MAT 1322 A. Winter 2013. Wednesday, February 6. 8h30–9h50

Midterm 1 Version A

Solution Key

Last name: ______________________________

First name: ______________________________

Student ID number: ________________________

- Exam duration: 80 minutes.
- Only calculators allowed by the Faculty of Sciences (Texas Instruments TI-30, TI-34 and Casio fx-260, fx-300) are authorized. Books and notes are not permitted.
- Solve each problem in the space provided. Use the back of the page for scratch work if necessary.
- Each problem requires a complete, clearly written solution. Partial marks will be assigned for significant, but incomplete, progress toward a solution. Zero credit will be given for unsupported answers unless otherwise noted.
- The exam is composed of 7 questions of between 4 and 10 points, for a total of 60 points, plus an 8^{th} bonus question worth 6 points.

1. (a) [4 pts] Why is $\int_0^5 \frac{x}{x^2-4}\, dx$ an improper integral?
If it is convergent, compute its value. If not, explain why it is divergent.

Solution. The integral is improper because the integrand $\frac{x}{x^2-4} = \frac{x}{(x-2)(x+2)}$ is undefined at the point $x = 2$, which is in the interval of integration $[0, 5]$. We have

$$\int_0^5 \frac{x}{x^2-4}dx = \int_0^2 \frac{x}{x^2-4}dx + \int_2^5 \frac{x}{x^2-4}dx$$

provided both integrals on the right hand side converge. Putting $u = x^2 - 4$, we find

$$\int \frac{x}{x^2-4}dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2-4| + C.$$

So

$$\int_0^2 \frac{x}{x^2-4}dx = \lim_{t\to 2^-}\int_0^t \frac{x}{x^2-4}dx = \lim_{t\to 2^-}\frac{1}{2}(\ln|t^2-4| - \ln(4)) = -\infty.$$

Since this integral is divergent, the given integral is divergent also.

(b) [4 pts] Determine if $\int_1^\infty \frac{1}{2\sqrt{x}+x^2}\, dx$ is convergent or divergent using an appropriate comparison test.

Solution. for all $x \geq 1$, we have $\sqrt{x} \leq x^2$, so

$$x^2 \leq 2\sqrt{x} + x^2 \leq 3x^2$$

whence

$$\frac{1}{3x^2} \leq \frac{1}{2\sqrt{x}+x^2} \leq \frac{1}{x^2}$$

and it follows that

$$\frac{1}{3} = \int_1^\infty \frac{dx}{3x^2} \leq \int_1^\infty \frac{dx}{2\sqrt{x}+x^2} \leq \int_1^\infty \frac{dx}{x^2} = 1.$$

In particular the integral is convergent, and its value is between $\frac{1}{3}$ and 1.

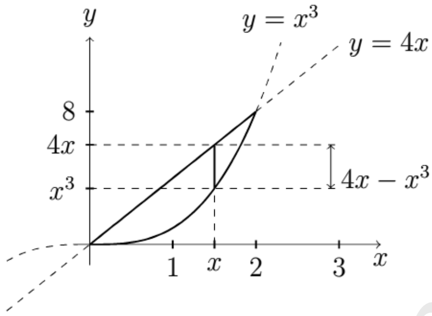
2. [10 pts] Let $\mathcal{R}$ be the region in the first quadrant bounded by the curves

$$y = 4x \quad \text{and} \quad y = x^3 .$$

(a) Sketch the region $\mathcal{R}$ and calculate its area.

Solution. We first determine the points of intersection of the two curves by solving the equation $4x = x^3$. We find $x = 0$ or $4 = x^2$, so $x = -2$, 0 or 2. Looking for those points in the first quadrant, we are left with $x = 0, 2$. So the points of intersection are $(0, 0)$ and $(2, 8)$.
The area of the region is

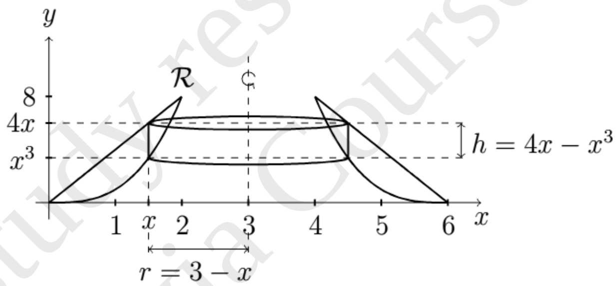
$$\int_0^2 (4x - x^3)\, dx = \left[2x^2 - \frac{x^4}{4}\right|_0^2 = \boxed{4}.$$



(b) Calculate the volume of the solid of revolution obtained by rotating the region $\mathcal{R}$ around the vertical line $x = 3$, and draw a typical layer which appears in the calculation of volume (disc, washer or cylindrical shell), labeling its dimensions.

