

Tutorial 2: Laplace Transforms and Transfer Functions

Q1. Using the definition of a Laplace transform, show that

(a) $e^{-bt} \sin(\omega t)$ is $\frac{\omega}{(s+b)^2 + \omega^2}$.

(b) $e^{-bt} \cos(\omega t)$ is $\frac{s+b}{(s+b)^2 + \omega^2}$.

Solution:

(a)

$$\begin{aligned} \mathcal{L} [e^{-bt} \sin(\omega t)] &= \int_0^{\infty} e^{-bt} \sin(\omega t) e^{-st} dt \\ &= \int_0^{\infty} \frac{e^{i\omega} - e^{-i\omega}}{2i} e^{-(s+b)t} dt \\ &= \frac{1}{2i} \left[\int_0^{\infty} e^{(s+b+i\omega)t} dt - \int_0^{\infty} e^{(s+b-i\omega)t} dt \right] \\ &= \frac{1}{2i} \left[\frac{1}{(s+b-i\omega)} - \frac{1}{(s+b+i\omega)} \right] \\ &= \frac{\omega}{(s+b)^2 + \omega^2} \end{aligned} \tag{1}$$

(b) Similarly,

$$\mathcal{L} [e^{-bt} \cos(\omega t)] = \frac{s+b}{(s+b)^2 + \omega^2} \tag{2}$$

Q2. The figure below shows a triangular pulse function.

- (a) From details in the drawing, calculate the pulse width, t_w .
- (b) Construct this function as the sum of simpler time elements, some perhaps translated in time, whose transforms can be found directly from the laplace transform table.
- (c) Find $U(s)$.
- (d) What is the area under the pulse?

Solution:

(a) Pulse width is obtained when $x(t) = 0$. Since $x(t) = h - at$, $h - at_w = 0$ or $t_w = h/a$.

(b) From the figure below it is easy to see that $x(t) = hS(t) - atS(t) + a(t - t_w)S(t - t_w)$.

(c) $X(s) = \frac{h}{s} - \frac{a}{s^2} + \frac{ae^{-st_w}}{s^2} = \frac{h}{s} + \frac{e^{-st_w} - 1}{s^2}$

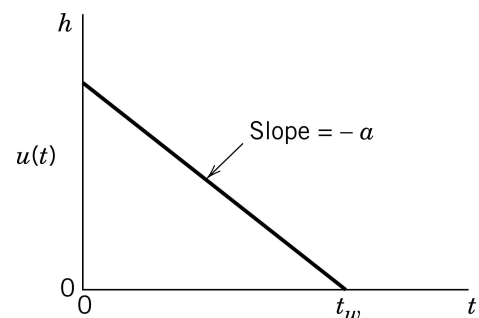


Figure 1: Question 2

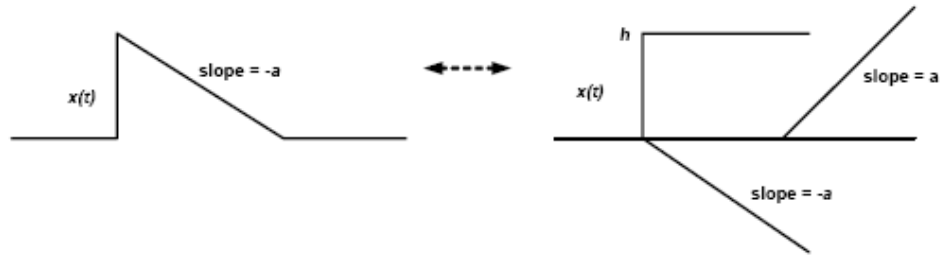


Figure 2: Question 2

(d) Area under pulse = $ht_w/2$

Q3. Expand each of the following s -domain functions into partial fractions:

(a) $Y(s) = \frac{6(s+1)}{s^2(s+1)}$

(b) $Y(s) = \frac{12(s+2)}{s(s^2+9)}$

(c) $Y(s) = \frac{(s+2)(s+3)}{(s+4)(s+5)(s+6)}$

(d) $Y(s) = \frac{1}{[(s+1)^2+1]^2(s+2)}$

Solution:

