrow \quad v_5 = 0\,v_1 - 2\,v_2 + 0\,v_3 \in \operatorname{span}\{v_1, v_2, v_3\}$

c) $2\,v_1 - 1\,v_2 + 1\,v_3 = 0$

$\Rightarrow \quad v_2 = 2\,v_1 + 1\,v_3 \in \operatorname{span}\{v_1, v_3\}$

LI

§8.2 Increasing LI sets

Theorem:

Let $\{v_1, \dots, v_m\}$ LI and consider $\operatorname{span}\{v_1, \dots, v_m\}$. Then,

$$\{v, v_1, \dots, v_m\} \text{ LI} \iff v \notin \operatorname{span}\{v_1, \dots, v_m\}$$

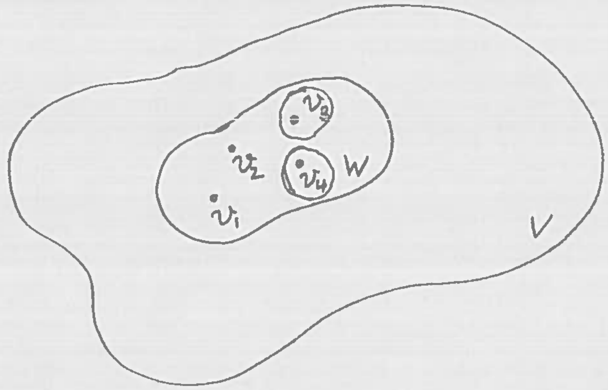
Proof

"$\Rightarrow$" By Observation (8), none of $v, v_1, \dots, v_m$ is contained in the span of the remaining ones. In particular, v is not.

"$\Leftarrow$" Consider $av + a_1 v_1 + \dots + a_m v_m = 0$.
If $a \neq 0$, then $v = -\frac{a_1}{a} v_1 - \dots - \frac{a_m}{a} v_m$,
$\Rightarrow v \in \text{span}\{v_1, \dots, v_m\}$. This is not possible,
so $a = 0$. But, since $\{v_1, \dots, v_m\}$ LI, also
$a_1 = \dots = a_m = 0$. $\square$

Message: We can increase any LI set as long as it does not already span the vector space.

Strategy



- $\{v_1, v_2\}$ LI
  but $\operatorname{span}\{v_1, v_2\} \neq W$
- $\{v_1, v_2, v_3\}$ add: $v_3 \notin \operatorname{span}\{v_1, v_2\}$
  but span $\neq W$
- $\{v_1, v_2, v_3, v_4\}$ add: $v_4 \notin \operatorname{span}\{v_1, v_2, v_3\}$
  with span $= W$

Usually, we consider the case $W = V$.

§ 8.3 Examples

first b) a) $\mathbb{P}_2$: $\{x^2, 1+2x\}$ LI

because $a \cdot x^2 + b \cdot (2x+1)$

$$= ax^2 + 2bx + b =: p(x) \overset{!}{\equiv} 0$$

"Should be zero everywhere"

Good reasoning for polynomials!

If $a \neq 0$, then $p(x)$ could have at most two roots. So, $a = 0$. Moreover, $p(0) = b$. So, $b = 0$. □

$x^3 \notin \operatorname{span}\{x^2, 1+2x\}$

similar argument.

Hence, $\{x^3, x^2, 1+2x\}$ LI.

b) $M_{22}(\mathbb{R})$: $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$ LI

<u>because</u> $a \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \overset{!}{=} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This is only possible for $a=b=0$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin \operatorname{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

similar argument.

Hence $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$ LI.

c) $\mathbb{R}^4$: $\{(1,2,1,1), (1,3,5,6)\}$ LI

$(1,0,0,0) \notin \operatorname{span}\{(1,2,1,1), (1,3,5,6)\}$

<u>because</u>

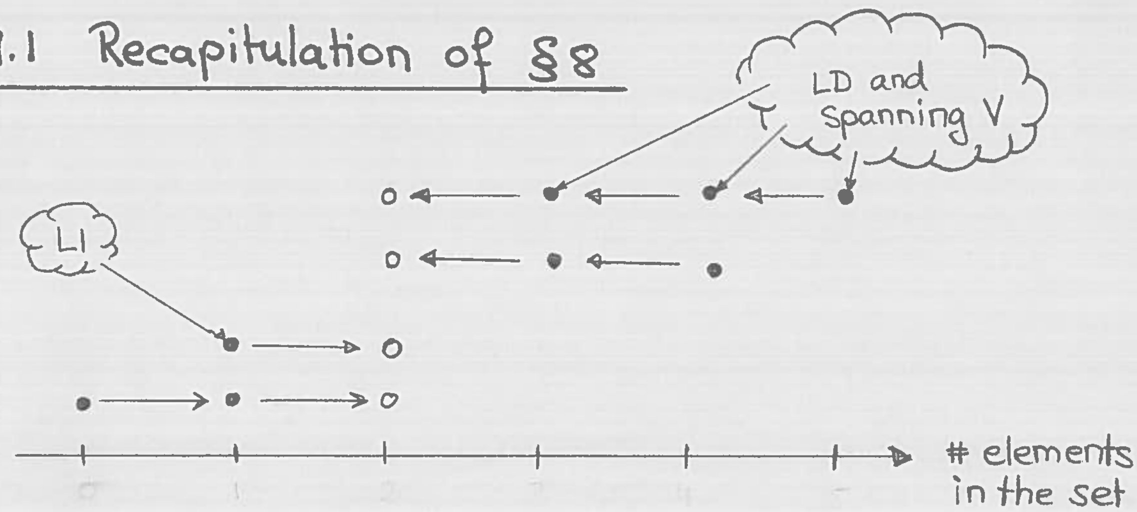
$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = a \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} + b \cdot \begin{pmatrix} 1 \\ 3 \\ 5 \\ 6 \end{pmatrix} \Leftrightarrow \begin{aligned} 1 &= a+b \\ 0 &= 2a+3b \\ 0 &= a+5b \\ 0 &= a+6b \end{aligned} \quad \lightning$$

Hence. $\{(1,2,1,1), (1,3,5,6), (1,0,0,0)\}$ LI.

<u>Difficulty:</u> Finding $v \notin \operatorname{span}\{\ldots\}$.

§9 Basis and dimension

§9.1 Recapitulation of §8



"o" = LI and spanning V

<u>Outlook</u> The # elements in the set "o" does not depend on which set we started and into which direction we went.

<u>Outlook</u> It is a property of V, and is called the "dimension" of V. Any set "o", i.e. any LI spanning set, is called a "basis" of V.

§9.2 Example in $\mathbb{R}^2$

first b)

a) $\emptyset \longrightarrow \{(1,0)\} \longrightarrow \{(1,0),(0,1)\}$

LI   LI   LI and spanning

<u>Sideremark</u>

Could we go further maintaining LI? — No!

Indeed, if $v \in \mathbb{R}^2$ then $v = a\cdot(1,0) + b\cdot(0,1)$,

$\Rightarrow a\cdot(1,0) + b\cdot(0,1) - 1\cdot v = 0, \Rightarrow \{(1,0),(0,1),v\}$ LD.

b) $\{(1,2),(3,4),(5,6)\} \longrightarrow \dots$

LD and spanning

$$\begin{pmatrix}5\\6\end{pmatrix} = a\begin{pmatrix}1\\2\end{pmatrix} + b\begin{pmatrix}3\\4\end{pmatrix} \rightsquigarrow \begin{matrix}5 = a+3b\\ 6 = 2a+4b\end{matrix} \rightsquigarrow \dots \rightsquigarrow \begin{matrix}a=-1\\ b=2\end{matrix}$$

So, $\begin{pmatrix}5\\6\end{pmatrix} = -\begin{pmatrix}1\\2\end{pmatrix} + 2\begin{pmatrix}3\\4\end{pmatrix}$.

$\dots \longrightarrow \{(1,2),(3,4)\}$

LI and spanning

Sideremark

Could we go further maintaining spannig? – No! Indeed, both span$\{(1,2)\}$ and span$\{(3,4)\}$ are lines through 0 and therefore $\neq \mathbb{R}^2$.

Hence

- $\dim(\mathbb{R}^2) = 2$
- $\{(1,0),(0,1)\}$ and $\{(1,2),(3,4)\}$ are both bases of $\mathbb{R}^2$.

§9.3 A general theorem

Theorem If a vector space V can be spanned by n vectors, then any LI subset has at most n vectors.

Message:

size of any LI set in V

$\leqslant$ size of any spanning set of V

Examples

a) $\mathbb{R}^3 = \text{span}\{(1,0,0),(0,1,0),(0,0,1)\}$

$\Rightarrow$ Any LI set in $\mathbb{R}^3$ has at most 3 vectors, any set in $\mathbb{R}^4$ with at least 4 vectors is LD.

b) $M_{22}(\mathbb{R}) = \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$\Rightarrow$ Any LI set in $M_{22}(\mathbb{R})$ has at most 4 vectors, any set in $M_{22}(\mathbb{R})$ with at least 5 vectors is LD.

c) $\mathcal{U} = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \,\middle|\, x,y \in \mathbb{R} \right\} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \subseteq \mathbb{R}^3$

$\Rightarrow$ Any LI set in $\mathcal{U}$ has at most 2 vectors, even though there are LI sets with 3 vectors in the surrounding space $\mathbb{R}^3$.

LECTURE 11

Announcement

For the test on Monday, you must bring your student card. You may be asked for it.

Hint for the test...

If you want to prove that some statement is correct, give a general argument. If you want to prove that some statement is wrong, construct an explicit counterexample (with numbers, ...).

to show that the statement or certain axioms fail!

Recall from last time:

size of any LI set in V

$\leq$ size of any spanning set in V

§9.4 Time for definitions

Definition

Let V be a vector space and $\{v_1, \ldots, v_m\}$ a set of vectors in V. $\{v_1, \ldots, v_m\}$ is called a basis of V if it is LI and it spans V.

So, what is a basis:

- LI spanning set of V
- biggest possible LI set in V
- smallest possible spanning set in V

Examples

a) $\{(1,0),(0,1)\}$ and $\{(1,2),(3,4)\}$ are both bases of $\mathbb{R}^2$.

b) $\{x^2, x, 1\}$ is a basis of $\mathbb{P}_2$. (Proof?!)

c) $\{(1,0)\}$ and $\{(1,0),(0,1),(1,1)\}$ are not bases of $\mathbb{R}^2$.

Theorem

If $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_n\}$ are two bases for a vector space V, then $m=n$.

Proof $\{v_1, \ldots, v_m\}$ LI, $\{w_1, \ldots, w_n\}$ spans V,
$\Rightarrow m \leq n$. Analogously, $n \leq m$. So, $m=n$. $\square$

Definition

If V has a finite basis $\{v_1, \ldots, v_m\}$, then the dimension of V is n, $\dim(V)=n$. If V doesn't have a finite basis, then it is infinite dimensional.

§9.5 Examples

a) $\dim(\mathbb{R}^2) = 2$

because $\{(1,0),(0,1)\}$ is a basis of $\mathbb{R}^2$.

b) $\dim(\mathbb{R}^3) = 3$, $\dim(\mathbb{R}^n) = n$ (see Example 67)

c) $\dim(M_{22}(\mathbb{R})) = 4$, $\dim(M_{mn}(\mathbb{R})) = mn$

(see Example 69)

d) $\dim(\mathbb{P}_2) = 3$, $\dim(\mathbb{P}_n) = n+1$

(see Example 68)

e) $\mathbb{P}$ and $\mathbb{F}(\mathbb{R})$ are infinite dimensional.

(see Examples 70-71)

f) $L = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \mid a,b,c \in \mathbb{R} \right\}$

has basis $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ (Proof!?)

whence $\dim(L) = 3$

first h)

g) $L = \{ (x+y, x+y, z) \mid x,y,z \in \mathbb{R} \}$

$= \operatorname{span}\{ (1,1,0), (1,1,0), (0,0,1) \}$

$= \operatorname{span}\{ (1,1,0), (0,0,1) \}$

Hence, $\dim(L) = 2$.

L1

h) $L = \{ (x,y,-x) \mid x,y \in \mathbb{R} \}$

$= \operatorname{span}\{ (1,0,-1), (0,1,0) \}$

Hence, $\dim(L) = 2$.

§ 10 Dimension theorems

The crowning of part II of the course!
Not part of test 2 ...

We already know:

Size of any LI set in V
$\leq \dim(V)$
$\leq$ size of any spanning set of V

Just recall that we may think of a basis as biggest possible LI set in V and as smallest possible spanning set of V.

§10.1 Obtaining bases

Recall that

(1) Every linearly independent subset of V can be extended to a basis of V:

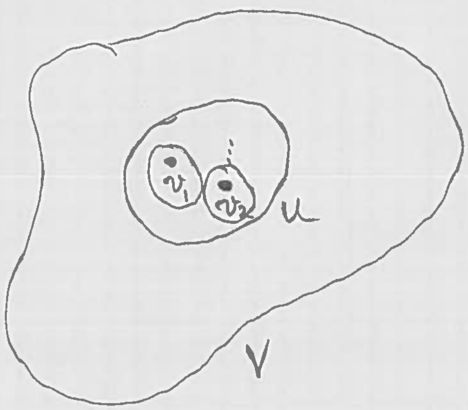
$$\{(1,0)\} \rightsquigarrow \{(1,0),(1,1)\}$$

(2) Every spanning set of V can be reduced to a basis of V:

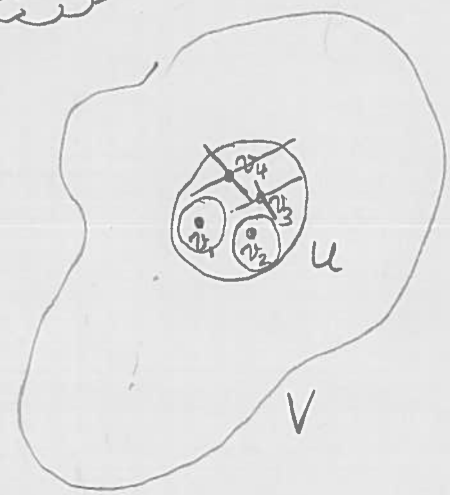
$$\{(1,0),(0,1),(1,1)\} \rightsquigarrow \{(1,0),(0,1)\}$$

If we know that $\dim(V) < \infty$ and want to find a basis for a subspace U, we could <u>either</u> start taking non-zero vectors from U forming larger and larger LI sets <u>or</u> start with a spanning set (finite) of U and cut it down.

a)



b)



§ 10.2 Checking for bases and dim of subspaces

Theorem Let V be a vector space with $\dim(V) = n < \infty$. Then,

(i) any LI set $\{v_1, \dots, v_n\}$ is a basis of V.