Solution. The easiest technique is the method of cylindrical shells. Rotating the portion of the region $\mathcal{R}$ between x and $x + \Delta x$ (for small Δx) about the line $x = 3$, we obtain a cylindrical shell of radius $r = 3 - x$, with height $h = 4x - x^3$ and thickness Δx (as shown in the figure). Its volume is

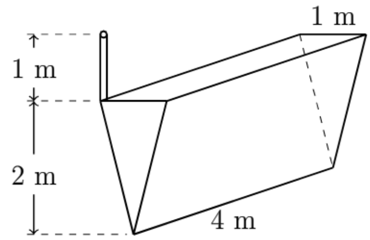
$$\Delta V \cong 2\pi r h \Delta x = 2\pi(3 - x)(4x - x^3)\Delta x$$



Thus, the total volume of the solid is

$$V = \int_0^2 2\pi(3 - x)(4x - x^3)dx = 2\pi \int_0^2 (12x - 4x^2 - 3x^3 + x^4)dx$$
$$= 2\pi \left[6x^2 - (4/3)x^3 - (3/4)x^4 + (1/5)x^5\right|_0^2 = \frac{232\pi}{15} \cong 48.59.$$

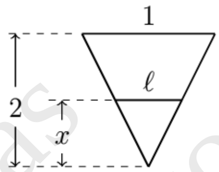
3. [8 pts] A reservoir in the form of a straight prism with triangular base is shown in the figure to the right. Its vertical faces are isosceles triangles of height 2 m and base 1 m, its length is 4 m, it is near the surface of the Earth, and it is full of water, which will be pumped to a height of 1 m above the reservoir.



Denote by x the height in meters measured **from the bottom of the reservoir**.

(a) What is, at first approximation, the volume of the layer of water between the heights x and $x + \Delta x$?

Solution. A horizontal section of the reservoir at height x is a rectangle of length 4 and width ℓ where ℓ is given by examination of similar triangles (figure to the right). We find



$$\frac{\ell}{x} = \frac{1}{2} \quad \text{so} \quad \ell = \frac{x}{2}.$$

So, (volume of the layer) $\cong$ (area of rectangle) $\times\ \Delta x = 4\ell\Delta x = 2x\Delta x$.

(b) What is, at first approximation, the work required to pump that layer of water to a height of 1 m above the reservoir? Recall that the density of water is 1000 kg/m^3, and gravitational acceleration at the surface of the earth is $g \cong 9.8$ m/s^2.

Solution. We must lift the layer up $3 - x$ meters. So the required work is

$$\begin{aligned} \text{Work} &= (\text{Force}) \times (\text{distance}) \\ &= 1000g(\text{volume}) \times (\text{distance}) \\ &\cong 9800(2x\Delta x)(3-x) = 19600x(3-x)\Delta x \end{aligned}$$

(c) What is, in Joules, the work required to pump all the water from the reservoir to a height of 1 m above the reservoir?

Solution. Since the reservoir is full of water, we integrate from $x = 0$ to $x = 2$:

$$W = \int_0^2 19600x(3-x)dx = 19600\int_0^2 (3x - x^2)dx = 19600\left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^2 \cong 65300 \text{ J.}$$

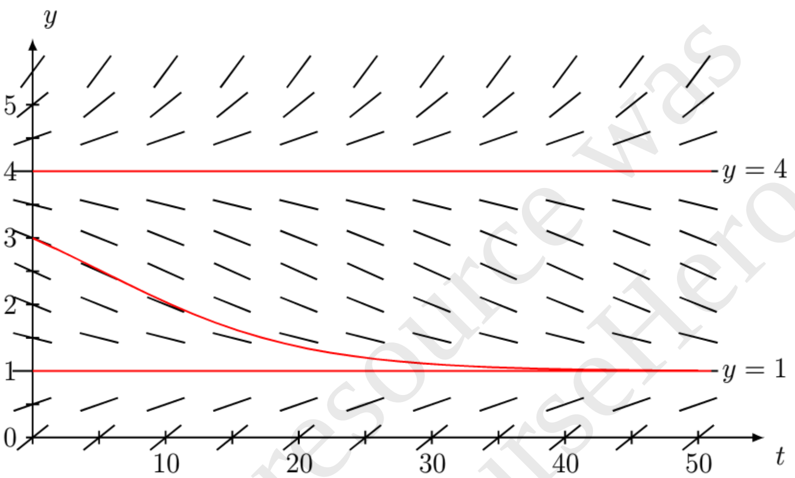
Questions 4 and 5 are short answer questions, for which you do not need to justify your answers.

4. [4 points] Consider the curve $x = t^2, \; y = e^{2t}, \quad 1 \le t \le 3$.
Write down the integral which gives the length of this curve. Simplify the integrand, but *do not calculate the integral.*

Response:

$$L = \int_1^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_1^3 \sqrt{(2t)^2 + (2e^{2t})^2}\,dt = \int_1^3 \sqrt{4t^2 + 4e^{4t}}\,dt\,.$$

5. [4 points] Below is the slope field for a differential equation $\frac{dy}{dt} = F(t, y)$.



(a) What are its equilibrium solutions (those of the form $y = C$)?

Response: $y = 1$ and $y = 4$.

(b) Draw, as well as you can, on the slope field, the graph of the particular solution with initial value $y(0) = 3$. Circle the answer below which is closest to $y(20)$.