(a)

$$Y(s) = \frac{6(s+1)}{s^2(s+1)} = \frac{6}{s^2} = \frac{\alpha_1}{s} + \frac{\alpha_2}{s^2}$$

$$\alpha_2 = s^2 \frac{6}{s^2} \Big|_{s=0} = 6 \quad \alpha_1 = 0$$

$$Y(s) = \frac{6}{s^2}$$

(b)

$$Y(s) = \frac{12(s+2)}{s(s^2+9)} = \frac{\alpha_1}{s} + \frac{\alpha_2 s + \alpha_3}{s^2+9}$$

Multiplying both sides by $s(s^2+9)$, we get $12(s+2) = \alpha_1(s^2+9) + (\alpha_2 s + \alpha_3)(s)$ or $12s + 24 = (\alpha_1 + \alpha_2)s^2 + \alpha_3 s + 9\alpha_1$. Equating coefficients of like powers of s ,

$$\begin{aligned} s^2 : \quad & \alpha_1 + \alpha_2 = 0 \\ s^1 : \quad & \alpha_3 = 12 \\ s^0 : \quad & 9\alpha_1 = 24 \end{aligned} \quad (3)$$

Solving simultaneously, $\alpha = \frac{8}{3}$, $\alpha_2 = -\frac{8}{3}$, $\alpha_3 = 12$.
Therefore,

$$Y(s) = \frac{8}{3} \frac{1}{s} + \frac{(-\frac{1}{8}s + 12)}{s^2 + 9}.$$

$$Y(s) = \frac{(s+2)(s+3)}{(s+4)(s+5)(s+6)} = \frac{\alpha_1}{s+4} + \frac{\alpha_2}{s+5} + \frac{\alpha_3}{s+6}$$

$$\alpha_1 = \frac{(s+2)(s+3)}{(s+5)(s+6)} \Big|_{s=-4} = 1$$

$$\alpha_2 = \frac{(s+2)(s+3)}{(s+4)(s+6)} \Big|_{s=-5} = -6$$

$$\alpha_3 = \frac{(s+2)(s+3)}{(s+4)(s+5)} \Big|_{s=-6} = 6$$

$$Y(s) = \frac{1}{s+4} - \frac{6}{s+5} + \frac{6}{s+6}$$

(d)

$$\begin{aligned} Y(s) &= \frac{1}{[(s+1)^2 + 1]^2(s+2)} = \frac{1}{(s^2 + 2s + 2)(s+2)} \\ &= \frac{\alpha_1 s + \alpha_2}{s^2 + 2s + 2} + \frac{\alpha_3 s + \alpha_4}{(s^2 + 2s + 2)^2} + \frac{\alpha_5}{s+2} \end{aligned}$$

Multiplying both sides by $(s^2 + 2s + 2)^2(s+2)$ gives

$$\begin{aligned} 1 &= \alpha_1 s^4 + 4\alpha_1 s^3 + 6\alpha_1 s^2 + 4\alpha_1 s + \alpha_2 s^2 + 6\alpha_2 s + 4\alpha_2 + \alpha_3 s^2 + 2\alpha_3 s \\ &\quad + \alpha_4 s + 2\alpha_4 + \alpha_5 s^4 + 4\alpha_5 s^3 + 8\alpha_5 s^2 + 8\alpha_5 s + 4\alpha_5 \end{aligned}$$

Equating coefficients of like powers of s ,

$$s^4: \quad \alpha_1 + \alpha_5 = 0$$

$$s^3: \quad 4\alpha_1 + \alpha_2 + 4\alpha_5 = 0$$

$$s^2: \quad 6\alpha_1 + 4\alpha_2 + \alpha_3 + 8\alpha_5 = 0$$

$$s: \quad 4\alpha_1 + 6\alpha_2 + 2\alpha_3 + \alpha_4 + 8\alpha_5 = 0$$

$$s^0: \quad 4\alpha_2 + 2\alpha_4 + 4\alpha_5 = 1$$

Solving simultaneously: $\alpha_1 = -1/4$, $\alpha_2 = 0$,
 $\alpha_3 = -1/2$, $\alpha_4 = 0$, $\alpha_5 = 1/4$.

$$Y(s) = \frac{-1/4s}{s^2 + 2s + 2} + \frac{-1/2s}{(s^2 + 2s + 2)^2} + \frac{1/4}{s+2}$$

Q4.