(ii) any spanning set $\{v_1, \dots, v_n\}$ is a basis of V.

Proof

(i) If $\{v_1, \dots, v_n\}$ didn't span V, pick $v \notin \text{span}\{v_1, \dots, v_n\}$. Then, $\{v, v_1, \dots, v_n\}$ is LI. In contradiction to:

$$\text{size of any LI set in } V \leq \dim(V).$$

(ii) If $\{v_1, \dots, v_n\}$ was LD, remove one element v_k without changing the span. Then, $\{v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_n\}$ is still spanning V. In contradiction to:

$$\dim(V) \leq \text{size of any spanning set of } V.$$

$\square$

<u>Theorem</u> Let V be a vector space with $\dim(V) = n < \infty$. Let U be a subspace of V. Then,

(i) $0 \leq \dim(U) \leq \dim(V)$

(ii) $\dim(U) = \dim(V) \iff U = V$

(iii) $\dim(U) = 0 \iff U = \{0\}$

Complete proof on p.105 of VSF! <u>For example:</u>

(ii) "$\Rightarrow$" Let $\dim(U) = \dim(V) = n < \infty$. Suppose that $U \subsetneqq V$. Pick any $v \in V$ with $v \notin U$. If $\{u_1, \dots, u_n\}$ is a basis of U, then $\{v, u_1, \dots, u_n\}$ is a LI set in V, by §8.2. In contradiction to:

$$\text{size of any LI set in } V \leq \dim(V) = n.$$

($\square$)

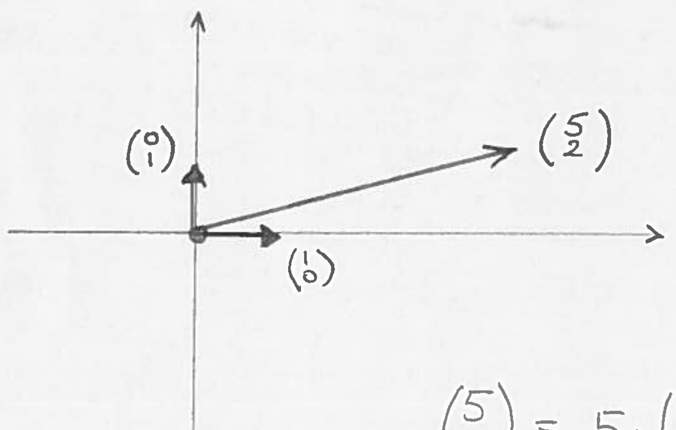
<u>Note:</u> If U is a subspace of V, $\dim(U) = m$. Then, any subspace of V which is contained in U is also a subspace of U and, as such, has dimension $\leq m$.

§10.3 Two examples

a) $\{(1,2),(3,4)\}$ is LI because none of the two vectors is a multiple of the other. But, $\dim(\mathbb{R}^2)=2$. So $\{(1,2),(3,4)\}$ is a basis by the first theorem.

b) Any subspace U of $M_{22}(\mathbb{R})$ has $0 \leq \dim(U) \leq 4$ because $\dim(M_{22}(\mathbb{R}))=4$. If $\dim(U)=4$, then U is the full vector space $M_{22}(\mathbb{R})$.

§10.4 The crowning – coordinats



ordered!

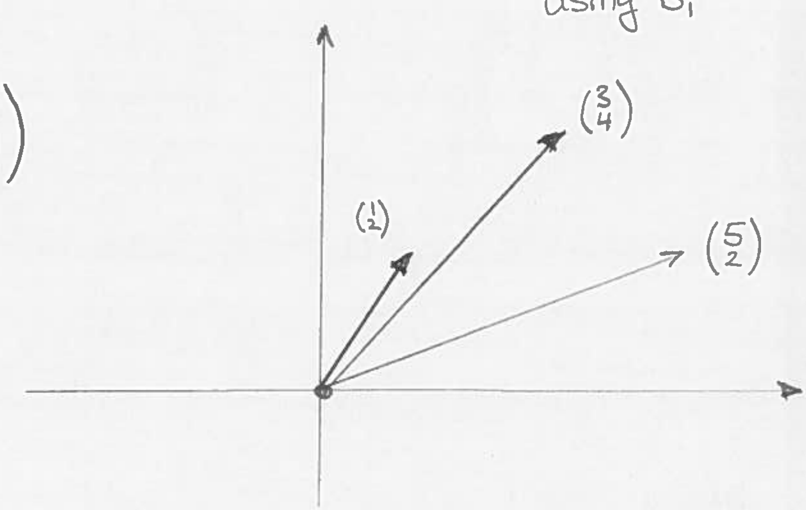
basis:
$B_1 = \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$

$$\begin{pmatrix}5\\2\end{pmatrix} = 5\cdot\begin{pmatrix}1\\0\end{pmatrix} + 2\cdot\begin{pmatrix}0\\1\end{pmatrix} = \begin{bmatrix}5\\2\end{bmatrix}_{B_1}$$

"coordinates" unique address using B_1

basis:
$B_2 = \left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}\right\}$

$$\begin{pmatrix}5\\2\end{pmatrix} = -7\cdot\begin{pmatrix}1\\2\end{pmatrix} + 4\cdot\begin{pmatrix}3\\4\end{pmatrix}$$
$$= \begin{bmatrix}-7\\4\end{bmatrix}_{B_2}$$



Why is that useful?

(i) "Understanding" presumably complicated spaces:

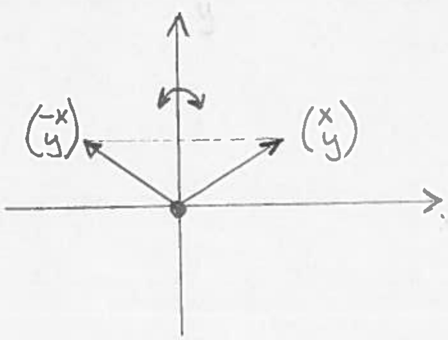
$\mathbb{P}_2$

basis: $\{x^2, x, 1\} =: B$ $\qquad\qquad$ $\mathbb{R}^3$

$7x^2 - 3x + 5$

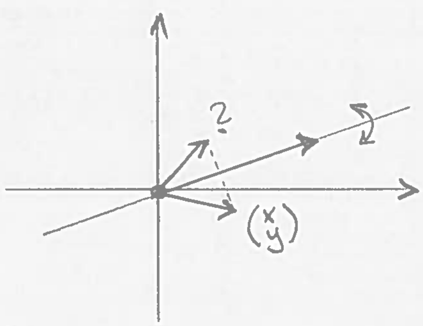
$$= \begin{bmatrix} 7 \\ -3 \\ 5 \end{bmatrix}_B \xleftrightarrow{\text{identify}} \begin{pmatrix} 7 \\ -3 \\ 5 \end{pmatrix}$$

(ii) Choosing "appropriate" coordinate systems:



Reflection at y-axis

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -x \\ y \end{pmatrix}$$



Reflection at axis spanned by $(5,2)$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \ ?$$

We may solve this by choosing a new basis $B := \{(5,2), (2,-5)\}$:

$$\begin{bmatrix} a \\ b \end{bmatrix}_B \mapsto \begin{bmatrix} a \\ -b \end{bmatrix}_B$$

Now, we only need to master the translation!

§10.5 Final theorem

Why is the address unique?

Theorem

Let $B = \{v_1, \dots, v_n\}$ be an ordered basis for a vector space V. Then, for every $v \in V$, there are unique $a_1, \dots, a_n \in \mathbb{R}$ such that:

$$v = a_1 v_1 + \dots + a_n v_n = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}_B$$

Proof sketch

Existence of $a_1, \dots, a_n \in \mathbb{R}$ is clear because B spans V. Suppose there were two addresses $a_1, \dots, a_n \in \mathbb{R}$ and $b_1, \dots, b_n \in \mathbb{R}$. Then:

$$a_1 v_1 + \dots + a_n v_n = b_1 v_1 + \dots + b_n v_n$$

$$\Rightarrow (a_1 - b_1) v_1 + \dots + (a_n - b_n) v_n = 0$$

$$\Rightarrow \forall i : \quad a_i - b_i = 0 \quad \text{(because } B \text{ is LI)}$$

$$\Rightarrow \forall i : \quad a_i = b_i$$

This is uniqueness. □

End of part II, now it is going to be much more explicit...

§ 11 Solving systems of linear equations

A good question from last week: "How to detect LI if the numbers are not that easy?"

§ 11.1 Motivating example

Is $\left\{ \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix} \right\}$ LD or LI ?

We are interested in solutions to:

$$a \cdot \begin{pmatrix}1\\2\\3\end{pmatrix} + b \cdot \begin{pmatrix}4\\5\\6\end{pmatrix} + c \cdot \begin{pmatrix}7\\8\\9\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Rewrite, but now each line separately:

$$\begin{aligned} a + 4b + 7c &= 0 \\ 2a + 5b + 8c &= 0 \\ 3a + 6b + 9c &= 0 \end{aligned}$$

"homogeneous linear system"
that is: all RHSs are 0

"coefficients"

This is a "linear system" with $m=3$ equations and $n=3$ unknowns / variables.

We are interested in the set S of all solutions (a,b,c) that satisfy the above equations simultaneously. The set S is called a "general solution" to the linear system.

LECTURE 13

§ 11.2 Three easy examples and important vocabulary

Recapitulate § 11.1 with slides!

including vocabulary

a)
$$\begin{aligned} a + 2b &= 7 \\ 2a - b &= 4 \end{aligned}$$
$$\begin{aligned} a + 2b &= 7 \\ -5b &= -10 \end{aligned}$$
$$b = 2$$
$$a = 3$$
$$\Rightarrow S = \{(3,2)\}$$

"general solution"

b)
$$\begin{aligned} a + 2b &= 7 \\ -a - 2b &= 1 \end{aligned}$$
$$\begin{aligned} a + 2b &= 7 \\ 0 &= 8 \end{aligned}$$
$$\lightning$$
$$\Rightarrow S = \{\} = \emptyset$$

c)
$$\begin{aligned} a + 2b &= 7 \\ 3a + 6b &= 21 \end{aligned}$$
$$\begin{aligned} a + 2b &= 7 \\ 0 &= 0 \end{aligned}$$
free parameter $b = t$

$a = 7 - 2t$

$$\Rightarrow S = \{(7-2t, t) \mid t \in \mathbb{R}\}$$
$$\Rightarrow S = \{(7,0) + t \cdot (-2,1) \mid t \in \mathbb{R}\}$$

"inhomogeneous linear system"
that is: not all RHSs are zero

"degenerate equations"

one solution

no solution

∞ many solutions

"inconsistent linear system"

"consistent linear system"

that is: at least one solution

Here, we are adding a multiple of one row to another, $R_2 - 2R_1 \to R_2$. This step does not change the general solution S. Details later!

§11.3 Observations

a) Homogeneous linear systems are always consistent because $(0,\dots,0) \in S$.

b) Any linear system with a degenerate inhomogeneous equation is inconsistent.

Learning vocabulary is important at this point!

Theorem Any linear system has either no, one, or infinitely many solutions.

Note that $x^2 = 64$, with its two solutions, is not a linear system!!!

§11.4 Elementary row operations

In our three easy examples, we added multiples of one row to another without changing the general solution. This allowed us to simplify the system so that we could finally obtain the general solution.

Similar moves:

1. Add a multiple of one row to another.
2. Interchange to rows.
3. Multiply a row by a non-zero scalar.

These moves are called "elementary row operations". Note that:

- Any solution before an ERo will still be a solution after an ERo.
- Every ERo is reversible.

Therefore, EROs don't change the general solution.

Example in shorthand notation

"augmented matrix"

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 2 & 5 & 8 & 0 \\ 3 & 6 & 9 & 0 \end{array}\right] \qquad \begin{array}{l} R_2 - 2R_1 \to R_2 \\ R_3 - 3R_1 \to R_3 \end{array}$$

"coefficient matrix"

"row-equivalent"

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & 0 \end{array}\right] \qquad -\tfrac{1}{3}R_2 \to R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -6 & -12 & 0 \end{array}\right] \qquad R_3 + 6R_2 \to R_3$$

leading 1's or pivots

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free parameter $c = t$

$b = -2t$

$a = -4b - 7t = 8t - 7t = t$

So,

$$S = \left\{ \begin{pmatrix} t \\ -2t \\ t \end{pmatrix} \middle| \, t \in \mathbb{R} \right\}$$

$$= \left\{ t \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \middle| \, t \in \mathbb{R} \right\}$$

§11.5 What are we aiming at: REF & RREF

Def A matrix (augmented or not) is in row echelon form ("REF") if:

(i) All zero rows are at the bottom.

(ii) The first non-zero entry in each row is a 1.

(iii) Each leading 1 is to the right of the leading 1's in the rows above.

