MAT 2384 A Assignment #1 Solutions

1. $2(1+x^2)yy' = 4x+1$, $y(0)=1$

the DE is separable:

$$2yy' = \frac{4x+1}{1+x^2}$$

so $$\int 2y\,dy = \int \left(\frac{4x}{1+x^2} + \frac{1}{1+x^2}\right)dx + C$$

which gives $y^2 = 2\ln(1+x^2) + \arctan x + C$ (implicit general solution)

then $y(0)=1 \Rightarrow (1)^2 = 2\ln(1+(0)^2) + \arctan(0) + C$

or $1 = 2\ln(1) + C = C$

and so $y^2 = 2\ln(1+x^2) + \arctan x + 1$

and the unique solution (in explicit form) is

$$\boxed{y = \left(2\ln(1+x^2) + \arctan x + 1\right)^{1/2}}$$

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2. $\cot x \sin y \, y' = 1$, $y(0)=0$

the DE is separable: $\sin y \, y' = \tan x$

and so $\int \sin y\,dy = \int \tan x\,dx + C$

we get $-\cos y = -\ln|\cos x| + c$

or $\cos y = \ln|\cos x| + C$

so $y = \arccos(\ln|\cos x| + C)$ (general solution)

$y(0)=0 \Rightarrow 0 = \arccos(\ln|\cos(0)| + C) = \arccos(C)$

thus $C=1$ and the unique solution is

$$\boxed{y = \arccos(\ln|\cos x| + 1)}$$

A15
A1
2

3. $(x\tan(y/x) - y)\,dx + x\,dy = 0, \quad y(1) = \pi/2$ (not separable)

$M(x,y) = x\tan(y/x) - y$ } both homogeneous
$N(x,y) = x$ } of degree 1

let $u = y/x$, ie $y = ux$ and $dy = u\,dx + x\,du$
and the DE will become

$(x\tan u - ux)\,dx + x(u\,dx + x\,du) = 0$

or $x\tan u\,dx - ux\,dx + xu\,dx + x^2\,du = 0$

or $x\tan u\,dx + x^2\,du$

which is separable : $\cot u\,du = -\frac{1}{x}\,dx$

and so $\int \cot u\,du = \int -\frac{1}{x}\,dx + C$

which gives $\ln|\sin u| = -\ln|x| + C$

exponentiate $\sin u = K/x$

and then $u = \arcsin(K/x)$

but $u = y/x$, so $y = x\arcsin(K/x)$ (general solution)

$y(1) = \pi/2 \Rightarrow \pi/2 = (1)\arcsin(K/1) = \arcsin(K)$

and so $K=1$ and the unique solution is

$\boxed{y = x\arcsin(1/x)}$

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4. $(2x+1)\,dx + (4y+2)\,dy = 0, \quad y(2) = 1$

DE is separable : $\int (4y+2)\,dy = -\int (2x+1)\,dx + C$

which gives $2y^2 + 2y = -x^2 - x + C$

or $2y^2 + 2y + x^2 + x = C$ (general solution)

$y(2) = 1 \Rightarrow 2(1)^2 + 2(1) + (2)^2 + (2) = C \Rightarrow C = 10$

$\therefore$ the unique solution is $\boxed{2y^2 + 2y + x^2 + x = 10}$

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5. $(x + 2y)\,dx - x\,dy = 0$, $y(1) = 5$ (not separable)

$M(x,y) = x + 2y$ } both homogeneous of
$N(x,y) = -x$ } deg. 1

let $y = ux$, $dy = u\,dx + x\,du$ and the DE becomes

$(x + 2ux)\,dx - x\,(u\,dx + x\,du) = 0$

$x\,dx + 2ux\,dx - ux\,dx - x^2\,du = 0$

or $x\,dx + ux\,dx - x^2\,du = 0$

or $x(1+u)\,dx - x^2\,du = 0$

which is separable : $\dfrac{du}{1+u} = \dfrac{1}{x}\,dx$

so integrate : $\displaystyle\int \frac{du}{1+u} = \int \frac{1}{x}\,dx + C$

we get $\ln|1+u| = \ln|x| + C$

exponentiate $1 + u = Kx$

and so $u = Kx - 1$

but $u = y/x$ and so $y = x(Kx - 1) = Kx^2 - x$ (gen sol'n)

$y(1) = 5 \Rightarrow 5 = K(1)^2 - (1) = K - 1 \Rightarrow K = 6$

and so the unique solution is $\boxed{y = 6x^2 - x}$

A15
A1
4

6. $f(x) = x^3 - 6x + 4$ $\quad$ $\left.\begin{array}{l} f(0) = 4 \\ f(1) = -1 \end{array}\right\}$ $\therefore$ sign change on $[0,1]$ so root in $[0,1]$

$f(x) = x^3 - 6x + 4 = 0 \Rightarrow x = \dfrac{x^3+4}{6}$

take $g(x) = \dfrac{x^3+4}{6}$, which will satisfy the fixed point condition

also $|g'(x)| = \left|\dfrac{3x^2}{6}\right| = \frac{1}{2}x^2 \leq \frac{1}{2} < 1$ on $[0,1]$

and so $x_{n+1} = g(x_n)$ will generate a convergent sequence

$x_0 = 0.5, \quad x_1 = g(0.5) = \dfrac{(0.5)^3+4}{6} = 0.6875$

$x_2 = g(x_1) = 0.720825$ $\quad$ $x_7 = g(x_6) = 0.732036$

$x_3 = g(x_2) = 0.729089$ $\quad$ $x_8 = g(x_7) = 0.732047$

$x_4 = g(x_3) = 0.731260$ $\quad$ $x_9 = g(x_8) = 0.732050$

$x_5 = g(x_4) = 0.731839$ $\quad$ $x_{10} = g(x_9) = 0.732051$

$x_6 = g(x_5) = 0.731994$ $\quad$ $x_{11} = g(x_{10}) = 0.732051$ $\therefore$ stop

(check. $f(0.732051) \approx -8.45 \times 10^{-7}$ so okay!)

$\therefore$ the root is $\boxed{0.732051}$

A15
A1
5

7. $f(x) = x^3 - 6x + 4$, so $f'(x) = 3x^2 - 6$

then $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{x_n^3 - 6x_n + 4}{3x_n^2 - 6} = \dfrac{2x_n^3 - 4}{3x_n^2 - 6}$

$x_0 = 0.5$, $x_1 = \dfrac{2(0.5)^3 - 4}{3(0.5)^2 - 6} = 0.714\,286$

$x_2 = 0.731\,898$

$x_3 = 0.732\,051$

$x_4 = 0.732\,051$ $\therefore$ stop root is $\boxed{0.732051}$

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8. if $\ln x = 9 - x^2$, then $f(x) = \ln x - 9 + x^2 = 0$

so $f'(x) = \frac{1}{x} + 2x$

and $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{\ln(x_n) - 9 + x_n^2}{\frac{1}{x_n} + 2x_n}$

$= \dfrac{10 - \ln(x_n) + x_n^2}{\frac{1}{x_n} + 2x_n}$

$x_0 = 2$, $x_1 = \dfrac{10 - \ln(2) + (2)^2}{\frac{1}{2} + 2(2)} = 2.9571$

$x_2 = 2.8246$

$x_3 = 2.8218$

$x_4 = 2.8218$ $\therefore$ stop

(check: $f(2.8218) \approx -7 \times 10^{-5}$ okay!)

$\therefore$ the solution is $\boxed{x = 2.8218}$