(a) For the integro-differential equation

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 2 \int_0^t e^{-\tau} d\tau$$

find $x(t)$.

(b) What is the value of $x(t)$ as $t \rightarrow \infty$?

Solution:

(a) Let us find the Laplace of integral of a function, $f(t)$.

$$\mathcal{L}\left[\int_0^t f(t^*) dt^*\right] = \int_0^\infty \left\{ \int_0^t f(t^*) dt^* \right\} e^{-st} dt \quad (4)$$

Integrating by parts and using the definition of a Laplace transform,

$$\begin{aligned} \mathcal{L}\left[\int_0^t f(t^*) dt^*\right] &= -\frac{1}{s} \left[e^{-st} \int_0^t f(t^*) dt^* \right]_0^\infty + \frac{1}{s} \int_0^\infty e^{-st} f(t) dt \\ &= \frac{1}{s} F(s) \end{aligned} \quad (5)$$

Therefore, Laplace transform of the original integro-differential equation is

$$\begin{aligned} s^2 X(s) + 3X(s) + 2X(s) &= \frac{2}{s(s+1)} \\ (s^2 + 3s + 1)X(s) &= \frac{2}{s(s+1)} \\ X(s) &= \frac{2}{s(s+1)^2(s+2)} \\ x(t) &= 1 - 2te^{-t} - e^{-2t} \end{aligned}$$

(b) $\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} sX(s) = 1$.

Q5. The dynamic model between an output variable y and an input variable u can be expressed by

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + y(t) = 4\frac{du(t-2)}{dt} - u(t-2)$$

(a) Will this system exhibit an oscillatory response after an arbitrary change in u ?

(b) What is the steady-state gain?

(c) For a step change in u of magnitude 1.5, what is $y(t)$?

Solution:

(a)

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + y(t) = 4\frac{du(t-2)}{dt} - u(t-2)$$

Taking the Laplace transform and assuming zero initial conditions,

$$s^2Y(s) + 3sY(s) + Y(s) = 4e^{-2s}sX(s) - e^{-2s}X(s)$$

Rearranging,

$$\frac{Y(s)}{U(s)} = G(s) = \frac{-(1-4s)e^{-2s}}{s^2+3s+1}$$

The standard form of the denominator is (we will learn this in Chapter 5): $\tau^2s^2 + 2\zeta\tau s + 1$. Therefore, $\tau = 1$, $\zeta = 1.5$.

(b) Steady-state gain $K = \lim_{s \rightarrow 0} G(s) = -1$.

(c) For a step change in x , $U(s) = \frac{1.5}{s}$ and $Y(s) = \frac{-(1-4s)e^{-2s}}{(s^2+3s+1)} \frac{1.5}{s}$. Therefore $y(t) = -1.5 + 1.5e^{-1.5t} \cosh(1.11t) + 7.38e^{-1.5t} \sinh(1.11t)$. Using MATLAB-Simulink, $y(t)$ with a delay of two units is shown below

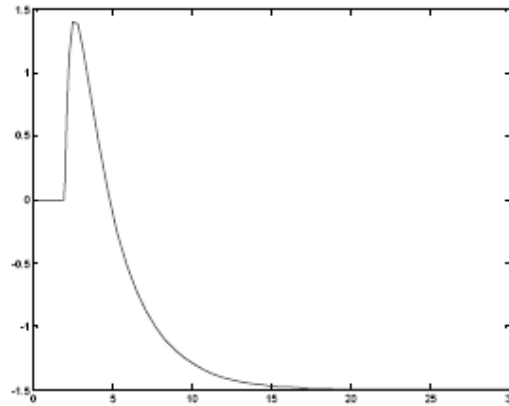


Figure 3: Question 5: Output variable for a step change in x of magnitude 1.5.