It is in reduced row echelon form ("RREF") if, in addition,

(iv) Each leading 1 is the only non-zero entry in its column.

Why is RREF even better? We can read off a general solution:

REF

$$\cdots \sim \left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 - 4R_2 \to R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

RREF

column without leading 1 ⇝ free parameter $c = t$

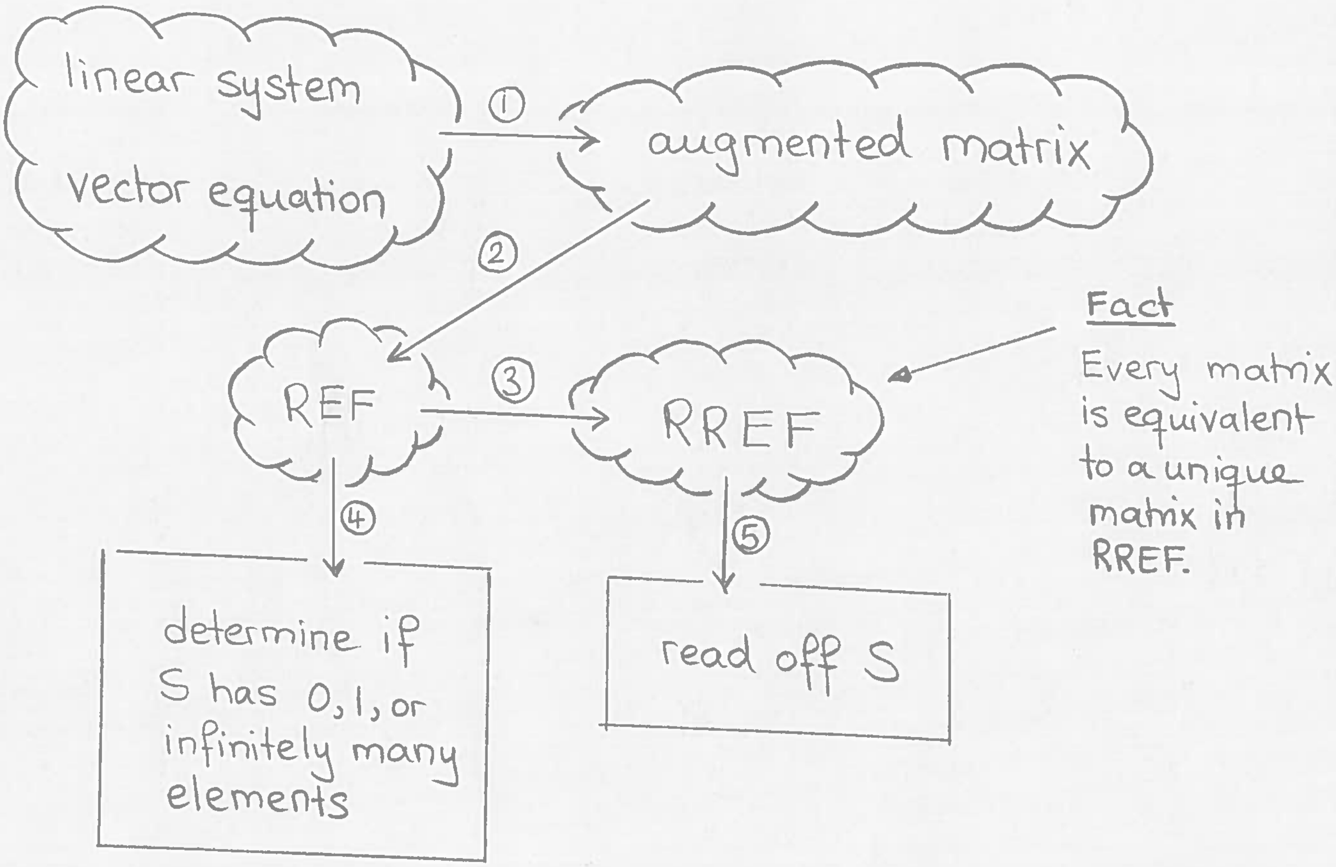
columns with leading 1 ⇝ no choice

$$\Rightarrow \begin{array}{l} a - t = 0 \\ b + 2t = 0 \end{array} \quad \Rightarrow \quad \begin{array}{l} a = t \\ b = -2t \end{array}$$

$$\Rightarrow S = \left\{ \begin{pmatrix} t \\ -2t \\ t \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}.$$

LECTURE 14

What are we currently doing?



How to do that?

① Copy coefficients and solution vector.
② & ③ Gauß elimination
④ & ⑤ Now...

from Reading Week

Show slides!

Recall sheet handed out last Lecture!

§11.6 Reading off a general solution

Note: $\left[\begin{array}{cccc|c} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 & 2 \end{array}\right]$ is in REF but not in RREF

Already see that the system is consistent!

$$\begin{array}{c} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 2 \end{array}\right] \\ \begin{array}{cccc} N & P & P & N \end{array} \end{array}$$ is in RREF

Solution is near!

Not necessary to write on board!

N = "non-pivot" = column without leading 1

⇝ free parameter!

P = "pivot" = column with leading 1

⇝ no choice!

Choose

$x_1 = s$
$x_4 = t$

$$\begin{aligned} x_2 + 3t &= 0 \\ x_3 &= 2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_2 &= -3t \\ x_3 &= 2 \end{aligned} \quad \Rightarrow \quad S = \left\{ \begin{pmatrix} s \\ -3t \\ 2 \\ t \end{pmatrix} \,\middle|\, s,t \in \mathbb{R} \right\}$$

Another example:

$$\left[\begin{array}{cccc|c} 1 & a & 0 & b & d \\ 0 & 0 & 1 & e & f \\ 0 & 0 & 0 & 0 & g \end{array}\right]$$

Call variables $x_1, \dots, x_4$!

a) $g \neq 0 \Rightarrow S = \{\} = \emptyset$

b) $g = 0 \Rightarrow$ $x_2 = s$, $x_4 = t$

$$\begin{aligned} x_1 + a \cdot s + b \cdot t &= d \\ x_3 + e \cdot t &= f \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} x_1 &= d - as - bt \\ x_3 &= f - et \end{aligned}$$

$$\Rightarrow \quad S = \left\{ \begin{pmatrix} d-as-bt \\ s \\ f-et \\ t \end{pmatrix} \,\middle|\, s,t \in \mathbb{R} \right\}$$

Yet another example:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right] \Rightarrow \begin{array}{l} x_1 = a \\ x_2 = b \\ x_3 = c \end{array} \Rightarrow S = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\}$$

Rules

① Row of type "$0\ 0 \cdots 0 \mid \underbrace{*}_{\neq 0}$" ? ("degenerate inhomogeneous equation")

<u>Yes:</u> $S = \emptyset$

<u>No:</u> Go on...

② Each column of the coefficient matrix has a leading 1 ?

<u>Yes:</u> S has 1 element

<u>No:</u> S has infinitely many elements

① and ② work with REF and RREF. From ③ onwards, we actually need RREF:

③ If S has 1 element, solution is the vector in the augmented column. Ignore zero-rows!

④ If S has infinitely many elements, the variables corresponding to columns without leading 1's become free parameters. The remaining variables can now be expressed in terms of the free parameters.

Practice

System with 4 equations in 3 variables that has:

0 solutions (inconsistent) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 4 \end{array}\right]$ 3 4

1 solution $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$ 3 3

infinitely many solutions $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ 2 2

Homogeneous system with 4 equations in 3 variables that has:

0 solutions (inconsistent) ——

1 solution $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ 3 3

infinitely many solutions $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ 2 2

always: rank(A) $\leq$ rank(A|b)

most of it already done!

§ 12 continues § 11

Example

<u>Definition</u> The rank of a matrix A, denoted by rank(A), is the number of leading 1's (=pivots) in any REF of A.

Theorem

Let $[A \mid b]$ be an augmented matrix. Then:

a) 0 solutions (inconsistent) $\iff$ $\operatorname{rank}(A) < \operatorname{rank}(A \mid b)$

b) 1 solution $\iff$ $\operatorname{rank}(A) = \operatorname{rank}(A \mid b)$ and $\operatorname{rank}(A) = \#$ cols of A

c) infinitely many solutions $\iff$ $\operatorname{rank}(A) = \operatorname{rank}(A \mid b)$ and $\operatorname{rank}(A) < \#$ cols of A

§ 13 Applications

This had been addressed on the Read and Practice Sheet and will be recalled here on slides...

In particular:

a) Testing scenarios

b) Traffic flow

Can also be found on pages 63 - 67 of these notes!

§ 13 Applications

a) For which $k \in \mathbb{R}$ is $\underbrace{\begin{pmatrix} -4 \\ 5 \\ 6 \end{pmatrix} \in \operatorname{span}\left\{ \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} -8 \\ 13 \\ k \end{pmatrix} \right\}}$?

This means:

$$\begin{pmatrix} -4 \\ 5 \\ 6 \end{pmatrix} = a \cdot \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} + b \cdot \begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix} + c \cdot \begin{pmatrix} -8 \\ 13 \\ k \end{pmatrix}$$ for some $a, b, c \in \mathbb{R}$

Idea: Translate into a linear system !

$$\left[\begin{array}{ccc|c} -2 & 4 & -8 & -4 \\ 1 & 1 & 13 & 5 \\ -1 & 4 & k & 6 \end{array}\right] \underset{\sim}{-\tfrac{1}{2}R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & -2 & 4 & 2 \\ 1 & 1 & 13 & 5 \\ -1 & 4 & k & 6 \end{array}\right]$$

$$\underset{\sim}{\begin{array}{c} R_2 - R_1 \to R_2 \\ R_3 + R_1 \to R_3 \end{array}} \left[\begin{array}{ccc|c} 1 & -2 & 4 & 2 \\ 0 & 3 & 9 & 3 \\ 0 & 2 & k+4 & 8 \end{array}\right] \underset{\sim}{\tfrac{1}{3}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -2 & 4 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 2 & k+4 & 8 \end{array}\right]$$

$$\underset{\sim}{R_3 - 2R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & -2 & 4 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & k-2 & 6 \end{array}\right]$$

<u>$k = 2$:</u> No solution!

<u>$k \neq 2$:</u> There is a solution!

<u>So</u>

For $k \neq 2$, we have:

$$\begin{pmatrix} -4 \\ 5 \\ 6 \end{pmatrix} \in \operatorname{span}\left\{ \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} -8 \\ 13 \\ k \end{pmatrix} \right\}$$

Remark

If we want to describe the general solution, we need the RREF. Here is our linear system for $k=4$:

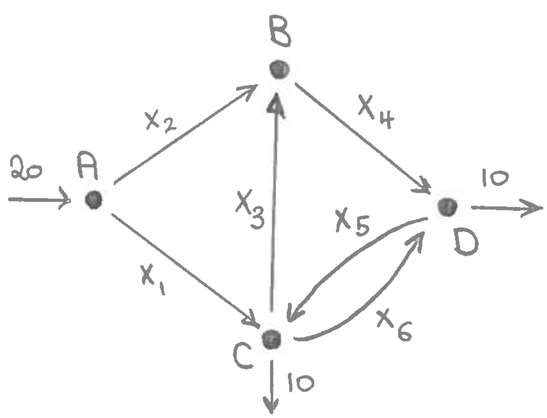
$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 6 \end{array}\right] \underset{\sim}{\tfrac{1}{2}R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -2 & 4 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\underset{\sim}{\begin{array}{c} R_2 - 3R_3 \to R_2 \\ R_1 - 4R_3 \to R_1 \end{array}} \left[\begin{array}{ccc|c} 1 & -2 & 0 & -10 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 3 \end{array}\right] \underset{\sim}{R_1 + 2R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -26 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

So

$$S = \left\{ \begin{pmatrix} -26 \\ -8 \\ 3 \end{pmatrix} \right\} \quad \text{for } k=4$$

b)



	Flow in		Flow out
A:	20	$=$	x_1+x_2
B:	x_2+x_3	$=$	x_4
C:	x_1+x_5	$=$	$10+x_3+x_6$
D:	x_4+x_6	$=$	$10+x_5$

$$\left[\begin{array}{cccccc|c} 1 & 1 & 0 & 0 & 0 & 0 & 20 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 1 & -1 & 1 & 10 \end{array}\right] \sim \dots \sim \begin{array}{c} \begin{array}{cccccc} & & r & & s & t \\ & & \downarrow & & \downarrow & \downarrow \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 0 & -1 & 0 & 1 & -1 & 10 \\ 0 & 1 & 1 & 0 & -1 & 1 & 10 \\ 0 & 0 & 0 & 1 & -1 & 1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

RREF

$$S = \left\{ \begin{pmatrix} 10+r-s+t \\ 10-r+s-t \\ r \\ 10+s-t \\ s \\ t \end{pmatrix} \middle| \; r,s,t \in \mathbb{R} \right\}$$

Our constraints:

$x_1, \dots, x_6 \in \mathbb{N} \cup \{0\}$

Now, suppose $\overline{CD}$ is closed in both directions! That is, $s=t=0$. Then we obtain:

$$S' = \left\{ \begin{pmatrix} 10+r \\ 10-r \\ r \\ 10 \\ 0 \\ 0 \end{pmatrix} \middle| \; r \in \mathbb{R} \right\}$$

Using our constraints, we obtain:

- $r \in \mathbb{N} \cup \{0\}$
- $0 \leq r \leq 10$

So, the maximal flow along $\overline{AC}$ is 20!

LECTURE 15

§14 Matrix Multiplication

We know that $M_{mn}(\mathbb{R})$ is a vector space. In particular, we can add any two $m \times n$ matrices and we can multiply any $m \times n$ matrix by a scalar.

