MAT1348: Consistency of Sets of Propositional Formulas

P. J. Scott , Winter 2017

1 Consistency of $\{A_1, \ldots, A_n\}$ using Truth Tables

Given a finite set of formulas of Propositional Logic, $\{A_1, \cdots, A_k\}$, with atoms $p_1, \cdots, p_n$, we say the formulas are *consistent* if the conjunction $A_1 \wedge A_2 \wedge \cdots \wedge A_k$ is satisfiable. Remember, a conjunction is true if and only if each of the conjuncts A_i is true. [1]

So, that means: $A_1 \wedge A_2 \wedge \cdots \wedge A_k$ *is satisfiable if in some row of the truth table, each of the formulas* $A_1, \cdots A_k$ *is simultaneously true on that row .* But a row of a truth table involves an assignment of **T**'s and **F**'s to the atoms $p_1, \cdots, p_n$. Hence, $\{A_1, \cdots, A_k\}$ is consistent if there is some assignment (= Valuation V) of **T**'s and **F**'s to the atoms p_i in some row of the truth table so that *on that row* all the formulas A_i have value **T**.

How to test if a finite set of Propositional formulas $\{A_1, \cdots, A_k\}$ is consistent:

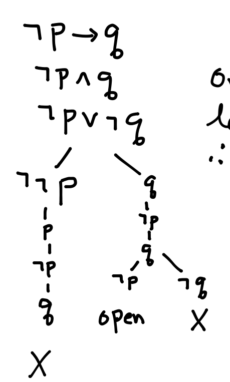
1. Write a Truth Table with final columns A_1, $A_2, \cdots, A_k$.
2. See if there is a row where all the A_i simultaneously have value **T**. (This is the same as saying the conjunction $(A_1 \wedge \cdots \wedge A_k)$ is **T**).
3. If the answer in (2) is YES, then the original set $\{A_1, \cdots, A_k\}$ is consistent. If the answer in (2) is NO, then the original set $\{A_1, \cdots, A_k\}$ is not consistent.

Example: Look at the set $\{(\neg p \to q), (\neg p \wedge q), (\neg p \vee \neg q)\}$. Write the truth table for these three formulas (as the last three columns of the table). You will see on the row in which we assign the values: $p := \mathbf{F}, q := \mathbf{T}$, all three formulas are **T**.

2 Consistency of $\{A_1, \ldots, A_n\}$ using truth trees

1. Put the formulas $A_1, \ldots, A_n$ at the root of the tree, and draw the complete truth tree.
2. If every branch (starting from the root) is closed, the set is not consistent.
3. If at least one branch is open, then this open branch corresponds to a valuation V making *all* the formulas on the branch **T**. In particular, at the root, $V(A_1) = V(A_2) = \cdots = V(A_n) = \mathbf{T}$. Hence the formulas are simultaneously true (under V) , so the set is consistent.

Example above



On the open branch, let $V(\neg p) = T$, $V(q) = T$.
$\therefore$ $V(p) = F$, $V(q) = T$
(gives same answer as above).

[1] Recall, conjunction is associative, e.g. $((A_1 \wedge A_2) \wedge A_3) \equiv (A_1 \wedge (A_2 \wedge A_3))$, so it doesn't matter how we parenthesize the expression $(A_1 \wedge \cdots \wedge A_k)$: all parenthesizations have the same truth value.