Q6. An Isothermal CSTR. Consider an isothermal continuous stirred tank reactor shown in figure 2. Use the following data in answering questions below: $F = 0.085 \text{ m}^3/\text{min}$; $V = 2.1 \text{ m}^3$; C_{A0} was initially at $0.925 \text{ mole}/\text{m}^3$ but suddenly changes to $1.85 \text{ mole}/\text{m}^3$ suddenly at $t = 0$. The chemical reaction is of first order, $r_A = -kC_A$ with $k = 0.040/\text{min}$. The heat of reaction is negligible and no heat is transferred to the surroundings.

(a) Develop a dynamic model using mass balance equations.

(b) Solve the dynamic model for C_A by integration.

(c) Solve the dynamic model for C_A using Laplace transforms.

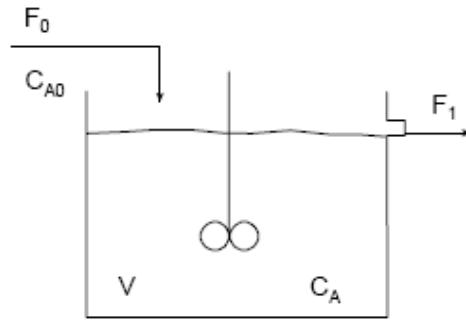


Figure 4: Question 6

- (d) Assuming that the reactant concentration should never go above 0.85 mole/m^3 . If an alarm sounds when the concentration reaches 0.83 mole/m^3 , would a person have enough time to respond?

Solution: Please try this problem on your own using what was done in the class in lecture 8.

Q7. Consider the following transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{5}{10s + 1}$$

- What is the steady-state gain?
- What is the time constant?
- If $U(s) = 2/s$, what is the value of the output $y(t)$ when $t \rightarrow \infty$?
- For the same $U(s)$, what is the value of the output when $t = 10$? What is the output when expressed as a fraction of the new steady-state value?
- If $U(s) = (1 - e^{-s})/s$, that is, the unit rectangular pulse, what is the output when $t \rightarrow \infty$?
- If $u(t) = \delta(t)$, that is, the unit impulse at $t = 0$, what is the output when $t \rightarrow \infty$?
- If $u(t) = 2\sin(3t)$, what is the value of the output when $t \rightarrow \infty$?

Solution:

- 5
- 10
- $Y(s) = \frac{10}{s(10s+1)}$. From the Final Value Theorem, $y(t) = 10$ when $t \rightarrow \infty$.
- $y(t) = 10(1 - e^{-t/10})$, then $y(10) = 6.32 = 63.2\%$ of the final value.

- (e) $Y(s) = \frac{5}{10s+1} \frac{1-e^{-s}}{s}$. From the final value theorem, $y(t) = 0$ when $t \rightarrow \infty$
- (f) $Y(s) = \frac{5}{(10s+1)} 1$. From the Final Value Theorem, $y(t) = 0$ when $t \rightarrow \infty$.
- (g) $Y(s) = \frac{5}{(10s+1)} \frac{6}{(s^2+9)}$ then $y(t) = 0.33e^{-0.1t} - 0.33\cos(3t) + 0.011\sin(3t)$. The sinusoidal input produces a sinusoidal output and $y(t)$ does not have a limit when $t \rightarrow \infty$.