Here is another operation:

$$m \times p \text{ matrix} \bullet p \times n \text{ matrix} = m \times n \text{ matrix}$$

$$m\updownarrow \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}_{\leftarrow p \rightarrow} \cdot \; p\updownarrow \underbrace{\begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & 1 \\ 0 & 7 & -2 \end{bmatrix}}_{\leftarrow n \rightarrow} = m\updownarrow \underbrace{\begin{bmatrix} -1 & 25 & -1 \\ -1 & 52 & 5 \end{bmatrix}}_{\leftarrow n \rightarrow}$$

dot product: 2nd row 3rd column

General rule: "row times column"

§14.1 Definition $A \in M_{mp}(\mathbb{R})$, $A = [a_{ij}]$, $B \in M_{pn}(\mathbb{R})$, $B = [b_{ij}]$. Then, $A \cdot B \in M_{mn}(\mathbb{R})$ with $A \cdot B = [c_{ij}]$ given by:

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} \qquad (\text{"product"})$$

Further examples:

a) $\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{A} \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ ← linear combination of the columns of A

c) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 2 & 7 \end{bmatrix}$ ← each column is a linear combination of the columns of A

First b)!

For a matrix A and a matrix $B = [b_1 | b_2 | \dots | b_n]$ we may thus say that:

$$A \cdot B = \left[Ab_1 \,\middle|\, Ab_2 \,\middle|\, \dots \,\middle|\, Ab_n \right]$$

b) "Relation to linear systems"

Problem Try to solve $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$!

In vector language: $x_1 \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

In linear systems:
$$\begin{aligned} x_1 + 2x_2 &= 5 \\ 3x_1 + 4x_2 &= 6 \end{aligned}$$

Solution $\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & 4 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -2 & -9 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 9/2 \end{array}\right]$

$\sim \left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 9/2 \end{array}\right]$ $\quad S = \left\{ \begin{pmatrix} -4 \\ 9/2 \end{pmatrix} \right\}$

$x_1 = -4$
$x_2 = 9/2$

d)

$$[1 \ 2] \cdot \underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -1 \end{bmatrix}}_{A} = [1 \ 5 \ 0] \leftarrow \text{linear combination of the rows of } A$$

§14.2 Strange properties

a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \end{bmatrix}$

$$A \cdot B = \begin{bmatrix} 5 & 6 & -1 \\ 11 & 12 & -3 \end{bmatrix}$$

-1 · 1st column of A
0 · 2nd column of A

B·A not defined!!!

b) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$A \cdot B = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}, \quad B \cdot A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

Interpretation as linear Combination!

Matrix multiplication is non-commutative!!! (*)

c) $A = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$ $n \times 1$ matrices (= column vectors)

$A^T B = 8$ (= A.B - dot product of vectors) Commutative

$B^T A = 8$ (= B.A dot product of vectors)

$$A B^T = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -1 & 2 \\ 0 & 0 & 0 \\ 8 & -2 & 4 \end{bmatrix}$$

$$B A^T = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 8 \\ -1 & 0 & -2 \\ 2 & 0 & 4 \end{bmatrix}$$

(= tensor products of vectors) non-commutative

d) $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

$$A \cdot B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O_{2\times 2}$$

<u>Note:</u> We can read $A \cdot B$ as follows: The first column of the product is $1 \cdot \binom{1}{-1} + 1 \cdot \binom{-1}{1} = \binom{0}{0}$, the second one is $2 \cdot \binom{1}{-1} + 2 \cdot \binom{-1}{1} = \binom{0}{0}$.

e) $A = \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 7 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

$$AC = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}, \quad BC = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix},$$

Compute in groups!

(**) No cancellation Even though $C \neq O_{2\times 2}$.

§14.3 Pleasant properties

We shall be using the following shorthand notation:

$$O_{m\times n} = m \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}_{\leftarrow n \rightarrow} \qquad I_m = m \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{\leftarrow m \rightarrow}$$

"zero matrix" "identity matrix"

Theorem Let A, B, C be matrices and $k \in \mathbb{R}$ be a scalar. Then, whenever defined:

a) $(AB)C = A(BC)$

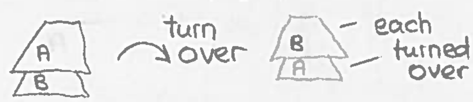
b) $A(B+C) = AB + AC$

c) $(B+C)A = BA + CA$

d) $k \cdot (AB) = (kA)B = A(kB)$

e) $(AB)^T = B^T A^T$

Two sheets of paper:



f) If A is $m \times n$, then $I_m A = A$ and $A I_n = A$.

g) If A is $m \times n$, then $A\, O_{n\times p} = O_{m\times p}$ and $O_{q\times m} A = O_{q\times n}$.

Proof of a)

$$A = \underset{m\times n}{[a_{ij}]}, \quad B = \underset{n\times p}{[b_{ij}]}, \quad C = \underset{p\times q}{[c_{ij}]}.$$

ij^{th} entry of $(AB)C$:

$$\sum_{k=1}^{p} \underbrace{\bigcirc}_{\substack{ik^{th} \\ \text{entry of } AB}} \cdot c_{kj} = \sum_{k=1}^{p} \left(\sum_{\ell=1}^{n} a_{i\ell} b_{\ell k} \right) \cdot c_{kj}$$

$=$

ij^{th} entry of $A(BC)$:

$$\sum_{\ell=1}^{n} a_{i\ell} \cdot \underbrace{\bigcirc}_{\substack{\ell j^{th} \text{ entry} \\ \text{of } BC}} = \sum_{\ell=1}^{n} a_{i\ell} \cdot \left(\sum_{k=1}^{p} b_{\ell k} \cdot c_{kj} \right)$$

$\square$

LECTURE 16

Recall from last time

With matrix multiplication we can do most simple algebraic operations but we shall keep in mind:

- Matrix multiplication is not commutative.
- In general, no cancellation is possible.

Example

numbers

$$(a+b)\cdot(a-b) = a^2 - b^2$$

matrices

$$(A+B)\cdot(A-B) = AA - AB + BA - BB = A^2 - AB + BA - B^2$$

§14.4 A bit of magic

Present the Cayley-Hamilton Theorem (p.150 ll. 1-18 of VSF) on slides!

§14.5 Block multiplication

"Treat submatrices as numbers!"

$$A = \left[\begin{array}{ccc|cc} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ \hline 0&0&0&0&1 \\ 0&0&0&0&0 \end{array}\right] = \left[\begin{array}{c|c} B & 0 \\ \hline 0 & C \end{array}\right]$$

with $B = \begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix}$ $C = \begin{bmatrix} 0&1 \\ 0&0 \end{bmatrix}$

$$A^2 = \left[\begin{array}{c|c} B & 0 \\ \hline 0 & C \end{array}\right] \cdot \left[\begin{array}{c|c} B & 0 \\ \hline 0 & C \end{array}\right] = \left[\begin{array}{c|c} B^2 & 0 \\ \hline 0 & C^2 \end{array}\right] = \left[\begin{array}{c|c} B & 0 \\ \hline 0 & 0 \end{array}\right]$$

$$A^{100} = \left[\begin{array}{c|c} B & 0 \\ \hline 0 & 0 \end{array}\right]$$

§14.6 Matrices and linear systems

The following are all equivalent:

a) linear system:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ x_1 - x_2 + x_3 &= 2 \\ x_2 - 3x_3 &= 0 \end{aligned}$$

b) matrix equation:

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 1 \\ 0 & 1 & -3 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}}_{b}$$

c) vector equation:

$$x_1 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + x_3 \cdot \begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$$

They all can be solved using Gauß elimination.
Let's have another look at b)!

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 1 \\ 0 & 1 & -3 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \left[c_1 \,\vdots\, c_2 \,\vdots\, c_3 \right] \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

<u>with</u> $c_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $c_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $c_3 = \begin{bmatrix} 3 \\ 1 \\ -3 \end{bmatrix}$

$$= \underbrace{x_1 c_1 + x_2 c_2 + x_3 c_3}_{\text{linear combination of the columns of } A}$$

Hence:

Ax is a linear combination of the columns of A.

This is a great perspective on linear systems, which allows us to say:

(i) $Ax = b$ consistent $\Leftrightarrow$ b is a linear combination of the columns of A

(ii) $Ax = 0$ has a unique solution $(x=0)$
$\Leftrightarrow$ the columns of A are linearly independent $\Leftrightarrow$ $\text{rank}(A) = \#$ columns of A

easy to see equivalence to "$Ax=0$ has a unique solution"

§14.7 Definition of column space

Let $A = \left[c_1 \mid c_2 \mid \cdots \mid c_n \right]$. Then,

$$\text{Col}(A) := \text{Span}\{c_1, c_2, \ldots, c_n\}.$$

If $A \in M_{mn}(\mathbb{R})$, then $\text{Col}(A)$ is a subspace of $\mathbb{R}^m$.

Nice check for linear independence

① Write vectors in columns of A.
② Run Gauß elimination.
③ Count leading ones: If a leading one in each column, $\Rightarrow$ LI. otherwise, $\Rightarrow$ LD.

$$\left\{ \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}2\\1\\3\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right\} \qquad \begin{bmatrix}1&2&0\\0&1&1\\1&3&1\end{bmatrix} \sim \begin{bmatrix}1&2&0\\0&1&1\\0&1&1\end{bmatrix} \sim \underbrace{\begin{bmatrix}1&2&0\\0&1&1\\0&0&0\end{bmatrix}}_{\text{REF}}$$

$\text{rank}(A) < \#$columns of A
$\Rightarrow$ LD.

We may write $\text{Col}(A) = \{Ax \mid x \in \mathbb{R}^n\}$. This is also known as the image of A, $\text{im}(A)$.

§15 Further spaces associated to matrices

§15.1 Definitions

a) Let $A = \begin{bmatrix} r_1 \\ \hline \vdots \\ \hline r_m \end{bmatrix}$, that is the matrix with rows $r_1, \ldots, r_m$.

Then, $\mathrm{Row}(A) := \mathrm{span}\{r_1, \ldots, r_m\}$ is the <u>row space</u> of A. Typically, we transpose the rows to obtain column vectors in $\mathbb{R}^n$.

b) Let $A \in M_{mn}(\mathbb{R})$. Then, $\mathrm{Null}(A) := \{x \in \mathbb{R}^n \mid Ax = 0\}$ is the <u>nullspace</u> or kernel of A. In the latter case, it is typically abbreviated by $\ker(A)$.

both subspaces of $\mathbb{R}^n$.

Example

$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

$$\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right\}$$
$$= \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} t \\ 0 \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}$$

not necessary!

$$\mathrm{Row}(A) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$$
$$= \left\{ \begin{pmatrix} t \\ 2t \\ 3t \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}$$

not necessary!

$$\mathrm{Null}(A) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \,\middle|\, \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

augmented matrix: $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ RREF

$$= \left\{ \begin{pmatrix} -2s-3t \\ s \\ t \end{pmatrix} \,\middle|\, s,t \in \mathbb{R} \right\}$$
$$= \left\{ s \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \,\middle|\, s,t \in \mathbb{R} \right\}$$
$$= \mathrm{span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

§15.2 Subspace test

<u>Lemma</u> $\mathrm{Null}(A)$ is a subspace of $\mathbb{R}^n$.

<u>Proof</u> We run the subspace test.

1. $0 \in \mathrm{Null}(A)$? — $A \cdot 0 = 0$, so $0 \in \mathrm{Null}(A)$.
2. closed under addition ? — Let $x, y \in \mathrm{Null}(A)$. Then $A(x+y) = Ax + Ay = 0 + 0 = 0$, $\Rightarrow x+y \in \mathrm{Null}(A)$.
3. closed under scalar multiplication ? — Let $x \in \mathrm{Null}(A)$ and $k \in \mathbb{R}$. Then $A(kx) = k(Ax) = k \cdot 0 = 0$, $\Rightarrow kx \in \mathrm{Null}(A)$.

$\square$

§15.3 Basis and dimension

$$A = \begin{bmatrix} 1 & 2 & 3 & -3 \\ 0 & 1 & 1 & -2 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

We want to find a basis of $\mathrm{Null}(A)$!

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & -3 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 3 & -3 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & -2 & -2 & 4 & 0 \end{array}\right]$$

maybe skip computation!

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 3 & -3 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\mathrm{Null}(A) = S = \left\{ \begin{pmatrix} -s-t \\ -s+2t \\ s \\ t \end{pmatrix} \middle| \; s,t \in \mathbb{R} \right\}$$

$$= \left\{ s \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \middle| \; s,t \in \mathbb{R} \right\}$$

$$= \mathrm{span} \underbrace{\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}}_{\text{This is already a basis !!!}}$$

Look at the 3rd and 4th entry!

Here, $\dim(\mathrm{Null}(A)) = 2$. Note, $\mathrm{rank}(A) = 2$.

Since $\dim(\mathrm{Null}(A)) = \#$ columns without leading one and $\mathrm{rank}(A) = \#$ columns with leading one, we obtain:

Rank-Nullity Theorem

$$\dim(\mathrm{Null}(A)) + \mathrm{rank}(A) = \#\text{columns of } A$$

§15.4 Inhomogeneous linear systems

Now, use A from §15.3 and let $b = \begin{bmatrix} 10 \\ 3 \\ 4 \end{bmatrix}$. Solve $Ax = b$, that is $\begin{bmatrix} 1 & 2 & 3 & -3 \\ 0 & 1 & 1 & -2 \\ 1 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 4 \end{bmatrix}$.