Choices: (A) 0.8   [(B) 1.3]   (C) 2.2   (D) 3   (E) 4.5

6. [8 pts] Paul makes a cup of hot chocolate at the boiling temperature of water (100 °C), then puts it outside to cool it down. Outside, the temperature is −15 °C, and 3 minutes later, when he checks his cup, its temperature is 40 °C.

(a) Let $T(t)$ be the temperature of the cup t minutes after Paul puts it outside. Supposing this temperature obeys Newton's law of cooling, give the differential equation which T satisfies, and solve it.

Solution. According to Newton's law of cooling, we have

$$\frac{dT}{dt} = k(T - (-15)) = k(T + 15)$$

where k is a constant. We deduce

$$\int \frac{dT}{T+15} = \int k\,dt \Rightarrow \ln|T+15| = kt + C$$
$$\Rightarrow T + 15 = Ae^{kt} \Rightarrow \boxed{T = -15 + Ae^{kt}}$$

where $A = \pm e^{C}$ is a constant. Since $T(0) = 100$, we get

$$100 = -15 + A \Rightarrow A = 115 \quad \text{thus} \quad \boxed{T(t) = -15 + 115e^{kt}}.$$

We also have $T(3) = 40$, so

$$40 = -15 + 115e^{3k} \Rightarrow e^{3k} = \frac{55}{115} = \frac{11}{23} \Rightarrow k = \frac{1}{3}\ln(11/23) \cong -0.2459$$
$$\Rightarrow \boxed{T(t) \cong -15 + 115e^{-0.2459t}}.$$

(b) If Paul leaves his cup outside, at what time t will his hot chocolate freeze? (Suppose that hot chocolate freezes at a temperature of 0 °C.)

Solution. We find t such that $T(t) = 0$. So we need to solve

$$0 = -15 + 115e^{-0.2459t} \Rightarrow e^{-0.2459t} = \frac{15}{115} = \frac{3}{23}$$
$$\Rightarrow t = \frac{\ln(3/23)}{-0.2459} \cong \boxed{8.28 \text{ min.}}$$

7. [8 pts] Solve the initial value problem $\dfrac{dy}{dt} = \dfrac{2t\sin(2t)}{y}, \quad y(0) = 2\,.$
Express the solution y as a function of t.

Solution: Separating the variables, we find

$$\int y\,dy = \int 2t\cos(2t)\,dt$$

To integrate the right side, proceed by parts, choosing $u = 2t$ and $dv = \cos(2t)dt$ so that $du = 2dt$ and $v = (1/2)\sin(2t)$. This gives

$$\begin{aligned}\int 2t\cos(2t)\,dt &= (2t)(1/2)\sin(2t) - \int (1/2)\sin(2t)2\,dt \\ &= t\sin(2t) - \int \sin(2t)\,dt = t\sin(2t) + (1/2)\cos(2t) + C\end{aligned}$$

Thus we find

$$\frac{y^2}{2} = t\sin(2t) + (1/2)\cos(2t) + C \Rightarrow y = \pm\sqrt{2t\sin(2t) + \cos(2t) + 2C}.$$

Since $y(0) = 2$, we conclude that $2 = \pm\sqrt{1+2C}$, so $2C = 3$ and

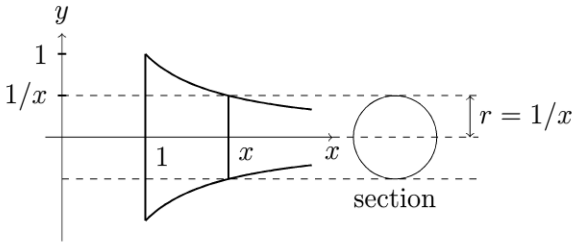
$$\boxed{y = \sqrt{2t\sin(2t) + \cos(2t) + 3}}$$

with a $+$-sign since $y(0) > 0$.

8. [6 pts **bonus**] Let $\mathcal{R}$ be the region of the plane defined by $1 \le x < \infty$ and $0 \le y \le \dfrac{1}{x}$. What is the volume of the solid of revolution obtained by rotating $\mathcal{R}$ about the x-axis? (This figure is called the "Horn of Gabriel" and was first studied by Torricelli, an Italian mathematician and physicist of the 17^{th} century, who invented the barometer in 1643.)

Solution. The cross-section of the solid at the point x is a disc of radius $r = 1/x$, as shown in the diagram below. Its area is $A(x) = \pi r^2 = \pi/x^2$, so the volume of the solid is

$$\begin{aligned}V &= \int_1^\infty \frac{\pi}{x^2}\,dx = \pi \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx \\ &= \pi \lim_{t\to\infty}\left[\frac{-1}{x}\right|_1^t = \pi \lim_{t\to\infty}\left(\frac{-1}{t} + 1\right) \\ &= \pi.\end{aligned}$$



(We could also invoke the p-test $\int_1^\infty \frac{dx}{x^p} = \frac{1}{p-1}$ if $p > 1$, putting $p = 2$.)