Q8. A simple surge tank with fixed valve on the outflow line is illustrated in figure 3. If the outflow rate is proportional to the square root of the liquid height an unsteady-state model for the level in the tank is given by

$$A \frac{dh}{dt} = q_i - C_v h^{1/2}$$

As usual, you can assume that the process initially is at steady state

$$\bar{q}_i = \bar{q} = C_v \bar{h}^{1/2}$$

- (a) Find the transfer function $H'(s)/Q'_i(s)$. Put the transfer function in standard gain/time constant form.
- (b) Find the transfer function $Q'(s)/Q'_i(s)$ and put it in standard form.
- (c) If the algebraic relation for the outflow rate is linear instead of square root, the level transfer function can be put into a first-order form.

$$\frac{H'(s)}{Q'_i(s)} = \frac{K^*}{\tau^*s + 1}$$

with $K^* = \bar{h}/\bar{q}_i$, $\tau^* = \bar{V}/\bar{q}_i$, and $\bar{V} = A\bar{h}$ is the initial steady-state volume. When written this way, τ^* is easily interpreted as the liquid residence time at the nominal operating conditions. What are equivalent expressions for K and τ in the part (a) level transfer function, that is, for the square root outflow relation?

Solution:

(a)

$$A \frac{dh}{dt} = q_i - C_v h^{1/2}$$

Let $f = q_i - C_v h^{1/2}$. Then using Taylor's series expansion,

$$f \approx \bar{q}_i - C_v \bar{h}^{1/2} + q_i - \bar{q}_i - \frac{1}{2} C_v \bar{h}^{-1/2} (h - \bar{h})$$

Therefore, $A \frac{dh'}{dt} = q'_i - \frac{C_v}{2\bar{h}^{1/2}} h'$ because $\bar{q}_i - C_v \bar{h}^{1/2} = 0$.

$$\begin{aligned} \left[sA + \frac{C_v}{2\bar{h}^{1/2}} \right] H'(s) &= Q'_i(s) \\ \frac{H'(s)}{Q'_i(s)} &= \frac{1}{sA + \frac{C_v}{2\bar{h}^{1/2}}} \\ \frac{H'(s)}{Q'_i(s)} &= \frac{2\bar{h}^{1/2}/C_v}{\frac{2A\bar{h}^{1/2}}{C_v}s + 1} \end{aligned}$$

where $K = \frac{2\bar{h}^{1/2}}{C_v}$ and $\tau = \frac{2A\bar{h}^{1/2}}{C_v}$.

By using Simulink-MATLAB, above solutions can be verified:

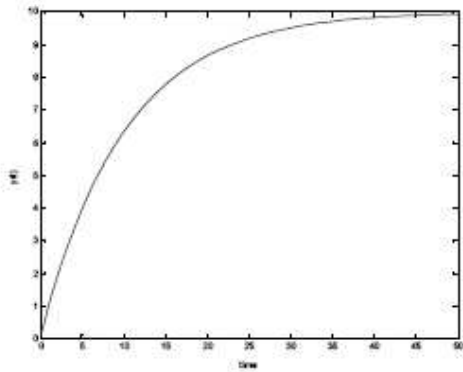


Fig S4.2a. Output for part c) and d)

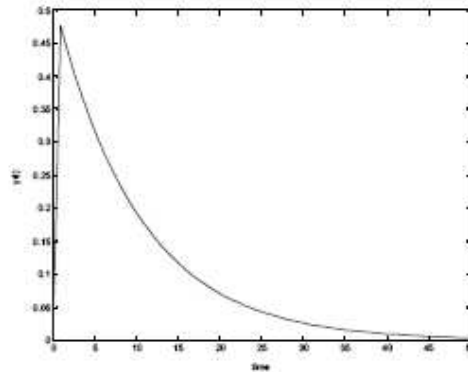


Fig S4.2b. Output for part e)

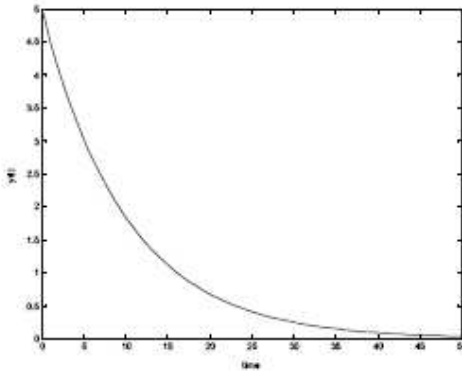


Fig S4.2c. Output for part f)