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & -3 & 10 \\ 0 & 1 & 1 & -2 & 3 \\ 1 & 0 & 1 & 1 & 4 \end{array}\right] \sim \underset{\text{same steps as in §15.3}}{\cdots} \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 4 \\ 0 & 1 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$S = \left\{ \begin{pmatrix} 4-s-t \\ 3-s+2t \\ s \\ t \end{pmatrix} \;\middle|\; s,t \in \mathbb{R} \right\}$$

$$= \left\{ \underbrace{\begin{pmatrix} 4 \\ 3 \\ 0 \\ 0 \end{pmatrix}}_{\text{particular solution}} + \underbrace{s \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}}_{\substack{\text{all Solutions to } Ax=0 \\ (= \mathrm{Null}(A))}} \;\middle|\; s,t \in \mathbb{R} \right\}$$

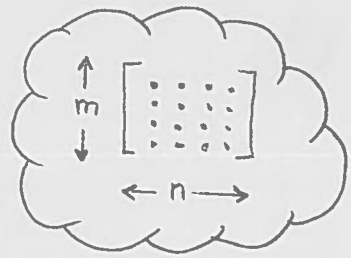
Theorem

If $Ax=b$ is consistent and v is a particular solution, that is $Av=b$, then the general solution is:

$$S = v + \mathrm{Null}(A).$$

§15.5 Summary of facts

a) Consistency of linear systems



Let $A \in M_{mn}(\mathbb{R})$ and $b \in \mathbb{R}^m$.

linear system $Ax=b$ is consistent

$\iff$ $b \in \mathrm{Col}(A)$ ← recall slides

$\iff$ $\mathrm{rank}(A) = \mathrm{rank}(A|b)$.

Moreover,

linear system $Ax=b$ is consistent for every $b \in \mathbb{R}^m$

$\iff$ Every $b \in \mathbb{R}^m$ is in $\mathrm{Col}(A)$

$\iff$ $\mathrm{Col}(A) = \mathbb{R}^m$

$\iff$ $\mathrm{rank}(A) = m$ $\quad (*)$

Proof sketch:

"$\Rightarrow$" If $\mathrm{rank}(A) < m$, the last row of the RREF of A is zero. Choose $b = (0, \dots, 0, 1)$. Do the elementary row operations backward, and you obtain an inconsistent system $Ax = b'$.

"$\Leftarrow$" If $\mathrm{rank}(A) = m$, then every row of the RREF has a leading one. So, the system must be consistent.

b) Number of solutions

Let $A \in M_{mn}(\mathbb{R})$ and $b \in \mathbb{R}^m$ so that $Ax=b$ is consistent.

linear system $Ax=b$ has a unique solution
$\Leftrightarrow$ no free parameters in the general solution
$\Leftrightarrow$ Every column of the RREF of A has a leading one.
$\Leftrightarrow$ linear system $Ax=0$ has the unique solution $x=0$
$\Leftrightarrow$ columns of A are linearly independent
$\Leftrightarrow$ $\mathrm{Null}(A)=\{0\}$ ← see §15.4
$\Leftrightarrow$ $\dim(\mathrm{Null}(A))=0$
$\Leftrightarrow$ $\mathrm{rank}(A)=n$ (**)

See rank nullity theorem

Message:

We can solve $Ax=b$ for every b iff $\mathrm{rank}(A)=m$.
Such a solution is unique iff $\mathrm{rank}(A)=n$.

§16 Finding bases

Let $W=\mathrm{span}\{v_1, v_2, \dots, v_k\}$. Find a basis of W!

§16.1 Row space algorithm

① Write vectors as rows into matrix.
② Transform into any REF (or into RREF if you like).
③ Non-zero rows form a basis of W.

Example

$$W = \operatorname{span}\left\{ \begin{pmatrix}1\\2\\2\\3\end{pmatrix}, \begin{pmatrix}2\\-2\\-8\\4\end{pmatrix}, \begin{pmatrix}1\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\2\\4\\1\end{pmatrix} \right\}$$

① $\underbrace{\begin{bmatrix}1 & 2 & 2 & 3\\ 2 & -2 & -8 & 4\\ 1 & 1 & 0 & 1\\ 0 & 2 & 4 & 1\end{bmatrix}}_{\text{Matrix}}$ ② $\sim \underbrace{\begin{bmatrix}1 & 0 & -2 & 0\\ 0 & 1 & 2 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}}_{\text{(R)REF}}$ ③ Basis of W:

$$\left\{ \begin{pmatrix}1\\0\\-2\\0\end{pmatrix}, \begin{pmatrix}0\\1\\2\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right\}$$

§16.2 Extending LI sets to a basis of $\mathbb{R}^n$ ← postponed to next lecture!

① Write vectors as rows into matrix.

② Transform into any REF (or into RREF if you like).

③ If the k-th column has no leading one, add the vector $\begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix}$ ← 1 in the k-th entry to the LI set. This yields a basis of $\mathbb{R}^n$.

Example from above

① & ② as above ③ Basis of $\mathbb{R}^n$:

$$\left\{ \begin{pmatrix}1\\0\\-2\\0\end{pmatrix}, \begin{pmatrix}0\\1\\2\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \right\}$$

§16.3 Reducing spanning set to a basis of W

There is a tiny little blemish in §16.1! The basis does not involve any of the initial vectors. What if we liked them and just wanted to reduce the spanning set until we obtain a basis? That's what the column space algorithm does!

1. Write vectors in <u>columns</u> of matrix.
2. Transform into any REF (or into RREF if you like).
3. Keep the vectors that correspond to columns with leading ones.

<u>Example</u>

1. $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & -2 & 1 & 2 \\ 2 & -8 & 0 & 4 \\ 3 & 4 & 1 & 1 \end{bmatrix}$

2. $\begin{array}{c} \begin{array}{cccc} 1^{st} & 2^{nd} & 3^{rd} & \\ \downarrow & \downarrow & \downarrow & \end{array} \\ \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1/6 & -1/3 \\ 0 & 0 & 1 & -1/5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$

3. <u>Basis of W:</u>

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ -8 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

<u>Corollary</u>

$\dim(\mathrm{Row}(A)) = \mathrm{rank}(A)$ by § 16.1

$\dim(\mathrm{Col}(A)) = \mathrm{rank}(A)$ by § 16.3

So, $\dim(\mathrm{Row}(A)) = \dim(\mathrm{Col}(A))$.

<u>Why does the row-space algorithm work?</u>

Consider the problem on pages 174 - 175 in our textbook, and the proof on page 176.

Next time in more detail!

LECTURE 18

both with slides

Evaluation

Warm up with W15-T4 #1 and F15-T3 #2

§ 16.4 Why does the column space algorithm work?

$$\mathrm{Col}\underbrace{\begin{bmatrix} 1 & 2 & -1 & 4 \\ 5 & 10 & -4 & 19 \\ 2 & 4 & -2 & 8 \end{bmatrix}}_{A} = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 10 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ -4 \\ -2 \end{pmatrix}, \begin{pmatrix} 4 \\ 19 \\ 8 \end{pmatrix} \right\}$$

We would like to have a basis!!!

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 5 & 10 & -4 & 19 \\ 2 & 4 & -2 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{RREF}$$

<u>Basis for the column space:</u> $\left\{ \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -4 \\ -2 \end{pmatrix} \right\}$

When we want to solve the following problem:

$$x_1 \cdot \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 2 \\ 10 \\ 4 \end{pmatrix} + x_3 \cdot \begin{pmatrix} -1 \\ -4 \\ -2 \end{pmatrix} + x_4 \cdot \begin{pmatrix} 4 \\ 19 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We obtain the following solution:

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 0 \\ 5 & 10 & -4 & 19 & 0 \\ 2 & 4 & -2 & 8 & 0 \end{array}\right] \sim \dots \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \text{RREF}$$

$$S = \left\{ \begin{pmatrix} -2s-3t \\ s \\ t \\ t \end{pmatrix} \,\middle|\, s,t \in \mathbb{R} \right\}$$

Choose $s := -1$ and $t := 0$

$$2 \cdot \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} - 1 \cdot \begin{pmatrix} 2 \\ 10 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow 2 \cdot \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \\ 4 \end{pmatrix}$$

Choose $s := 0$ and $t := -1$

$$3 \cdot \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} - 1 \cdot \begin{pmatrix} -1 \\ -4 \\ -2 \end{pmatrix} - 1 \cdot \begin{pmatrix} 4 \\ 19 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad 3 \cdot \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} - 1 \cdot \begin{pmatrix} -1 \\ -4 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 19 \\ 8 \end{pmatrix}$$

So, we can remove the second and the fourth vector without changing the span. The remaining vectors are LI. because:

$$a \cdot \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} + b \cdot \begin{pmatrix} -1 \\ -4 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 5 & -4 & 0 \\ 2 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \text{ RREF}$$

1st and 3rd column of the above RREF

§16.5 About the way we define the null space

Let A be the matrix from §16.4!

$\text{Null}(A) = \{ x \in \mathbb{R}^4 \mid Ax = 0 \}$ — using this definition, we can easily check if a vector is in Null(A) or not.

$(1,0,0,0) \notin \text{Null}(A)$

because:

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 5 & 10 & -4 & 19 \\ 2 & 4 & -2 & 8 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$(2,-1,0,0) \in \text{Null}(A)$

because:

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 5 & 10 & -4 & 19 \\ 2 & 4 & -2 & 8 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

That's by far easier to verify than considering a spanning set and asking if a vector is a linear combination of the vectors in the spanning set...

Let us describe spans as null spaces!

$$\mathrm{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$$

$$= \left\{ s\cdot\begin{pmatrix}1\\2\\3\end{pmatrix} + t\cdot\begin{pmatrix}1\\0\\1\end{pmatrix} \;\middle|\; s,t\in\mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} \;\middle|\; s\cdot\begin{pmatrix}1\\2\\3\end{pmatrix} + t\cdot\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix} \text{ for some } s,t\in\mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} \;\middle|\; \begin{bmatrix}1&1\\2&0\\3&1\end{bmatrix}\cdot\begin{bmatrix}s\\t\end{bmatrix} = \begin{bmatrix}x\\y\\z\end{bmatrix} \text{ for some } s,t\in\mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} \;\middle|\; \left[\begin{array}{cc|c}1&1&x\\2&0&y\\3&1&z\end{array}\right] \text{ is consistent} \right\}$$

$$= \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} \;\middle|\; \left[\begin{array}{cc|c}1&1&x\\0&-2&y-2x\\0&-2&z-3x\end{array}\right] \text{ is consistent} \right\}$$

$$= \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} \;\middle|\; \left[\begin{array}{cc|c}1&1&x\\0&-2&y-2x\\0&0&z-y-x\end{array}\right] \text{ is consistent} \right\}$$

$$= \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} \;\middle|\; -x-y+z=0 \right\}$$

$$= \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} \;\middle|\; \begin{bmatrix}-1&-1&1\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\end{bmatrix} \right\}$$

$$= \mathrm{Null}\begin{bmatrix}-1&-1&1\end{bmatrix}$$

§ 16.6 Extending LI sets to a basis of $\mathbb{R}^n$

Let A be the matrix from § 16.4!

Our computations show that A has RREF:

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, a basis for the row space is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$. If we want to extend this basis of Row(A) to a basis of $\mathbb{R}^4$, we have to add two more vectors. The easiest way is to add vectors $(0 \cdots 0 1 0 \cdots 0)$ where the 1 is in a column in which the RREF didn't have a leading 1. In our case,

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Then,

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{RREF}$$

There is a leading 1 in every column, so the vectors are LI, and hence form a basis of $\mathbb{R}^4$.

§ 16.7 Final Remark

$\dim(\text{Row}(A)) = \text{rank}(A)$

$\dim(\text{Col}(A)) = \text{rank}(A)$

$\dim(\text{Null}(A)) = \#\text{columns of } A - \text{rank}(A)$

§17 Bases and invertible matrices

§17.1 Finding bases in general vector spaces

Problem Find a basis of the subspace W of $\mathbb{P}_3$:

$$W = \operatorname{span}\{3 + x + 4x^2 + 2x^3,\ 2 + 4x + 6x^2 + 8x^3,\ 1 + 3x + 4x^2 + 6x^3,\ -1 + 2x + x^2 + 4x^3\}$$

Choose an ordered basis of $\mathbb{P}_3$, say $B = \{1, x, x^2, x^3\}$, and work with the coordinate vectors, which live in $\mathbb{R}^4$.

$$\operatorname{Span}\left\{ \begin{bmatrix} 3\\1\\4\\2 \end{bmatrix}_B, \begin{bmatrix} 2\\4\\6\\8 \end{bmatrix}_B, \begin{bmatrix} 1\\3\\4\\6 \end{bmatrix}_B, \begin{bmatrix} -1\\2\\1\\4 \end{bmatrix}_B \right\}$$

Run the row space algorithm.

$$\begin{bmatrix} 3 & 1 & 4 & 2 \\ 2 & 4 & 6 & 8 \\ 1 & 3 & 4 & 6 \\ -1 & 2 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 6 \\ 0 & -2 & -2 & -4 \\ 0 & -8 & -8 & -16 \\ 0 & 5 & 5 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 6 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{RREF}$$

Read off a basis and translate back to $\mathbb{P}_3$.

$$\left\{ \begin{bmatrix} 1\\0\\1\\0 \end{bmatrix}_B, \begin{bmatrix} 0\\1\\1\\2 \end{bmatrix}_B \right\} \rightsquigarrow \{1 + x^2,\ x + x^2 + 2x^3\}.$$

§ 17.2 Extending LI sets to bases in general vector spaces

Extend $\{1+x^2,\ x+x^2+2x^3\}$ to a basis of $\mathbb{P}_3$.

Again, work with the respective coordinate vectors:

$$\left\{ \begin{bmatrix}1\\0\\1\\0\end{bmatrix}_B, \begin{bmatrix}0\\1\\1\\2\end{bmatrix}_B \right\} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

We need to add:

$$\begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

So, a basis of $\mathbb{P}_3$ is:

$$\left\{ \begin{bmatrix}1\\0\\1\\0\end{bmatrix}_B, \begin{bmatrix}0\\1\\1\\2\end{bmatrix}_B, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}_B, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}_B \right\}$$

This means in terms of polynomials:

$$\{1+x^2,\ x+x^2+2x^3,\ x^2,\ x^3\}.$$

Can be skipped, has already been discussed at the end of lecture 18!

§17.3 Motivating invertible matrices

From the row and column space algorithm, we can easily see:

$$\left.\begin{array}{l} \dim(\mathrm{Row}(A)) = \#\text{ leading ones in any REF of } A \\ \dim(\mathrm{Col}(A)) = \#\text{ leading ones in any REF of } A \end{array}\right\} \begin{array}{l} = \mathrm{rank}(A) \\ = \mathrm{rank}(A^T) \; ! \end{array}$$

Moreover, in §15.5, we stated some facts. Let's recall some of them:

① Let A be an $m \times n$ matrix.

$Ax = b$ consistent for every $b \in \mathbb{R}^m$

$\Leftrightarrow \mathrm{Col}(A) = \mathbb{R}^m$ ← otherwise we could find a bad $b \in \mathbb{R}^m$

("the columns span all of $\mathbb{R}^m$")

$\Leftrightarrow \mathrm{rank}(A) = m$

② Now, let $b \in \mathbb{R}^m$ such that $Ax = b$ is consistent.

$Ax = b$ has a unique solution

$\Leftrightarrow$ "the columns of A are LI"

$\Leftrightarrow \mathrm{rank}(A) = n$ ← no free parameters

Now, we consider the special case that A is an $n \times n$ matrix.

$\text{rank}(A) = n$

$\Leftrightarrow$ "the columns of A form a basis of $\mathbb{R}^n$"

$\Leftrightarrow$ $Ax = b$ is consistent and has a unique solution ("precisely one solution")

$\Leftrightarrow$: A is "invertible".

Idea: We can construct an inverse A^{-1} so that

$$Ax = b \Leftrightarrow A^{-1}Ax = A^{-1}b \Leftrightarrow x = A^{-1}b$$

Example below:

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$$

§18 Matrix inverses

Definition If A is an $n \times n$ matrix and B is an $n \times n$ matrix such that $AB = I$ and $BA = I$, then B is called an inverse of A, written $B = A^{-1}$. In this case, A is called invertible.

Example $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

§18.1 Finding the inverse of a 2x2 matrix

Lemma Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then:

If $ad - bc \neq 0$, then $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

If $ad - bc = 0$, then A is not invertible. $(\square)$

We won't give a proof because we will obtain a more general formula later today!

Lemma Suppose, A is an invertible $n \times n$ matrix. Then, any linear system $Ax = b$ is consistent and has a unique solution.

Proof Consistent $A \cdot \underbrace{(A^{-1}b)}_{\text{Solution}} = (AA^{-1})b = Ib = b.$

Unique If $Ax = b$ and $Ay = b$, then

$$x = A^{-1}Ax = A^{-1}b = A^{-1}Ay = y. \qquad \square$$

The converse is also true. Whenever any linear system $Ax = b$ is consistent and has a unique solution, we can construct A^{-1}.

Before doing that, we record some more nice facts about invertible matrices.

§18.2 Algebraic properties of inverses

Lemma Let $k \neq 0$ be a scalar, p an integer, A, C invertible $n \times n$ matrices. Then, the following matrices are also invertible:

(i) A^{-1} : $(A^{-1})^{-1} = A$

(ii) A^{p} : $(A^{p})^{-1} = (A^{-1})^{p}$

(iii) A^{T} : $(A^{T})^{-1} = (A^{-1})^{T}$

(iv) kA : $(kA)^{-1} = \frac{1}{k} A^{-1}$

(v) AC : $(AC)^{-1} = C^{-1}A^{-1}$

sock and shoe rule

Let's prove (v):

① $(AC)\cdot(C^{-1}A^{-1}) = A(CC^{-1})A^{-1} = AIA^{-1}$
$= AA^{-1} = I$

② $(C^{-1}A^{-1})\cdot(AC) = C^{-1}(A^{-1}A)C = C^{-1}IC$
$= C^{-1}C = I$ $\square$

§18.3 Finding the inverse

Let A be an $n \times n$ matrix. Let us assume that any linear system $Ax=b$ is consistent and has a unique solution. We would like to construct an $n \times n$ matrix B such that:

$$A \cdot \underbrace{\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}}_{B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In order to find the first column of B, solve $Ax = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ because $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ is the first column of I. In order to find the second column of B, solve $Ax = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$. And so on.

this is the only possible choice, so the inverse matrix is unique!

Example $A = \begin{bmatrix}1 & 1\\2 & 3\end{bmatrix}$

First column of B: $\left[\begin{array}{cc|c}1 & 1 & 1\\2 & 3 & 0\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 1 & 1\\0 & 1 & -2\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 0 & 3\\0 & 1 & -2\end{array}\right]$

Second column of B: $\left[\begin{array}{cc|c}1 & 1 & 0\\2 & 3 & 1\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 1 & 0\\0 & 1 & 1\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 0 & -1\\0 & 1 & 1\end{array}\right]$

So, $B = \begin{bmatrix}3 & -1\\-2 & 1\end{bmatrix}$. And, indeed, $AB = I$.

In order to avoid doing the same computation twice, we typically work with superaugmented matrices:

$$\left[\begin{array}{cc|cc}1 & 1 & 1 & 0\\2 & 3 & 0 & 1\end{array}\right] \sim \left[\begin{array}{cc|cc}1 & 1 & 1 & 0\\0 & 1 & -2 & 1\end{array}\right] \sim \left[\begin{array}{cc|cc}1 & 0 & 3 & -1\\0 & 1 & -2 & 1\end{array}\right].$$

Algorithm

Want to find A^{-1}:

$$[A \mid I] \sim \ldots \sim [I \mid A^{-1}]$$

rank(A) < n

If A cannot be transformed into I, then $Ax = b$ is not always consistent and A is not invertible.

Stop! Until now, we only now that $AB = I$. Why does also $BA = I$ hold (which we need to have in order to talk about an inverse)?

Lemma If $AB = I$, then $BA = I$.

Proof

postponed to next class!

① $\mathrm{rank}(B) = n$, because:

$$Bx = 0 \Rightarrow Ix = (AB)x = A(Bx) = A0 = 0.$$

So, $\mathrm{Null}(B) = \{0\}$, $\Rightarrow \dim(\mathrm{Null}(B)) = 0$,
$\Rightarrow \mathrm{rank}(B) = n - 0 = n$.

② $\mathrm{rank}(B) = n$ implies that any $Bx = b$ is consistent and has a unique solution. So, we can find a matrix C such that $BC = I$.

$$C = (AB)C = A(BC) = A.$$

So, $BA = I$. □

Conclusion:

$\mathrm{rank}(A) = n$

$\Leftrightarrow$ any $Ax = b$ is consistent and has a unique solution

$\Leftrightarrow$ A is invertible with $[A \mid I] \sim \dots \sim [I \mid A^{-1}]$

$\mathrm{rank}(A) < n$

$\Leftrightarrow$ A is not invertible

§18.4 Example

a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 4 \end{bmatrix}$ $\quad \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -4 & 1 \end{array}\right]$ $\lightning$

not invertible

b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & -3 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

$A^{-1} = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ typo in VSF ☺

LECTURE 20

Recapitulation from last time ...

Let A be an $n \times n$ matrix.

Definition A is invertible if there is an $n \times n$ matrix B such that $AB = I$ and $BA = I$.

Lemma

A is invertible $\Longrightarrow$ Every linear system $Ax = b$ is consistent and has a unique solution.

Proof sketch: $Ax = b \Rightarrow A^{-1}Ax = A^{-1}b \Rightarrow x = A^{-1}b$.

Lemma The converse is also true:

Every linear system $Ax=b$ is consistent and has a unique solution. $\Rightarrow$ A is invertible

Proof:

Let b_i be the ith column of I and let x_i be the unique vector with $Ax_i = b_i$. Then,

$$A \cdot [x_1 | x_2 | \cdots | x_n] = [Ax_1 | Ax_2 | \cdots | Ax_n]$$

$$= [b_1 | b_2 | \cdots | b_n] = I$$

So, we know that $B := [x_1 \mid x_2 \mid \cdots \mid x_n]$ satisfies $AB = I$. We still need to check that $BA = I$.

Claim If $AB = I$, then $BA = I$.

① $\text{rank}(B) = n$ because:

$$Bx = 0 \Rightarrow ABx = A0 \Rightarrow x = 0$$

So, $\text{Null}(B) = \{0\}$, $\Rightarrow \dim(\text{Null}(B)) = 0$,
$\Rightarrow \text{rank}(B) = n - \dim(\text{Null}(B)) = n$.

② Since $\text{rank}(B) = n$, we know that every linear system $Bx = b$ is consistent and has a unique solution.

The same argument as before shows that there is an $n \times n$ matrix C with $BC = I$.

But, $C = (AB)C = A(BC) = A$. So, $BA = I$.

This completes the proof, and gives us a method to construct inverse matrices... $\square$

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & -3 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Example</u>

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 4 \end{bmatrix} \qquad \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -4 & 1 \end{array}\right] \quad \lightning$$

NOT INVERTIBLE

$\operatorname{rank}(A) = 2$!

So, not every linear system $Ax = b$ is consistent !

Too many different notions ? Let's get an overview ...

§18.5 Invertible matrix theorem

Theorem Let A be an $n \times n$ matrix. Then, the following are equivalent:

(1) $\text{rank}(A) = n$

(2) $Ax = 0$ has only the trivial solution

(3) $Ax = b$ is consistent for every $b \in \mathbb{R}^n$

(4) $Ax = b$ has a unique solution for every $b \in \mathbb{R}^n$

(5) The RREF of A is I.

(6) $\text{Null}(A) = \{0\}$

(7) $\text{Col}(A) = \mathbb{R}^n$

(8) $\text{Row}(A) = \mathbb{R}^n$

(9) $\text{rank}(A^T) = n$

(10) The columns of A are linearly independent.

(11) The rows of A are linearly independent.

(12) The columns of A span $\mathbb{R}^n$

(13) The rows of A span $\mathbb{R}^n$

(14) The columns of A form a basis of $\mathbb{R}^n$

(15) The rows of A form a basis of $\mathbb{R}^n$

(16) A is invertible.

(17) A^T is invertible.

Note:
$\text{Null}(A) = \ker(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$
$\text{Col}(A) = \text{im}(A) = \{Ax \mid x \in \mathbb{R}^n\}$

This theorem is important, and it is very likely that you will be asked to translate! For example, F15-T4 6 b:

Express "$Ax=0$ has infinitely many solutions" in terms of:

- $\operatorname{rank}(A)$ ——— $\operatorname{rank}(A) \neq n$ / $\operatorname{rank}(A) < n$
- the RREF of A ——— is $\neq I$ / has a row of zeros
- the columns of A ——— are linearly dependent

<u>Note</u>

$$\operatorname{rank}(A) = n \iff \operatorname{rank}(A^T) = n$$

This is non-trivial but follows from the fact that:

$$\begin{aligned} \operatorname{rank}(A) &= \dim(\operatorname{Row}(A)) \leftarrow \text{row space algorithm} \\ &= \dim(\operatorname{Col}(A)) \leftarrow \text{column space algorithm} \\ &= \dim(\operatorname{Row}(A^T)) \\ &= \operatorname{rank}(A^T) \qquad \square \end{aligned}$$

New part!

§ 19 Orthogonality — § 19.1

<u>Recall</u> $u, v \in \mathbb{R}^n$.

$u \perp v$ ("orthogonal") $\iff \underset{\uparrow\ \text{dot product}}{u \cdot v} = 0$

Properties of the dot product

(i) $u \cdot u \geq 0$ and $u \cdot u = 0 \Leftrightarrow u = 0$

(ii) $u \cdot v = v \cdot u$ } "symmetric"

(iii) $(au + bv) \cdot w = a\, u \cdot w + b\, v \cdot w$

(iv) $u \cdot (av + bw) = a\, u \cdot v + b\, u \cdot w$

(iii), (iv) } "bilinear"

Example

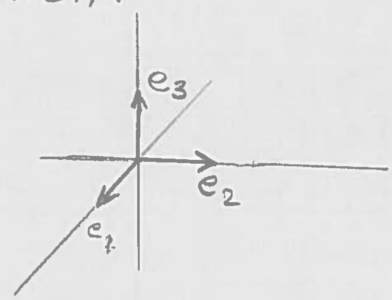
$$(1,-1,0,1) \cdot (1,1,1,0) = 1 \cdot 1 + (-1) \cdot 1 + 0 \cdot 1 + 1 \cdot 0 = 0$$

orthogonal

Let us make a nice observation:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

are pairwise orthogonal



Let $v \in \mathbb{R}^3$. We would like to express

$$v = a e_1 + b e_2 + c e_3$$

to find out a, take the dot product with e_1:

$$\Rightarrow \quad e_1 \cdot v = a \underbrace{e_1 \cdot e_1}_{1} + b \underbrace{e_1 \cdot e_2}_{0} + c \underbrace{e_1 \cdot e_3}_{0}$$

$$\Rightarrow \quad e_1 \cdot v = a$$ ☺

Similarly, $e_2 \cdot v = b$ and $e_3 \cdot v = c$.

That's easy! In fact, we can generalize this observation.