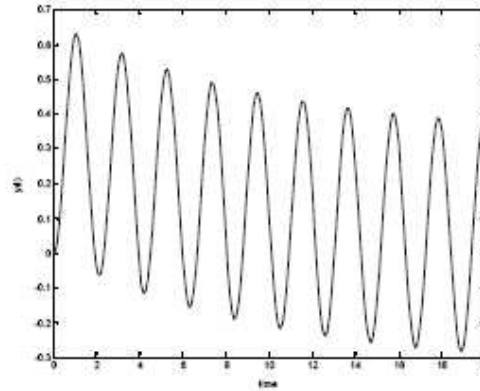


Fig S4.2d. Output for part g)

Figure 5: Question 7

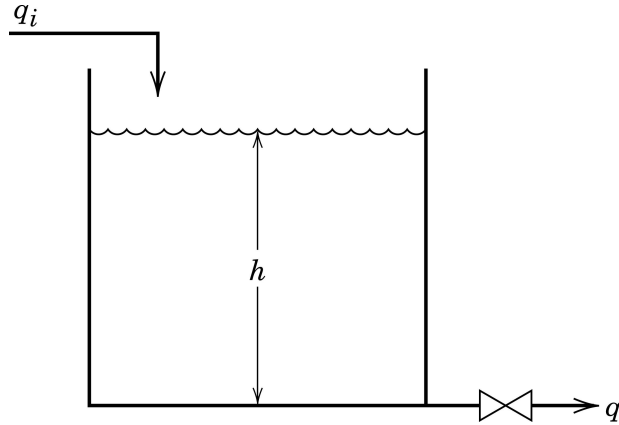


Figure 6: Question 8

(b) Because $q = C_v h^{1/2}$,

$$q' = C_v \frac{1}{2} \bar{h}^{-1/2} h' = \frac{C_v}{2\bar{h}^{1/2}} h' = \frac{1}{K} h'$$

$$\therefore \frac{Q'(s)}{H'(s)} = \frac{1}{K}, \quad \frac{Q'(s) H'(s)}{H'(s) Q_i'(s)} = \frac{1}{K} \frac{K}{\tau s + 1}$$

and

$$\frac{Q'(s)}{H'(s)} = \frac{1}{\tau s + 1}$$

(c) For a linear outflow relation

$$A \frac{dh}{dt} = q_i - C_v^* h \quad \text{Note that } C_v^* \neq C_v$$

$$A \frac{dh'}{dt} = q_i' - C_v^* h'$$

$$A \frac{dh'}{dt} + C_v^* h' = q_i'$$

$$\frac{A}{C_v^*} \frac{dh'}{dt} + h' = \frac{1}{C_v^*} q_i'$$

Multiplying numerator and denominator by \bar{h} on each side yields

$$\frac{A\bar{h}}{C_v^*\bar{h}} \frac{dh'}{dt} + h' = \frac{\bar{h}}{C_v^*\bar{h}} q_i'$$

or

$$\frac{\bar{V}}{\bar{q}_i} \frac{dh'}{dt} + h' = \frac{\bar{h}}{\bar{q}_i} q_i'$$

$$\tau^* = \frac{\bar{V}}{\bar{q}_i} \quad K^* = \bar{h} \bar{q}_i$$

To put τ and K in comparable terms for the square root outflow form of the transfer function, multiply numerator and denominator of each by $\bar{h}^{1/2}$.

$$K = \frac{2\bar{h}^{1/2} \bar{h}^{1/2}}{C_v \bar{h}^{1/2}} = \frac{2\bar{h}}{C_v \bar{h}^{1/2}} = \frac{2\bar{h}}{\bar{q}_i} = 2K^*$$

$$\tau = \frac{2A\bar{h}^{1/2} \bar{h}^{1/2}}{C_v \bar{h}^{1/2}} = \frac{2A\bar{h}}{C_v \bar{h}^{1/2}} = \frac{2\bar{V}}{\bar{q}_i} = 2\tau^*$$

Thus level in the square root outflow TF is twice as sensitive to changes in q_i and reacts only 1/2 as fast (two times more slowly) since $\tau = 2\tau^*$.