§ 19.2 Orthogonal sets of vectors

Definition A set of vectors $\{v_1, \dots, v_m\} \in \mathbb{R}^n$ is called orthogonal if:

(i) $v_i \cdot v_j = 0$ for all $1 \leq i < j \leq m$

(ii) $v_i \neq 0$ for all $1 \leq i \leq m$

The set is called orthonormal if, in addition:

(iii) $v_i \cdot v_i = 1$ for all $1 \leq i \leq m$

$\Leftrightarrow \|v_i\|^2 = 1$
$\Leftrightarrow \|v_i\| = 1$

Notes

① We can always make an orthogonal set orthonormal by:

$$\{v_1, \dots, v_m\} \rightsquigarrow \left\{ \frac{v_1}{\|v_1\|}, \dots, \frac{v_m}{\|v_m\|} \right\}$$

② There is a Pythagorean property of orthogonal vectors:

$$\begin{aligned} \|v_1 + \dots + v_m\|^2 &= (v_1 + \dots + v_m)(v_1 + \dots + v_m) \\ &= v_1 v_1 + \dots + v_m v_m \\ &= \|v_1\|^2 + \dots + \|v_m\|^2 \end{aligned}$$

LECTURE 21

Example F14-T5 5 ("Problem sheet"):

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad u_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$\{u_1, u_2, u_3\}$ orthogonal set?

$$\left.\begin{array}{ll} u_1 \cdot u_2 = 0 \ \checkmark & u_1 \neq 0 \ \checkmark \\ u_1 \cdot u_3 = 0 \ \checkmark & u_2 \neq 0 \ \checkmark \\ u_2 \cdot u_3 = 0 \ \checkmark & u_3 \neq 0 \ \checkmark \end{array}\right. \quad \Rightarrow \text{YES!}$$

<u>Theorem</u> Orthogonal sets are linearly independent.

<u>Proof</u> $\{v_1, \dots, v_m\}$ orthogonal set.

$$a_1 v_1 + \dots + a_m v_m = 0 \qquad | \, v_i \cdot$$

$$\Rightarrow v_i \cdot (a_1 v_1 + \dots + a_m v_m) = v_i \cdot 0$$

$$\Rightarrow a_1 \underbrace{v_i \cdot v_1}_{=0} + \dots + a_i \underbrace{v_i \cdot v_i}_{\neq 0} + \dots + a_m \underbrace{v_i \cdot v_m}_{=0} = 0$$

$$\Rightarrow a_i = 0$$

$\square$

<u>Corollary</u> An orthogonal spanning set is a basis.

<u>Theorem</u> Let $\{w_1, \dots, w_m\}$ be an orthogonal basis of a subspace W of $\mathbb{R}^n$. Then, any $w \in W$ can be written as:

$$w = \frac{w_1 \cdot w}{w_1 \cdot w_1} \cdot w_1 + \dots + \frac{w_m \cdot w}{w_m \cdot w_m} w_m$$

coordinates with respect to $\{w_1, \dots, w_m\}$

Proof

Since $\{w_1, \dots, w_m\}$ is a basis of W, there are $a_1, \dots, a_m \in \mathbb{R}$ such that:

$$w = a_1 w_1 + \dots + a_m w_m \qquad | \, w_i \cdot$$

$$\Rightarrow \quad w_i \cdot w = a_i \; w_i \cdot w_i$$

$$\Rightarrow \quad a_i = \frac{w_i \cdot w}{w_i \cdot w_i} = \frac{w_i \cdot w}{\|w_i\|^2} \qquad \square$$

Example

Express $\begin{pmatrix}1\\6\\7\end{pmatrix}$ as linear combination of $\left\{\begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}\right\}$.

orthogonal basis

$$\begin{pmatrix}1\\6\\7\end{pmatrix} = a_1 \cdot \begin{pmatrix}1\\2\\1\end{pmatrix} + a_2 \cdot \begin{pmatrix}1\\0\\-1\end{pmatrix} + a_3 \cdot \begin{pmatrix}1\\-1\\1\end{pmatrix}$$

$$a_1 = \frac{\begin{pmatrix}1\\2\\1\end{pmatrix} \cdot \begin{pmatrix}1\\6\\7\end{pmatrix}}{\begin{pmatrix}1\\2\\1\end{pmatrix} \cdot \begin{pmatrix}1\\2\\1\end{pmatrix}} = \frac{20}{6} = \frac{10}{3}, \dots$$

§ 19.3 Orthogonal projections

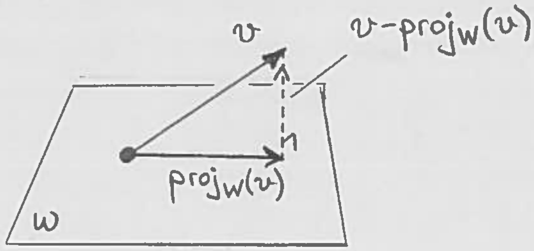
We can also apply the above formula to vectors not in W.

Definition

Let W be a subspace of $\mathbb{R}^n$ with orthogonal basis $\{w_1, \dots, w_m\}$. Then, for any $v \in \mathbb{R}^n$:

$$\operatorname{proj}_W(v) = \frac{w_1 \cdot v}{w_1 \cdot w_1} w_1 + \dots + \frac{w_m \cdot v}{w_m \cdot w_m} w_m$$

is the orthogonal projection of v onto W.



Remark $\mathrm{proj}_W(v)$ does not depend on the orthogonal basis chosen. (see part 4 below)

Theorem Let W be a subspace of $\mathbb{R}^n$, $v \in \mathbb{R}^n$. Then,

(1) $\mathrm{proj}_W(v) \in W$

(2) $v - \mathrm{proj}_W(v)$ is orthogonal to every vector in W

(3) $\mathrm{proj}_W(v)$ is the best approximation to v by vectors in W (!!!)

(4) $\mathrm{proj}_W(v)$ is the only vector with (1) and (2).

Proof in textbook, pp. 209 - 211. For (2):

$$\left(\sum_{k=1}^{m} a_k w_k\right) \cdot \left(v - \sum_{\ell=1}^{m} \frac{w_\ell \cdot v}{w_\ell \cdot w_\ell} \cdot w_\ell\right)$$

$$= a_k \cdot \left(\sum_{k=1}^{m} w_k \cdot v - \sum_{k=1}^{m} \sum_{\ell=1}^{m} \frac{w_\ell \cdot v}{w_\ell \cdot w_\ell} \, w_k \cdot w_\ell\right)$$

$$= a_k \left(\sum_{k=1}^{m} w_k \cdot v - \sum_{k=1}^{m} w_k \cdot v\right) = 0$$

Probably not in class!

Homepage!

Review since Lecture 15!

§ 19.4 Gram-Schmidt algorithm

This section shows the existence of orthogonal bases and gives a way to transform any basis into an orthogonal one.

Theorem Let $\{u_1, \dots, u_m\}$ be any basis of U. Then, define:

(1) $w_1 := u_1$

(2) $w_2 := u_2 - \mathrm{proj}_{V_1}(u_2)$ where $V_1 = \mathrm{span}\{w_1\}$

$$= u_2 - \frac{w_1 \cdot u_2}{w_1 \cdot w_1} w_1$$

(3) $w_3 := u_3 - \mathrm{proj}_{V_2}(u_3)$ where $V_2 = \mathrm{span}\{w_1, w_2\}$

$$= u_3 - \frac{w_1 \cdot u_3}{w_1 \cdot w_1} w_1 - \frac{w_2 \cdot u_3}{w_2 \cdot w_2} w_2$$

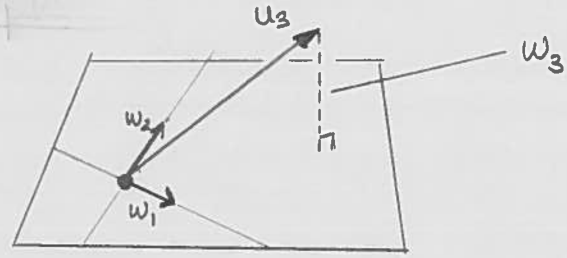
That's only possible because $w_1 \perp w_2$!

$\vdots$

(m) $w_m := u_m - \mathrm{proj}_{V_{m-1}}(u_m)$ where $V_m = \mathrm{span}\{w_1, \dots, w_{m-1}\}$

$$= u_m - \frac{w_1 \cdot u_m}{w_1 \cdot w_1} w_1 - \dots - \frac{w_{m-1} \cdot u_m}{w_{m-1} \cdot w_{m-1}} w_{m-1}$$

This is an orthogonal basis for U. We could scale each vector w_i to $\frac{w_i}{\|w_i\|}$ in order to obtain an orthonormal basis.



Example

$$U = \text{span}\left\{ \begin{pmatrix}1\\-1\\2\end{pmatrix}, \begin{pmatrix}2\\0\\2\end{pmatrix}, \begin{pmatrix}5\\5\\3\end{pmatrix} \right\}$$

This set is LI, hence a basis of U.

$$w_1 = \begin{pmatrix}1\\-1\\2\end{pmatrix}$$

$$w_2 = \begin{pmatrix}2\\0\\2\end{pmatrix} - \frac{\begin{pmatrix}1\\-1\\2\end{pmatrix}\cdot\begin{pmatrix}2\\0\\2\end{pmatrix}}{\begin{pmatrix}1\\-1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix}} \cdot \begin{pmatrix}1\\-1\\2\end{pmatrix} = \begin{pmatrix}2\\0\\2\end{pmatrix} - 1 \cdot \begin{pmatrix}1\\-1\\2\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$w_3 = \begin{pmatrix}5\\5\\3\end{pmatrix} - \frac{\begin{pmatrix}1\\-1\\2\end{pmatrix}\cdot\begin{pmatrix}5\\5\\3\end{pmatrix}}{\begin{pmatrix}1\\-1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix}} \begin{pmatrix}1\\-1\\2\end{pmatrix} - \frac{\begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}5\\5\\3\end{pmatrix}}{\begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}} \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}5\\5\\3\end{pmatrix} - \begin{pmatrix}1\\-1\\2\end{pmatrix} - 5\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$= \begin{pmatrix}-1\\1\\1\end{pmatrix}$$

Practice Find an orthogonal basis for

Maybe skip?

$$U = \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x+y-z=0 \right\}$$

$$\dots = \text{span}\left\{ \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right\} = \dots = \text{span}\left\{ \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}-1/2\\1\\1/2\end{pmatrix} \right\}$$

$$= \text{span}\left\{ \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\2\\1\end{pmatrix} \right\}$$

LECTURE 22

Recall "Gram-Schmidt" by discussing the example from p. 108 of these notes on slides.

Part V

§21 DETERMINANTS

Motivate this construction by showing the slides for the second DGD (= after Lecture 3).

Remark The way to compute the determinant of a 3×3 matrix presented there, may be of disadvantage in the future. Therefore, I strongly recommend to use the following way only!

§21.1 Definition

Let A be an $n\times n$ matrix.

$\boxed{n=1}$ $A=[a]$ $|A| = \det(A) = a$

$\boxed{n=2}$ $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $|A| = \det(A) = ad-bc$

"area of the parallelogram"

$\boxed{n\geq 3}$ $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ We are going to define $\det(A)$ using determinants for smaller n.

Let A_{ij} be the $(n-1)\times(n-1)$ matrix obtained from A by removing the i-th row and the j-th column.

Alternative 1

Expansion along any row

Pick the i-th row of A. Whatever i you like.

$$\det(A) = (-1)^{i+1} a_{i1} \det(A_{i1}) + (-1)^{i+2} a_{i2} \det(A_{i2}) + \dots + (-1)^{i+n} a_{in} \det(A_{in})$$

Alternative 2

Expansion along any column

Pick the j-th column of A. Whatever j you like.

$$\det(A) = (-1)^{1+j} a_{1j} \det(A_{1j}) + (-1)^{2+j} a_{2j} \det(A_{2j}) + \dots + (-1)^{n+j} a_{nj} \det(A_{nj})$$

Theorem The result does not depend on the alternative and the value chosen for i or j.

Example

$$A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The "+" or "−" can be determined by the chessboard trick:

$$\begin{bmatrix} 2^+ & 3^- & 4^+ & 5^- \\ 1^- & 0^+ & 0^- & 1^+ \\ 0^+ & 1^- & 0^+ & 1^- \\ 0^- & 0^+ & 1^- & 0^+ \end{bmatrix}$$

Expansion along first row:

$$\det(A) = 2 \det\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} - 3 \det\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + 4 \det\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} - 5 \det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, we reduced the problem to determinants of 3×3 matrices.

$$\det\begin{bmatrix}0&0&1\\1&0&1\\0&1&0\end{bmatrix} = 0\cdot\det(\cdots) - 0\cdot\det(\cdots) + 1\cdot\det\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

1st row expansion

$$= 1\cdot(1-0) = 1$$

$$\det\begin{bmatrix}1&0&1\\0&0&1\\0&1&0\end{bmatrix} = 1\cdot\det\begin{bmatrix}0&1\\1&0\end{bmatrix} - 0\cdot\det(\cdots) + 1\cdot\det\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

$$= 1\cdot(0-1) + 1\cdot(0-0) = -1$$

$$\det\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = 1\cdot\det\begin{bmatrix}1&0\\0&1\end{bmatrix} - 0\cdot\det(\cdots) + 0\cdot\det(\cdots)$$

$$= 1\cdot(1-0) = 1$$

$$\det\begin{bmatrix}1&0&1\\0&1&1\\0&0&0\end{bmatrix} = 0\cdot\det(\cdots) - 0\cdot\det(\cdots) + 0\cdot\det(\cdots)$$

3rd row expansion

$$= 0$$

So, if a matrix A has a row of zeros, then $\det(A) = 0$.

<u>Hence:</u>

$$\det(A) = 2\cdot 1 - 3\cdot(-1) + 4\cdot 0 - 5\cdot 1 = 0$$

We could have reached this result more easily by expanding along rows or columns with many zeros.

$$\det(A) \underset{\substack{\uparrow \\ \text{4th row} \\ \text{expansion}}}{=} -\det\begin{bmatrix} 2 & 3 & 5 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\underset{\substack{\uparrow \\ \text{2nd row} \\ \text{expansion}}}{=} -\left(-\det\begin{bmatrix} 3 & 5 \\ 1 & 1 \end{bmatrix} - \det\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \right)$$

$$= -2 + 2 = 0 \quad . \quad ☺$$

§21.2 Properties

Proposition Let A be an $n \times n$ matrix.

(1) If A has a row or column of zeros, $\det(A) = 0$.

(2) $\det(A^T) = \det(A)$

(3) If A is a triangular matrix, i.e. either all entries below or all entries above the main diagonal are zero, then $\det(A)$ is the product of the diagonal entries.

Example for (3)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \qquad \det(A) \underset{\substack{\uparrow \\ \text{1st column} \\ \text{expansion}}}{=} 1 \cdot \det\begin{bmatrix} 4 & 5 \\ 0 & 6 \end{bmatrix} = 1 \cdot 4 \cdot \det[6]$$

$$= 1 \cdot 4 \cdot 6$$

Example for (1) has occured before !

§21.3 Determinants and elementary row operations

In light of Property (3) we could also try to apply ERo's in order to transform our matrix A in triangular form and read off det(A) easily. But, ERO's change the determinant, and we have to keep track of what we are doing.

Proposition Let A be an $n \times n$ matrix and let B is obtained from A by applying an ERO. Then:

ERo	det
• interchange two rows	$\det(B) = -\det(A)$
• multiply one row by r	$\det(B) = r\det(A)$
• add a multiple of one row to another	$\det(B) = \det(A)$

The proposition remains true if we replace "rows" by "columns".

Example

$$\det\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 5 \\ 4 & 0 & 3 \end{bmatrix} = -\det\begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & 0 \\ 4 & 0 & 3 \end{bmatrix} = -\det\begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & 0 \\ 0 & 0 & -17 \end{bmatrix}$$

$$= 34$$

LECTURE 23

Recapitulation of determinants and their behaviour with respect to elementary row operations on slides.

<u>Important</u> The same holds when we replace "row" by "column".

§ 21.4 Awesome properties of determinants

<u>Theorem</u> Let A and B be $n \times n$ matrices. Then,

(1) $\det(rA) = r^n \det(A)$

(2) $\det(AB) = \det(A) \cdot \det(B)$

(3) $\det(A) \neq 0 \iff A$ is invertible

<u>in this case</u> $\det(A^{-1}) = \dfrac{1}{\det(A)}$

Property (3) does not only give a new item to our translation theorem and a way to check if vectors are linearly independent. No, it is also not hard to prove:

ERO's may change the determinant but do not change the fact that the determinant is different from zero or not. Let B be the RREF of A. Then,

- A invertible $\Rightarrow B = I \Rightarrow \det(B) = 1$

  $\Rightarrow \det(B) \neq 0 \Rightarrow \det(A) \neq 0$

- A not invertible $\Rightarrow$ B has a row of zeros $\Rightarrow \det(B) = 0 \Rightarrow \det(A) = 0$ $\square$

Application

$$A = \begin{bmatrix} x & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

Find all $x \in \mathbb{R}$ such that A is not invertible.

$$\det(A) = x \cdot \det\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} + \det\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = 6x - 2$$

So, $\det(A) = 0 \Leftrightarrow 6x - 2 = 0 \Leftrightarrow x = \frac{1}{3}$.

§22 EIGENVALUES AND EIGENVECTORS

$$A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$$

$$A \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$A \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix} = 4 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

When multiplying A to this vector, we scale it!

$$A \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

§22.1 Definition

Let A be an $n \times n$ matrix. If $\lambda \in \mathbb{R}$ is a scalar and $x \in \mathbb{R}^n$ a non-zero vector such that $Ax = \lambda x$ then λ is called an eigenvalue and x is called an eigenvector of A.

§ 22.2 How to find eigenvalues

We want to find all $\lambda \in \mathbb{R}$ such that there is a non-zero vector $x \in \mathbb{R}^n$ with:

$$Ax = \lambda x \Leftrightarrow Ax = \lambda I x \Leftrightarrow Ax - \lambda I x = 0$$
$$\Leftrightarrow (A - \lambda I) x = 0$$

Ha! The homogeneous system $(A - \lambda I)x = 0$ has a non-trivial (= non-zero) solution iff:

$$A - \lambda I \text{ is not invertible}$$
$$\Leftrightarrow \det(A - \lambda I) = 0$$

Definition $\chi_A(\lambda) := \det(A - \lambda I)$ "characteristic polynomial of A"

$$\boxed{\text{So, } \lambda \text{ is an eigenvalue of } A \text{ iff } \chi_A(\lambda) = 0.}$$

Let's try this for our initial example:

$$A - \lambda I = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix} - \lambda \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3-\lambda & -1 \\ -2 & 2-\lambda \end{bmatrix}$$

$$\chi_A(\lambda) = \det \begin{bmatrix} 3-\lambda & -1 \\ -2 & 2-\lambda \end{bmatrix} = (3-\lambda)(2-\lambda) - 2$$
$$= \lambda^2 - 5\lambda + 4$$
$$= (\lambda - 1)\cdot(\lambda - 4)$$

So, $\lambda = 1$ and $\lambda = 4$ are the only eigenvalues.

Moreover, an $n \times n$ matrix has at most n eigenvalues!

§22.3 How to find eigenvectors

That's something we already know. If λ is an eigenvalue, then the corresponding eigenvectors are the non-trivial (= non-zero) solutions of:

$$(A - \lambda I)\, x = 0$$

That is, the non-trivial (= non-zero) elements of:

$$\ker (A - \lambda I) \qquad \text{"kernel" / "null space"}$$

<u>Definition</u> Let λ be an eigenvalue of A. Then the subspace

$$E_\lambda = \ker (A - \lambda I)$$

is called the λ-eigenspace of A. Its non-trivial elements are the eigenvectors of A.

Let's try this for our initial example:

$\boxed{\lambda = 1}$ $$E_1 = \ker \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} = \left\{ \begin{pmatrix} \frac{1}{2}s \\ s \end{pmatrix} \;\middle|\; s \in \mathbb{R} \right\}$$

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ -2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad = \left\{ s \cdot \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \;\middle|\; s \in \mathbb{R} \right\}$$

"row equivalent"

$$= \left\{ s \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \;\middle|\; s \in \mathbb{R} \right\}$$

<u>In particular:</u> $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \in E_1$.

$\boxed{\lambda = 4}$ $\quad E_4 = \ker\begin{bmatrix} -1 & -1 \\ -2 & -2 \end{bmatrix} = \left\{ \begin{pmatrix} -s \\ s \end{pmatrix} \,\middle|\, s \in \mathbb{R} \right\}$

$$\left[\begin{array}{cc|c} -1 & -1 & 0 \\ -2 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad = \left\{ s \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \,\middle|\, s \in \mathbb{R} \right\}$$

in particular: $\begin{pmatrix} 1 \\ -1 \end{pmatrix} \in E_4$.

ATTENTION $\quad$ 0 does never count as an eigenvector!

But it may be an eigenvalue!

§ 22.4 Towards an eigenvector basis of $\mathbb{R}^n$

In our initial example, we had:

$\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$ basis of E_1

$\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ basis of E_4

maybe postpone! - definitely!

Theorem Any set consisting of eigenvectors corresponding to distinct eigenvalues is linearly independent.

Corollary If we take the union of bases of all eigenspaces, we obtain a linearly independent set in $\mathbb{R}^n$.

In our case, $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ is even a basis of $\mathbb{R}^2$.

<u>This has a great consequence:</u>

- $P := \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$

- $\det(P) = 3 \Rightarrow P$ is invertible

  $P^{-1} = \begin{bmatrix} 1/3 & 1/3 \\ -2/3 & 1/3 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 3 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 1 & -2/3 & 1/3 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 1/3 & 1/3 \\ 0 & 1 & -2/3 & 1/3 \end{array}\right]$$

- $$P^{-1}AP = \begin{bmatrix} 1/3 & 1/3 \\ -2/3 & 1/3 \end{bmatrix} \cdot \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$
  $$= \begin{bmatrix} 1/3 & 1/3 \\ -8/3 & 4/3 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$
  $$= \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

<u>Definition</u> If there is a basis of eigenvectors, then A is called diagonalizable.

LECTURE 24

Recall from last time

$$A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$$

We seek to find all $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^2$, $x \neq 0$, such that $Ax = \lambda x$. In this case, λ is an <u>eigenvalue</u> and x is an <u>eigenvector</u>.

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -1 \\ -2 & 2-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 4 = (\lambda - 1)\cdot(\lambda - 4)$$

eigenvalues $\lambda = 1$ and $\lambda = 4$

$\boxed{\lambda = 1}$ $\quad \ker(A - 1\cdot\lambda) = \left\{ s \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \;\middle|\; s \in \mathbb{R} \right\}$

<u>Basis</u> $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

$\boxed{\lambda = 4}$ $\quad \ker(A - 4\lambda) = \left\{ s \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \;\middle|\; s \in \mathbb{R} \right\}$

<u>Basis</u> $\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$

<u>Definition</u> An $n \times n$ matrix A is <u>diagonalizable</u> if there is a basis of $\mathbb{R}^n$ consisting entirely of eigenvectors of A.

Already from §23:

Proof below!

In our case, we have such a basis: $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$.

So, our A is diagonalizable. Let P be the matrix whose columns form the basis consisting entirely of eigenvectors:

$$P = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

In order to (double) check that the columns are LI, we observe that $\det(P) = 3$.

So, P is invertible.

Then:

$$AP = \left[A\begin{bmatrix}1\\2\end{bmatrix} \;\vdots\; A\begin{bmatrix}-1\\1\end{bmatrix} \right] = \left[1\cdot\begin{bmatrix}1\\2\end{bmatrix} \;\vdots\; 4\cdot\begin{bmatrix}-1\\1\end{bmatrix} \right] = P \cdot \begin{bmatrix}1 & 0\\0 & 4\end{bmatrix}$$

$$\Rightarrow \quad P^{-1}AP = \begin{bmatrix}1 & 0\\0 & 4\end{bmatrix}$$

diagonal matrix with eigenvalues in its diagonal (in the same order as the eigenvectors in P)

Example on slides!

§ 22.5 Problematic cases

a) Not enough real roots:

not diagonalizable (over the reals)

$$A = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix} \qquad \det(A-\lambda I) = \begin{vmatrix}-\lambda & -1\\1 & -\lambda\end{vmatrix} = \lambda^2+1$$

No real root, cannot be factorized!!!

We won't ask these things in the final!

Hey, recall the first lecture! We could work with complex numbers instead:

$$\det(A-\lambda I) = (\lambda-i)(\lambda+i) \qquad (\text{"Fundamentalsatz"})$$

$\boxed{\lambda=i}$ $\ker\begin{bmatrix}-i & -1\\1 & -i\end{bmatrix} = \left\{ s\cdot\begin{pmatrix}i\\1\end{pmatrix} \,\middle|\, s\in\mathbb{C} \right\}$. Basis $\left\{\begin{pmatrix}i\\1\end{pmatrix}\right\}$

$\boxed{\lambda=-i}$ $\ker\begin{bmatrix}i & -1\\1 & i\end{bmatrix} = \left\{ s\cdot\begin{pmatrix}-i\\1\end{pmatrix} \,\middle|\, s\in\mathbb{C} \right\}$. Basis $\left\{\begin{pmatrix}-i\\1\end{pmatrix}\right\}$

$$P := \begin{bmatrix}i & -i\\1 & 1\end{bmatrix} \qquad P^{-1}AP = \begin{bmatrix}i & 0\\0 & -i\end{bmatrix}$$

☺

b) Not enough eigenvectors:

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \qquad \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 = (\lambda-2)^2$$

algebraic multiplicity 2

$\boxed{\lambda = 2}$ $\ker \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \left\{ s \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \,\middle|\, s \in \mathbb{R} \right\}$ Basis $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

geometric multiplicity 1

So, we can't find a basis of $\mathbb{R}^2$ consisting entirely of eigenvectors. Not diagonalizable!

§ 23 Final conclusions about diagonizability

Definition The multiplicity of λ as a root of the characteristic polynomial is called the algebraic multiplicity of λ. The dimension of the eigenspace E_λ is called the geometric multiplicity of λ.

① Theorem Let λ be an eigenvalue of an $n \times n$ matrix A. Then:

$$\boxed{1 \leq \text{geom. mult. of } \lambda \leq \text{alg. mult. of } \lambda}$$

By the way, record that if $\lambda = 0$ is an eigenvalue of λ, then $\dim(E_0) \geq 1$, $\Rightarrow \dim \ker(A - 0 \cdot I) \geq 1$, $\Rightarrow \dim \ker(A) \geq 1$, $\Rightarrow A$ is not invertible.

Theorem Eigenvectors corresponding to distinct eigenvalues are always LI.

Corollary

1. If an $n \times n$ matrix has n distinct eigenvalues, it is always diagonalizable.
2. If $\lambda_1, \dots, \lambda_k$ are the distinct eigenvalues of A, then:

$$\boxed{\begin{aligned} &A \text{ diagonalizable} \\ &\Leftrightarrow \text{the sum of the geom. mult. of } \lambda_1, \dots, \lambda_k \text{ is } n \\ &\Leftrightarrow \dim(E_{\lambda_1}) + \dots + \dim(E_{\lambda_k}) = n \end{aligned}}$$

What to do ?!

1. Compute $\det(A - \lambda I)$ and find all roots (=eigenvalues). This needs a decomposition into linear factors. If that's impossible A is not diagonalizable over the reals.

   Three methods for factorization in my solution to W15-Final #14.

2. Find a basis for each eigenspace.
3. If there are n vectors in all $\Rightarrow$ diagonalizable
   If there are $< n$ vectors in all $\Rightarrow$ not diagonalizable
4. In the former case, write vectors into the columns of P. Then $P^{-1}AP = D$.

Conclude with slides !